

Complex Dynamics of a Gause Predator-prey Model: Impact of the Triple Effects of Fear, Cooperative Hunting and Fractional Order

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ABSTRACT

A Gause-type predator-prey framework with the triple impacts of fear, cooperative hunting, and fractional order is presented in the current work. We study how fractional order, cooperative hunting intensity, and degree of fear affect the model's dynamics. Since these fractional-order models describe typical dynamical frameworks in the applied sciences, their inclusion and the related long-term memory greatly characterise the model's behaviour. It is shown that solutions exist, are unique, and are bounded. Additionally, the resilience of the positive equilibrium and the presence of Hopf bifurcation are explored. According to the study, coexisting equilibria may be impacted by memory, cooperative hunting intensity, and fear level. We also found that any increase in hunting cooperation and fear levels leads to the disappearance of the predator size. Thus, fear and cooperation are crucial for maintaining the stability of ecosystems. In conclusion, the primary findings are confirmed using numerical simulations utilising the MATLAB software, and their biological consequences are examined. It is shown that the non-integer order derivative stabilises and may be helpful in managing interspecies cohabitation.

Keywords: Predator-prey framework, fear effect, cooperative hunting, fractional derivative, stability, Hopf bifurcation.

Mathematics Subject Classification (MSC): 34D20, 34D23, 9210, 93D05.

1. INTRODUCTION

The investigation of relationships among predators and prey is a prominent focus in ecological mathematics research, since these interactions are the most important factors in determining species behaviour, community species composition, and their dynamics in natural ecosystems. A useful technique for getting a thorough grasp of the dynamic properties of the system is mathematical modelling. A large number of biologists and mathematicians have studied predator-prey interactions in great detail in order to shed light on the implications of these connections on ecosystems. Lotka and Volterra are credited with the first groundbreaking studies of predator-prey interactions (Lotka, 1926, Volterra, 1926), which often result in stable coexistence and population

cycles. Since then, in order to represent real experimental phenomena or biological processes under a range of scenarios, the Lotka–Volterra framework has been continually enhanced by adding different growth function types and functional responses based on different populations (Arancibia-Ibarra et al., 2021; Beddington, 1975; Cantrell and Cosner, 2001; DeAngelis, 1975; Hassell and Varley, 1969; Holling, 1959; Holling, 1965; Tripathi et al, 2016).

A typical mathematical scheme of a predator-prey framework is the one elucidated by Gause, which is structured as follows (Kot, 2011)

$$\begin{aligned}\frac{du}{dt} &= ru \left(1 - \frac{u}{k}\right) - v\varphi(\cdot), \\ \frac{dv}{dt} &= cv\varphi(\cdot) - mv,\end{aligned}\tag{1}$$

where u and v symbolise the amount of prey and predators, correspondingly, r , the per capita inherent amount of increase of prey, k , the maximal allowable amount of prey, c , the process of converting amount, m , the predator level of mortality, and $\varphi(\cdot)$ the functional response, which denotes to the predator's amount of prey intake, are all represented. Numerous parameters, including handling duration, encounter rate, and population density, often influence the functional response. Population dynamics modelling has made substantial use of Holling II functional response $\varphi(u) = \frac{eu}{1+ehu}$, one of the functional responses. where h is the processing time and e is the encounter rate. This is a bounded nonlinear function that increases monotonically.

Cooperation is a widespread habit exhibited in numerous biological groups. It makes an extensive effect on the population growth rate within an ecological community. When focusing on the predator-prey relationship, several processes have been discovered that enable the prey population to reproduce, spawn, find food, and defend themselves. Each of these has the potential to trigger a demographic Allee effect (Courchamp et al, 2008). Among other interactions, hunting cooperation in predators may also have Allee effects; varying degrees of this cooperative behaviour affect both species' chances of survival and alter the permanency of the ecological system. Based on the aforementioned Holling type-II functional response, Alves and Hilker (Alves and Hilker, 2017) proposed the following predator-dependent functional response to represent predators' cooperative hunting:

$\varphi(u, v) = \frac{(e+av)u}{1+h(e+av)u}$. The quantity a indicates the degree to which predators hunt

cooperatively. The rationale for the modelling is that the predators' encounter rate with their prey is no longer the same due to their cooperative hunting. However, it is important to note that it is a function that increases as predator population size rises. This function is similar to the encounter-driven functional response proposed by Berec (Berec, 2010). The comparison experiments established that the inclusion of cooperative hunting in framework (1) results in much more intricate and diverse dynamics, as shown in both Lotka-Volterra type and Holling II type functional responses.

The second premise of our argument is that predators reduce the rate of reproduction among prey populations. We multiply the prey birth pace (r) by a reducing function of predator density

$\phi(\theta, v) = \frac{1}{(1+\theta v)}$ in order to account for the fear phenomena. The parameter $\theta(>0)$ indicates the

measure of fear that causes a decline in the reproduction pace of the prey. As a consequence, the

model (1), which depicts cooperative conduct from predators while hunting, causes fear in the prey population, which lowers the prey's birth rate because of the fear of potential predation. Next, the model (1) becomes into

$$\begin{aligned}\frac{du}{dt} &= \frac{ru}{(1+\theta v)} \left(1 - \frac{u}{k}\right) - \frac{(e+av)u}{1+h(e+av)u} v, \\ \frac{dv}{dt} &= c \frac{(e+av)u}{1+h(e+av)u} v - mv.\end{aligned}\quad (2)$$

Employing non-integer order differential equations for modelling prey-predator frameworks may provide several advantages. Non-integer order frameworks' nonlocal feature is contextual, which means they rely on both their current and prior states. A traditional integer-order framework's conversion efficiency to a non-integer-order framework depends on the differentiation order ζ . However, the solutions' behaviour may be drastically altered by a marginal change in ζ (Yousef and Bozkurt, 2019). Although integer-order differential equations (IDEs) are technically incapable of representing complicated biological processes including non-linear behaviour and long-term memory, fractional-order differential equations (FDEs) are able to do so. In order to account for the delay period, often known as the memory effect, Bozkurt used a fractional order differential equation method to develop the glioblastoma multiform (GBM)-immune system (IS) interaction in (Bozkurt, 2014). In numerous publications (Barman et al, 2021; Choi et al, 2014; Dijlali and Ghanbari 2021; Edelstein-Keshet, 2005; Hong et al, 2017; Odibat and Momani, 2008; Ozturk and Ozkose, 2020; Wei et al, 2010; Wen, 2005), the application of non-integer derivatives to model biological processes, including genetic traits, species memory acquired throughout the life cycle, freedom to order the derivative, and many others, has been illustrated. We enhance model (2) to a fractional-order form in light of this information so as to illustrate the research in a more natural setting which corresponds to the requirements of the food chain life cycle. Consequently, the non-integer order ecological framework, including

$$\begin{aligned}{}^c D_t^\zeta u(t) &= \frac{ru}{(1+\theta v)} \left(1 - \frac{u}{k}\right) - \frac{(e+av)u}{1+h(e+av)u} v, \\ {}^c D_t^\zeta v(t) &= c \frac{(e+av)u}{1+h(e+av)u} v - mv.\end{aligned}\quad (3)$$

From an ecological perspective, the changing aspects of predator-prey interactions in the context of cooperative hunting and the subsequent emergence of anti-predator behaviour within the population are very consequential. Researchers have lately focused a great deal of emphasis on the roles that hunting cooperation and anti-predator conduct play in the predator-prey system. However, research on how hunting cooperation and fear interact with a fractional order predator-prey framework has not yet been done. Using the non-integer order predator-prey framework (3), the current research aims to explore the threefold influence of fractional order, predator foraging cooperation, and predator-induced fear on the prey population.

Highlights

- The investigation of non-integer order predator-prey framework considers the collective impact of predators' fear and cooperative hunting.

- By using Matignon's criterion, we ascertain the equilibrium points of the framework and analyse the prerequisites that govern their occurrence and resilience.
- To identify the behaviours of the system, an effective numerical technique is used.
- The mathematical conclusions are explained using various kinds of graphical representations.

The article is organised in the following structure. The primary requirements for fractional calculus are covered in Section 2, since they are important for understanding the primary findings of this model. In Section 3, it is shown that the solutions are bounded, exist, and are unique. In Section 4, the model's equilibria and local stability analysis are examined, and the presence of Hopf bifurcation criterion is given. Numerical simulations are used in Section 5 to demonstrate the theoretical findings. Finally, the work is concluded in Section 6.

2. PRELIMINARIES

This section aims to cover certain fundamental features of non-integer order differential equations, since they will be useful in subsequent parts.

Definition 1 (Diethelm and Ford, 2002) When $n-1 < \zeta < n$, n is a natural number and for a function f belonging to $L^1(R^+)$, the Caputo non-integer derivative of order $\zeta > 0$ is provided by $D_t^\zeta f(t) = \frac{1}{\Gamma(n-\zeta)} \int_0^t \frac{f^n(\vartheta)}{(t-\vartheta)^{\zeta+1-n}} d\vartheta$, where $f(t)$ is absolutely continuous derivative up to the order $n-1$.

Lemma 1. Given $0 < \zeta < 1$, A, B are members of $\mathcal{R}$, $A > 0, t_0 \geq 0$ and $u(t)$ is a continuous functions on $[t_0, \infty)$ fulfilling $D^\zeta u(t) \leq -Au(t) + B, u(t_0) = u_0$. Consequently, for every given t the following is true: $u(t) \leq \left(u_0 - \frac{B}{A}\right) E_\zeta \left[-A(t-t_0)^\zeta\right] + \frac{B}{A}$, in which $E_\zeta(\cdot)$ represents the Mittag–Leffler function, and is given by $E_\zeta = \sum_{i=0}^{\infty} \frac{x^i}{\Gamma(\zeta i + 1)}$.

Theorem 1 (Matignon, 1996) Let $\frac{d^\zeta x}{dt^\zeta} = Bx; x(0) = x_0$ be the non-integer order linear differential equation with $0 < \zeta \leq 1, x \in R^n$ and B is an arbitrary matrix of size $n \times n$. Consequently, possible scenarios exist:

- Only when all of the latent values $\lambda_j, (j=1, 2, \dots, n)$ of B meet $|\arg(\lambda_j)| > \frac{\zeta\pi}{2}$, then the solution $x=0$ is asymptotically stable.
- All of B 's latent values $\lambda_j, (j=1, 2, \dots, n)$ that meet $|\arg(\lambda_j)| > \frac{\zeta\pi}{2}$ and those $|\arg(\lambda_j)| = \frac{\zeta\pi}{2}$ must possess the identical geometric and algebraic multiplicity for the solution $x=0$ to be stable.

Theorem 2 (Petráš, 2011) Let $\frac{d^\zeta x}{dt^\zeta} = f(x); x(0) = x_0$, be the non-integer order differential equation with $0 < \zeta \leq 1$ and $x \in R^n$. The framework's equilibrium points, which can be found by solving

$f(x) = 0$, are locally asymptotically stable if all of the latent values λ_j of the community matrix $J = \frac{\partial f}{\partial x}$

that are found at the equilibrium points meet the criterion as $|\arg(\lambda_j)| > \frac{\xi\pi}{2}$, $j = 1, 2, \dots, n$.

3. BOUNDEDNESS AND EXISTENCE, UNIQUENESS

This section looks at the system's solution's boundedness, existence, and uniqueness (3).

3.1. Boundedness

We show the system's solution (3) is both bounded and positive. Let

$$\mathcal{R}_+^2 = \{U(t) \in \mathcal{R}^2 : U(t) \geq 0\} \text{ and } U(t) = \begin{pmatrix} u(t) \\ v(t) \end{pmatrix}.$$

Theorem 3. *The solution of the non-integer order framework (3) is bounded in $\mathcal{R}_+^2$. Furthermore, population density stays within a nonnegative range.*

Proof. Assume the function $S(t) = u(t) + \frac{1}{c}v(t)$ (4)

and w be a positive constant. Applying the non-integer derivative to each side of (4) and making use of (3) leads to the conclusion that

$$\begin{aligned} D_t^\zeta S(t) + wS(t) &= D_t^\zeta u(t) + \frac{1}{c}D_t^\zeta v(t) + w\left(u(t) + \frac{1}{c}v(t)\right) \\ &= \frac{ru}{(1+\theta v)}\left(1 - \frac{u}{k}\right) - \frac{(e+av)u}{1+h(e+av)u}v + \frac{(e+av)u}{1+h(e+av)u}v - \frac{m}{c}v + w\left(u(t) + \frac{1}{c}v(t)\right) \\ &\leq -\frac{r}{k}\left(u - \frac{k(r+w)}{2r}\right)^2 + \frac{k(r+w)^2}{4r} \quad \text{where } w < 1. \\ D_t^\zeta S(t) + wS(t) &\leq \frac{k(r+w)^2}{4r}, \end{aligned}$$

Using Lemma1, it follows that

$$\begin{aligned} S(t) &\leq \left(S(t_0) - \frac{k(r+w)^2}{4r}\right)E_\zeta(-wt^\zeta) + \frac{k(r+w)^2}{4r}, \\ &\leq S(t_0)E_\zeta(-wt^\zeta) + \frac{k(r+w)^2}{4r}\left(1 - E_\zeta(-wt^\zeta)\right). \end{aligned} \quad (5)$$

As $t \rightarrow \infty$, it may be deduced that $S(t) \leq \frac{k(r+w)^2}{4r}$. As a result, all (3) solutions that begin with $\mathcal{R}_+^2$ are limited to the domain Λ that follows,

$$\Lambda = \left\{ (u(t), v(t)) \in \mathcal{R}_+^2 : u(t) + \frac{1}{c}v(t) \leq \frac{k(r+w)^2}{4r} + \delta, \forall \delta > 0 \right\} \text{ which is positively invariant.}$$

3.2. Existence and Uniqueness

Here, we use the fixed-point approach to determine the conditions under which the solution to the non-integer order prey-predator framework (3) must exist and be unique inside the selected area $\Phi \times (t_0, T], T < \infty$, where $\Phi = \{(u, v) \in \mathcal{R}^2 : \max(|u|, |v|) \leq M\}$. The boundedness of the solution ensures that M exists. The fractional integral form of the model may be used to rewrite it, giving

$$\begin{aligned} u(t) - u(t_0) &= \frac{1}{\Gamma(\zeta)} \int_{t_0}^t (t-\eta)^{\zeta-1} \left(\frac{ru(\eta)}{1+\theta v(\eta)} \left(1 - \frac{u(\eta)}{k} \right) - \frac{(e+av(\eta))u(\eta)}{1+h(e+av(\eta))u(\eta)} v(\eta) \right) d\eta, \\ v(t) - v(t_0) &= \frac{1}{\Gamma(\zeta)} \int_{t_0}^t (t-\eta)^{\zeta-1} \left(c \frac{(e+av(\eta))u(\eta)}{1+h(e+av(\eta))u(\eta)} v(\eta) - mv(\eta) \right) d\eta. \end{aligned} \quad (6)$$

Denoting by Ω_1 and Ω_2 the following kernels

$$\begin{aligned} \Omega_1(t, u(t), v(t)) &= \frac{ru(t)}{(1+\theta v(t))} \left(1 - \frac{u(t)}{k} \right) - \frac{(e+av(t))u(t)}{1+h(e+av(t))u(t)} v(t), \\ \Omega_2(t, u(t), v(t)) &= c \frac{(e+av(t))u(t)}{1+h(e+av(t))u(t)} v(t) - mv(t), \end{aligned} \quad (7)$$

the theorem mentioned below is valid.

Theorem 4. Let $M_1 = \left(\frac{r}{(1+\theta M)} + \frac{2rM}{k(1+\theta M)} + \frac{(e+aM)M}{(1+h(e+aM)M)^2} \right)$ and

$$M_2 = \left(\frac{cM((1+heM)(e+2aM)+a^2hM^3)}{(1+h(e+aM)M)^2} + m \right). \text{ The kernels } \Omega_i(t, u, v), (i=1, 2) \text{ comply with the}$$

Lipschitz and contraction criteria in the area $\Phi \times (t_0, T]$ for $0 < M_i < 1, (i=1, 2)$.

Proof. Let $u(t_1) = u_1$ and $u(t_2) = u_2$ be two functions for the kernel Ω_1 and $v(t_1) = v_1, v(t_2) = v_2$ be two functions that correspond to the kernel Ω_2 . As a result,

$$\begin{aligned} \|\Omega_1(t, u_1, v) - \Omega_1(t, u_2, v)\| &= \left\| \frac{ru_1}{(1+\theta v)} \left(1 - \frac{u_1}{k} \right) - \frac{(e+av)u_1}{1+h(e+av)u_1} v - \frac{ru_2}{(1+\theta v)} \left(1 - \frac{u_2}{k} \right) + \frac{(e+av)u_2}{1+h(e+av)u_2} v \right\| \\ &\leq \left(\frac{r}{(1+\theta M)} + \frac{2rM}{k(1+\theta M)} + \frac{(e+aM)M}{(1+h(e+aM)M)^2} \right) \|u_1 - u_2\| \\ &\leq M_1 \|u_1 - u_2\| \end{aligned} \quad (8)$$

and

$$\|\Omega_2(t, u, v_1) - \Omega_2(t, u, v_2)\| = \left\| c \frac{(e+av_1)u}{1+h(e+av_1)u} v_1 - mv_1 - c \frac{(e+av_2)u}{1+h(e+av_2)u} v_2 + mv_2 \right\|$$

$$\begin{aligned}
 &\leq \left(\left(\frac{cM}{(1+h(e+aM)M)^2} \right) \left((1+heM)(e+2aM) + a^2hM^3 \right) + m \right) \|v_1 - v_2\| \\
 &\leq \left(\frac{cM((1+heM)(e+2aM) + a^2hM^3)}{(1+h(e+aM)M)^2} + m \right) \|v_1 - v_2\| \\
 &\leq M_2 \|v_1 - v_2\|,
 \end{aligned} \tag{9}$$

$$\text{where } M_1 = \left(\frac{r}{(1+\theta M)} + \frac{2rM}{k(1+\theta M)} + \frac{(e+aM)M}{(1+h(e+aM)M)^2} \right),$$

$$M_2 = \left(\frac{cM((1+heM)(e+2aM) + a^2hM^3)}{(1+h(e+aM)M)^2} + m \right). \text{ Consequently, kernels } \Omega_1 \text{ and } \Omega_2 \text{ satisfy the Lipschitz}$$

requirements, and if $0 < M_i < 1$, ($i=1,2$), then M_1 and M_2 are corresponding contractions of Ω_1 and Ω_2 . Let us examine the kernels Ω_1 and Ω_2 , assuming that the requirements (8) and (9) are met. Following that, (6) might be written

$$\begin{aligned}
 u(t) &= u(t_0) + \frac{1}{\Gamma(\zeta)} \int_{t_0}^t (t-\eta)^{\zeta-1} \Omega_1(\eta, u, v) d\eta, \\
 v(t) &= v(t_0) + \frac{1}{\Gamma(\zeta)} \int_{t_0}^t (t-\eta)^{\zeta-1} \Omega_2(\eta, u, v) d\eta.
 \end{aligned} \tag{10}$$

The model (10) has the following beginning conditions and recurrence form, are as follows:

$$u_0(t) = u(t_0), v_0(t) = v(t_0) \tag{11}$$

and

$$\begin{aligned}
 u_n(t) &= u(t_0) + \frac{1}{\Gamma(\eta)} \int_{t_0}^t (t-\eta)^{\zeta-1} \Omega_1(\eta, u_{n-1}, v) d\eta, \\
 v_n(t) &= v(t_0) + \frac{1}{\Gamma(\eta)} \int_{t_0}^t (t-\eta)^{\zeta-1} \Omega_2(\eta, u, v_{n-1}) d\eta.
 \end{aligned} \tag{12}$$

Next, we define the terms' sequential difference as

$$\begin{aligned}
 \psi_{1n}(t) &= u_n(t) - u_{n-1}(t) = \frac{1}{\Gamma(\zeta)} \int_{t_0}^t (t-\eta)^{\zeta-1} [\Omega_1(\eta, u_{n-1}, v) - \Omega_1(\eta, u_{n-2}, v)] d\eta, \\
 \psi_{2n}(t) &= v_n(t) - v_{n-1}(t) = \frac{1}{\Gamma(\zeta)} \int_{t_0}^t (t-\eta)^{\zeta-1} [\Omega_2(\eta, u, v_{n-1}) - \Omega_2(\eta, u, v_{n-2})] d\eta,
 \end{aligned} \tag{13}$$

$$\text{where } u_n(t) = \sum_{i=0}^n \psi_{1i}(t), v_n(t) = \sum_{i=0}^n \psi_{2i}(t). \tag{14}$$

We may deduce from conditions (8) through (9) that, using the norm of (13),

$$\begin{aligned}\|\psi_{1n}(t)\| &= \|u_n(t) - u_{n-1}(t)\| \leq \frac{M_1}{\Gamma(\zeta)} \int_{t_0}^t (t-\eta)^{\zeta-1} \|\psi_{1(n-1)}(\eta)\| d\eta, \\ \|\psi_{2n}(t)\| &= \|v_n(t) - v_{n-1}(t)\| \leq \frac{M_2}{\Gamma(\zeta)} \int_{t_0}^t (t-\eta)^{\zeta-1} \|\psi_{2(n-1)}(\eta)\| d\eta.\end{aligned}\quad (15)$$

The subsequent theorem is valid.

Theorem 5. Suppose that (8) through (9) are true. The unique solution to the non-integer model presented in (3) exists if $\frac{M_i T^\zeta}{\Gamma(\zeta+1)} < 1$, $i = 1, 2$,

(16)

Proof. Suppose that the functions $u(t)$ and $v(t)$ are bounded along with the kernels Ω_1 and Ω_2 satisfy the Lipschitz condition. From (14) and (15), it follows that

$$\begin{aligned}\|\psi_{1n}(t)\| &\leq \|u_{(0)}(t)\| \left\{ \frac{M_1 T^\zeta}{\Gamma(\zeta+1)} \right\}^n, \\ \|\psi_{2n}(t)\| &\leq \|v_{(0)}(t)\| \left\{ \frac{M_2 T^\zeta}{\Gamma(\zeta+1)} \right\}^n.\end{aligned}\quad (17)$$

This implies the presence of possible solutions. In addition, to establish that the values (17) satisfy the equation (3), we consider

$$\begin{aligned}u(t) - u(t_0) &= u_{(n)}(t) - l_{1(n)}(t), \\ v(t) - v(t_0) &= v_{(n)}(t) - l_{2(n)}(t),\end{aligned}\quad (18)$$

where $l_{1(n)}(t), l_{2(n)}(t)$ represent the terms that remain. We demonstrate that the terms in equation (18) fulfil the conditions $\|l_{1(n)}(t)\| \rightarrow 0$ and $\|l_{2(n)}(t)\| \rightarrow 0$. We now proceed to examine the conditions

$$\begin{aligned}\|l_{1(n)}(t)\| &\leq \left\| \frac{1}{\Gamma(\zeta)} \int_{t_0}^t (t-\eta)^{\zeta-1} [\Omega_1(\eta, u, v) - \Omega_1(\eta, u_{(n-1)}, v)] d\eta \right\| \\ &\leq \frac{1}{\Gamma(\zeta)} \int_{t_0}^t (t-\eta)^{\zeta-1} \|\Omega_1(\eta, u, v) - \Omega_1(\eta, u_{(n-1)}, v)\| d\eta, \\ &\leq \frac{M_1 T^\zeta}{\Gamma(\zeta+1)} \|u - u_{(n-1)}\|.\end{aligned}\quad (19)$$

By employing recursive approaches, we obtain

$$\|l_{1(n)}(t)\| \leq \left\{ \frac{T^\zeta}{\Gamma(\zeta+1)} \right\}^{n+1} M_1^{n+1}.$$

When $n \rightarrow \infty$, it is evident that $l_{1(n)}(t) \rightarrow 0$. Likewise, it can be deduced that $l_{2(n)}(t) \rightarrow 0$.

While establishing that the solution to model (3) is unique, we will assume that there is another solution to the system, denoted as $\hat{u}(t)$ and $\hat{v}(t)$. Subsequently

$$\begin{aligned}\|u(t) - \hat{u}(t)\| &\leq \frac{1}{\Gamma(\zeta)} \int_{t_0}^t (t-\eta)^{\zeta-1} \|\Omega_1(\eta, u, v) - \Omega_1(\eta, \hat{u}, v)\| d\eta, \\ \|v(t) - \hat{v}(t)\| &\leq \frac{1}{\Gamma(\zeta)} \int_{t_0}^t (t-\eta)^{\zeta-1} \|\Omega_2(\eta, u, v) - \Omega_2(\eta, u, \hat{v})\| d\eta.\end{aligned}\quad (20)$$

Given the Lipschitz condition, it can be deduced that

$$\begin{aligned}\|u(t) - \hat{u}(t)\| &\leq \frac{M_1 t^\zeta}{\Gamma(\zeta+1)} \|u(t) - \hat{u}(t)\|, \\ \|v(t) - \hat{v}(t)\| &\leq \frac{M_2 t^\zeta}{\Gamma(\zeta+1)} \|v(t) - \hat{v}(t)\|,\end{aligned}\quad (21)$$

and hence

$$\begin{aligned}\|u(t) - \hat{u}(t)\| \left(1 - \frac{M_1 t^\zeta}{\Gamma(\zeta+1)}\right) &\leq 0, \\ \|v(t) - \hat{v}(t)\| \left(1 - \frac{M_2 t^\zeta}{\Gamma(\zeta+1)}\right) &\leq 0,\end{aligned}\quad (22)$$

Considering (16), it is evident that $\|u(t) - \hat{u}(t)\| = 0$ and $\|v(t) - \hat{v}(t)\| = 0$.

4. EQUILIBRIA AND STABILITY ANALYSIS

System (3)'s stationary points may be found by taking $D_t^\zeta u(t) = 0$, $D_t^\zeta v(t) = 0$.

$$\begin{aligned}\frac{ru}{(1+\theta v)} \left(1 - \frac{u}{k}\right) - \frac{(e+av)u}{1+h(e+av)u} v &= 0, \\ c \frac{(e+av)u}{1+h(e+av)u} v - mv &= 0.\end{aligned}$$

The system admits the trivial equilibrium $E_0 \equiv (0, 0)$, the predator-free equilibrium $E_1 \equiv (k, 0)$ and the interior equilibrium $E^* \equiv (u^*, v^*)$, where $u^* = \frac{m}{(e+av)(c-mh)} > 0$ if $c > mh$ and v^* is a positive root of the equation

$$a_0 v^5 + a_1 v^4 + a_2 v^3 + a_3 v^2 + a_4 v + a_5 = 0, \quad (23)$$

where

$$\begin{aligned}a_0 &= a^3 k \theta (c - mh), a_1 = a^2 k (a + 3e\theta) (c - mh), a_2 = 3aek (a + e\theta) (c - mh), \\ a_3 &= ke^2 (c - mh) (3a + e\theta) - a^2 r k h m, a_4 = k (c - mh) (e^3 - ar) - amr (2hke - mh - 1), \text{ and} \\ a_5 &= erm (mh - khe + 1) - erk (c - mh).\end{aligned}$$

Since all the parameters in this case are positive, a_0, a_1 , and a_2 are also positive. Therefore, depending on the sign of a_3, a_4 , and a_5 , equation (23) may have up to three positive roots.

4.1 Local Stability

Using the latent values of the Jacobian matrix and applying the Matignon Local Stability Theorem in Petras (Petráš, 2011), we establish that the stationary points of the framework (3) are locally resilient. Assume that system (3)'s equilibrium point is E^* . For E^* to be locally asymptotically resilient, the Matignon Local Stability Theorem states that every latent value λ_i of the community matrix

$$J(E^*) = \begin{pmatrix} \frac{r}{1+\theta v} \left(1 - \frac{2u}{k}\right) - \frac{(e+av)v}{(1+h(e+av)u)^2} & -\frac{ru\theta}{(1+\theta v)^2} \left(1 - \frac{u}{k}\right) - \frac{u((e+av)(1+h(e+av)u)+av)}{(1+h(e+av)u)^2} \\ \frac{c(e+av)v}{(1+h(e+av)u)^2} & \frac{cu((e+av)(1+h(e+av)u)+av)}{(1+h(e+av)u)^2} - m \end{pmatrix}$$

must satisfies $|\arg(\lambda_i)| > \frac{\zeta\pi}{2}$.

Theorem 6. The trivial equilibrium point $E_0(0,0)$, remains a saddle point.

Proof. The community matrix for $E_0(0,0)$ is $J(E_0) = \begin{pmatrix} r & 0 \\ 0 & -m \end{pmatrix}$.

The latent values of $J(E_0)$ are obtained as $\lambda_1 = r > 0$ and $\lambda_2 = -m < 0$. Of these two values, one is positive with $|\arg(\lambda_1)| = 0$ and other is non-positive with $|\arg(\lambda_2)| = \pi$. As a result, the first latent value conforms to the criteria $|\arg(\lambda_i)| > \frac{\zeta\pi}{2}$, while the second fulfils $|\arg(\lambda_i)| < \frac{\zeta\pi}{2}$. Hence, E_0 remains a saddle point.

Theorem 7. The equilibrium point $E_1(k,0)$ is locally asymptotically stable if $c < \frac{m(1+hek)}{ke}$.

Proof. The community matrix for E_1 is $J(E_1) = \begin{pmatrix} -r & -\frac{ke}{(1+hek)} \\ 0 & \frac{cke}{(1+hek)} - m \end{pmatrix}$

The latent values of the community matrix are $\lambda_1 = -r$ and $\lambda_2 = \frac{cke}{(1+hek)} - m$, which is negative if $c < \frac{m(1+hek)}{ke}$. Thus, we get $|\arg(\lambda_1)| = |\arg(\lambda_2)| = \pi$, hence requirement that $|\arg(\lambda_i)| > \frac{\zeta\pi}{2}$, $i = 1, 2$ is attained. Therefore, E_1 is asymptotically stable if $c < \frac{m(1+hek)}{ke}$.

Theorem 8. The positive equilibrium point $E^* = (u^*, v^*)$ is locally asymptotically stable, as long as any of the following criteria are true.

i) $\text{trace}(J(E^*)) \leq 0$,

ii) $\text{trace}(J(E^*)) > 0$, $\text{trace}^2(J(E^*)) - 4\det(J(E^*)) < 0$ and

$$\sqrt{\text{trace}^2(J(E^*)) - 4\det(J(E^*))} > \text{trace}(J(E^*)) \tan\left(\frac{\zeta\pi}{2}\right).$$

Proof. The community matrix at $E^* = (u^*, v^*)$ can obtained for system (3) is

$$J(E^*) = \begin{pmatrix} -\frac{ru^*}{k(1+\theta v^*)} - \frac{hq^2 u^* v^*}{(1+hu^*)^2} & -\frac{q\theta u^* v^*}{(1+\theta v^*)(1+hu^*)} - \frac{u^* q(1+hu^*) - au^* v^*}{(1+hu^*)^2} \\ \frac{cq v^*}{(1+hu^*)^2} & \frac{cau^* v^*}{(1+hu^*)^2} \end{pmatrix} \quad (24)$$

where, $j_{11} = -\frac{ru^*}{k(1+\theta v^*)} - \frac{hq^2 u^* v^*}{(1+hu^*)^2}$, $j_{12} = -\frac{q\theta u^* v^*}{(1+\theta v^*)(1+hu^*)} - \frac{u^* q(1+hu^*) - au^* v^*}{(1+hu^*)^2}$,

$j_{21} = \frac{cq v^*}{(1+hu^*)^2}$, $j_{22} = \frac{cau^* v^*}{(1+hu^*)^2}$ and $q = (e + av^*)$.

Now, the latent equation of the community matrix is $\lambda^2 - T\lambda + D = 0$ and its roots are given by

$$\lambda_{1,2} = \frac{T \pm \sqrt{T^2 - 4D}}{2}, \quad (25)$$

where T and D are the trace and determinant of the community matrix $J(E^*)$ given by

$$T = -\frac{ru^*}{k(1+\theta v^*)} - \frac{hq^2 u^* v^*}{(1+hu^*)^2} + \frac{cau^* v^*}{(1+hu^*)^2},$$

$$D = \left(-\frac{ru^*}{k(1+\theta v^*)} - \frac{hq^2 u^* v^*}{(1+hu^*)^2} \right) \left(\frac{cau^* v^*}{(1+hu^*)^2} \right) - \left(\frac{cq v^*}{(1+hu^*)^2} \right) \left(-\frac{q\theta u^* v^*}{(1+\theta v^*)(1+hu^*)} - \frac{u^* q(1+hu^*) - au^* v^*}{(1+hu^*)^2} \right)$$

The investigation of the resilience of E^* will be categorised into the subsequent four scenarios:

(1) When $T^2 - 4D \geq 0$ along with $T < 0$ imply that λ_i ($i=1,2$) are negative real numbers. Therefore,

$|\arg \lambda_i| = \pi > \frac{\zeta\pi}{2}$. The interior equilibrium is asymptotically stable.

(2) When $T^2 - 4D \geq 0$ along with $T \geq 0$ indicate that at least one of λ_1, λ_2 is positive. Consequently, $|\arg \lambda_i| = 0 < \frac{\zeta\pi}{2}$ for at least one of the elements in λ_i , resulting in an unstable interior equilibrium point.

(3) When $T^2 - 4D < 0$, then the latent values are couple of complex conjugates $\lambda_{1,2} = \frac{T}{2} \pm i \frac{\sqrt{4D - T^2}}{2}$.

(a) If $T > 0$ then $\operatorname{Re}(\lambda_1) = \operatorname{Re}(\lambda_2) > 0$ and $\operatorname{Im}(\lambda_1) = -\operatorname{Im}(\lambda_2) = \frac{\sqrt{4D - T^2}}{2} > 0$. Then, when $\frac{\sqrt{4D - T^2}}{T} > \tan \frac{\zeta\pi}{2}$, the interior equilibrium is asymptotically stable. In any other case, it will cause instability.

(b) If $T < 0$ then $\operatorname{Re}(\lambda_1) = \operatorname{Re}(\lambda_2) < 0$. Hence $|\arg \lambda_i| > \frac{\pi}{2}$ and the equilibrium is asymptotically resilient.

(4) When $T = 0$ and $T^2 - 4D = -4D < 0$, then $\operatorname{Re}(\lambda_1) = \operatorname{Re}(\lambda_2) = 0$. Hence $|\arg \lambda_i| = \frac{\pi}{2} > \frac{\zeta\pi}{2}$, $i = 1, 2$. The equilibrium will be asymptotically stable under these conditions.

4.2 Existence of Hopf bifurcation

Even with a simple system (two or three species), mathematical models of nonlinear populations interacting display rich and complicated dynamical characteristics. Oscillation is among the most prevalent forms of dynamical behaviour in population dynamics. The Hopf bifurcation occurs when the system exhibits oscillating behaviour or if a limit cycle is present. The emergence or fading of a periodic trajectory due to a local alteration in an equilibrium point's stability characteristics is identified as a Hopf bifurcation. In this instance, we investigate the potential for Hopf bifurcation to occur in relation to the bifurcating parameter ζ around the positive equilibrium point E^* .

Let's examine a non-integer order system

$$D_t^\zeta z(t) = f(\phi, z) \quad (26)$$

where $\zeta \in [0, 1)$, $z \in \mathbb{R}^2$ and let E^* be an interior stationary point. The consideration of a non-integer order Hopf bifurcation in (Petráš 2011, Abdelouahab, 2012) leads us to believe that system (26) undergoes a Hopf bifurcation when the amount of ϕ crosses the value ϕ^* about the equilibria E^* .

i) The latent equation possesses pair of complex conjugate latent values

$$\lambda_{1,2}(\phi) = \nu(\phi) \pm i\pi(\phi),$$

$$\text{ii) } \nu_{1,2}(\zeta, \phi^*) = 0, \text{ where } \nu_{1,2} = \frac{\zeta\pi}{2} - |\arg(\lambda_i(\phi))|,$$

$$\text{iii) } \left. \frac{\partial \nu_{1,2}}{\partial \phi} \right|_{\phi=\phi^*} \neq 0.$$

5. NUMERICAL SIMULATIONS

Here, with the use of MATLAB software, we will talk about the numerical outcomes of system (3). Our earlier findings are confirmed by the simulations we display.

Illustration 1. In order to illustrate the influence that derivatives of non-integer order have on the behaviour of the proposed framework, we use $\zeta = 0.85, 0.96$ and 0.98 . The parameter values that we use are $r = 0.2, k = 0.976, \theta = 0.031, e = 0.246, a = 0.913, h = 0.8, c = 0.954, m = 0.104$. It was observed that there was stability in the dynamic behaviour of both species as the fractional-order derivative ζ approached zero. Obviously, decreasing the non-integer order ζ leads to an even quicker dampening of the trajectories' oscillations, exhibiting the model's long-term memory's strong stabilising impact.

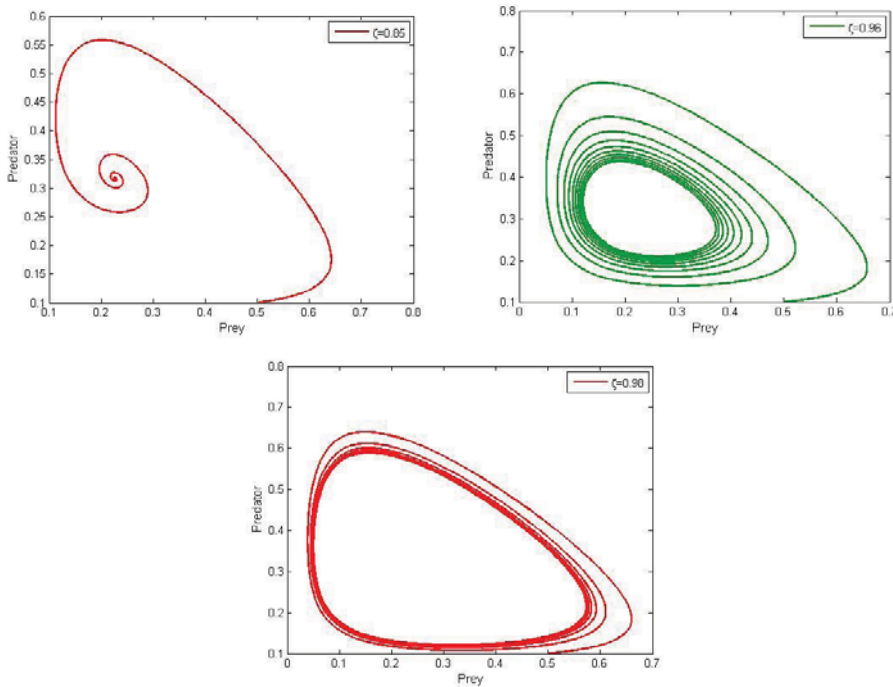


Figure 1. The numerical approximation trajectories of the framework (3)'s solutions for various ζ levels (0.85, 0.96, 0.98). It is evident that fractional operators have a stabilising effect.

Illustration 2. With the parameter values $r = 0.249, k = 0.951, \theta = 0.981, e = 0.146, h = 0.57, c = 0.754, m = 0.065$, we are now aiming to investigate the framework's behaviour using hunting cooperation (a). As we can see in Figure 2, the system becomes unstable as hunting cooperation increases. Furthermore, populations of predators and prey exhibit periodic fluctuations. The system experiences a loss of stability via Hopf bifurcation when the parameter a reaches a value of 0.6. The density of predators increases during periods of poor hunting cooperation and decreases during periods of strong hunting cooperation, as a result of predators overconsuming their prey. Due to

predators' improved ability to feed on prey, the biological ramifications for predator population growth are greater for minimal amounts of a ; however, for high amounts of a , the predator population declines as prey density falls.

Illustration 3. Next, we will examine how fear and the fractional order effect influence the dynamics of the framework (3). We observe that system without fractional order ($\zeta = 1$) is unstable for the including and not including dread effect, remaining parameter values are similar to illustration 2. But scheme is steady for $\zeta = 0.95$ in both cases i.e., including and not including dread effect as shown in Figure 3. Our findings indicate that stabilising focus will supplant population oscillations when the cost of fear is increased. It can be inferred that the steadiness of the scheme is significantly influenced through both the fear impact and the non-integer order.

6. CONCLUSIONS

Understanding the changing behaviour of ecological frameworks can be greatly enhanced by examining the intriguing phenomenon of animal foraging. For the purpose of increasing their densities, predators and prey use various tactics. Predators have developed cooperative hunting behaviours and often reside in groups within natural habitats. In addition to increasing their chances of successfully capturing food, cooperative predators are able to target larger or quicker prey than they are. When the frequency of predator attacks rises, prey animals also exhibit anti-predator behaviours. Prey populations exhibit a range of anti-predator reactions as a result of their fear of potential predators, including as adjustments to habitat utilisation, foraging habits, increased awareness, and physiological changes. It is well known from the literature that when prey animals detect a threat of predators, they might change their feeding locations to safer spots, give up their regions of greatest intake, become more alert, and modify their mating habits. Such a survival approach decreases the pace of reproduction within prey populations. Predator cooperation instills dread in the prey population, which lowers the prey population's birth rate.

This study presents a novel Gause-type predator-prey model including fractional order, cooperative hunting, and fear as triple effects. The linearization method was employed to ascertain the presence and exclusivity of a non-negative solution, as well as to investigate the local permanency of the positive equilibrium point. It has been determined under what circumstances a Hopf bifurcation will occur. In conclusion, a few chosen illustrations of how the order of derivation (ζ), fear, and cooperative hunting influence the dynamic behaviour of prey and predator density have been supported by numerical simulations. As far as we are concerned, the results of this research may provide some guidance for exploring the dynamics of other systems that describe occurrences in life.

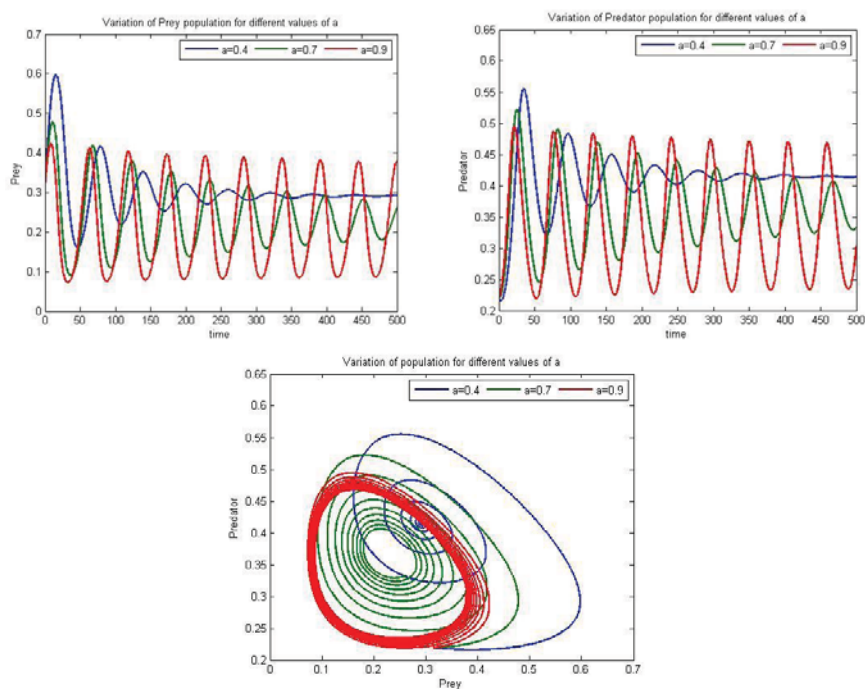


Figure 2. Hunting cooperation destabilize the predator-prey dynamics of the framework (3)

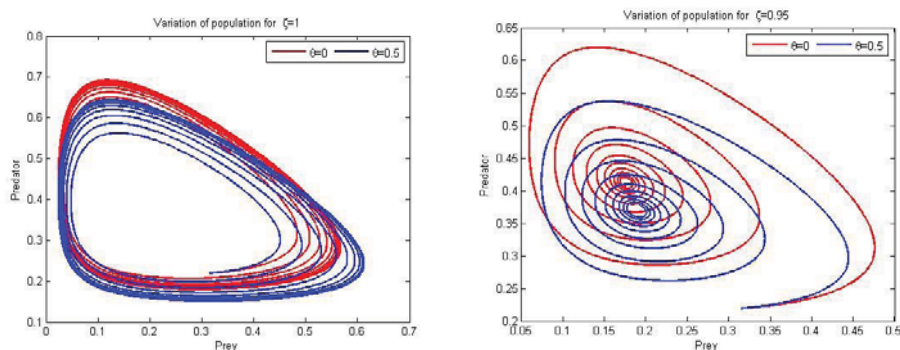


Figure 3. Fear effect and fractional order stabilize the predator-prey dynamics of the system (3)

Declaration of competing interest

The authors state that none of the work presented in this study may have been influenced by any known conflicting financial interests or personal relationships.

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