

Interactive consequence of prey refuge in a tri-trophic food chain model

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ABSTRACT

The present article appertains to a tri-trophic food chain model of the interacting population, including partial prey refuge and intra-specific competition among the predators. The precursory issues, including boundedness and dissipativeness of the system, are established. The stability analyses corresponding to the feasible equilibria of the present system are carried out systematically with special emphasis on global stability. The system appears to undergo Hopf-bifurcation depending on the appropriate set of model parameter values. In the absence of any intra-specific competition, the system exposes chaotic behaviour which is duly confirmed by Lyapunov exponents. On the contrary, such a chaotic response disappears for a specific value of self-interaction. Attention is also focused on the consequences of prey refuge graphically, which possesses a stabilizing effect and can control the system's chaotic dynamics, leading to period-doubling and limit cycles. Finally, numerical simulations are performed for the purpose of model validation regarding its applicability in the realm of ecological food chain systems.

Keywords: Food chain model, Intra-specific competition, Prey refuge, Hopf-bifurcation, Lyapunov function, Chaos, Lyapunov Exponent.

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1 Introduction

In modern times, mathematical modelling of the food chain system emerges to evolve into the focus of attraction to the researchers in the domain of theoretical ecology and applied mathematics. It helps to elevate the level of understanding the process of predation and the stability phenomenon of the ecosystem including its long-term response as well (Sarkar, Ali and Guin, 2021a), (Sarkar, Ali and Guin, 2021b). A food chain is a network connected linearly between prey and predator species. The testimony

of chaos is widespread until now although its manifestation is exceptional in natural population. Various efforts are made to transform the chaotic dynamics into stability and oscillatory phenomenon owing to their scarcity in biological systems. The pioneering work on chaos in a tri-trophic food chain model system undertaken by Hastings and Powell (Hastings and Powell, 1991) has pointed out that chaotic dynamics could exist in natural food webs. The Rosenzweig-MacArthur food chain model revealed a sequence of pairs of bifurcations as noted in (Kuznetsov, De Feo and Rinaldi, 2001). Zhao and Lv (Lv and Zhao, 2008) showed all the cases of stable steady state, periodicity and chaotic behaviour as well from the outcomes of their three species food chain model. Both the occurrences of Hopf-bifurcation and phase transition scenario around positive interior equilibrium in the presence of delay are studied successfully by Chen and Sun (Chen, Yu and Sun, 2007). Boer et al. (Boer, Kooi and Kooijman, 1999) investigated a tri-trophic food chain model to uncover both the local and the global bifurcations around the equilibria and also the limit cycles phenomena of the system. In view of all these studies, the present one is based on that of Hastings and Powell model (Hastings and Powell, 1991) describing the dynamics of prey-dependent food chain model using the following set of differential equations

$$\frac{dX}{dT} = rX \left(1 - \frac{X}{K}\right) - \frac{M_1XY}{A_1 + X}, \quad (1.1a)$$

$$\frac{dY}{dT} = \frac{E_1M_1XY}{A_1 + X} - D_1Y - \frac{M_2YZ}{A_2 + Y}, \quad (1.1b)$$

$$\frac{dZ}{dT} = \frac{E_2M_2YZ}{A_2 + Y} - D_2Z, \quad (1.1c)$$

$$X(0) \geq 0, Y(0) \geq 0, Z(0) \geq 0, \quad (1.1d)$$

where X , Y and Z represent the prey, the middle-predator and the top-predator species respectively at time T . All other parameters, namely r , K , M_i , A_i , E_i ($0 < E_i < 1$) and D_i ($i = 1, 2$) designate respectively the growth rate of prey in absence of predator, the carrying capacity of prey, the maximal predator growth rates, half-saturation constants, the predators' efficiency and the predators death rates for the system (1.1). All these model parameters are assumed to be positive throughout the investigation.

The complexity of the outcomes of the food chain model (1.1) has been extensively studied by researchers (Hastings and Powell, 1991), (Huxel and McCann, 1998) in terms of the occurrence of Hopf-bifurcation and the stability analysis corresponding to its temporal dynamics. Abrams and Roth (Abrams and Roth, 1994) made an effort to investigate the dynamical response of a tri-trophic food chain model in the environment of Holling type II functional response for the consumer species. The prime objective of this study is to demonstrate the emergence of chaotic situation in a class of ecological models. The global stability of the model around its positive interior equilibrium using a general Lyapunov function has been shown by Chiu and Hsu (Chiu and Hsu, 1998). The model (1.1) was revisited by Klebanoff and Hastings (Klebanoff and Hastings, 1994) in order to examine the dynamics of a tri-trophic food chain employing bifurcation theory so as to demonstrate the existence of chaos in the neighborhood of the equilibrium position in the absence of top-predator species. Recently, Haque et al. (Haque, Ali and Chakravarty, 2013) revised the model (1.1) with the inclusion of an additional term specifying intra-specific competition among the predator and top-predator species. Pielou (Pielou, 1969) modified the well-known Lotka-Volterra model with the incorporation of an intra-specific competition term in the predator species. Subsequently, Haque (Haque, 2009) investigated further a ratio-dependent prey-predator interacting species model with the incorporation of an intra-specific competition term in the predator population in order to illustrate how an intra-specific competition among predators could be favourable for the concerned model. The researchers propounded from their investigations that the predators' self-interaction term possesses a stabilizing effect.

Apart from these, refuge is a salient concept in ecology that helps reducing the risk of predation to a considerable extent. Consequently, the existence of refuge not only has importance on prey species but also on predators as well. Several investigations (Ko and Ryu, 2006), (Kar, 2005), (González-

Olivares and Ramos-Jiliberto, 2003), (Huang, Chen and Zhong, 2006) have already been carried out by highlighting the impact of prey refuge to indicate firstly that the refuge has a stabilizing effect on the prey-predator interaction and secondly the prey species can be confined from extinction by employing this policy. The model under proposition takes into account the techniques of dynamical systems (Murray, 2000), (Perko, 2013), (Lynch, 2004) in order to achieve the objectives of the present model system.

The rest of the article is organized as follows. Section 2 includes the formulation of the mathematical model. Certain preliminary results regarding boundedness and dissipativeness are presented in the respective Sections 3 and 4. Section 5 records all the feasible equilibria of the model system. The stability analyses including the local and the global stability around all the possible equilibria are carried out in Section 6. Section 7 employs a series of numerical simulations in order to reveal that the model undertaken can exhibit complexities in temporal dynamics. Finally, some concluding remarks supplemented by ecological relevance are provided in the closing Section 8.

2 Mathematical model construction and its analysis

Looking at the importance of self-interaction, which has a significant role in the prey-predator system, we were influenced to modify the famous Rosenzweig-MacArthur model of tri-trophic food chain model equations (1.1) by incorporating intra-specific competitions among predators. In addition to all, we take one more attempt to analyze the system behaviour in the presence of prey refuge. Consequently, the updated prey-dependent tri-trophic food chain model takes the form

$$\frac{dX}{dT} = rX \left(1 - \frac{X}{K}\right) - \frac{M_1(1 - m_1)XY}{A_1 + (1 - m_1)X}, \quad (2.1a)$$

$$\frac{dY}{dT} = \frac{E_1 M_1(1 - m_1)XY}{A_1 + (1 - m_1)X} - D_1 Y - \frac{M_2(1 - m_2)YZ}{A_2 + (1 - m_2)Y} - H_1 Y^2, \quad (2.1b)$$

$$\frac{dZ}{dT} = \frac{E_2 M_2(1 - m_2)YZ}{A_2 + (1 - m_2)Y} - D_2 Z - H_2 Z^2, \quad (2.1c)$$

$$X(0) \geq 0, \quad Y(0) \geq 0, \quad Z(0) \geq 0, \quad (2.1d)$$

where m_1, m_2 are the amounts of prey refuge of X and Y and H_i ($i = 1, 2$) stand for predators self-interaction factors respectively. Notice that ecological meaningful prey refuge parameters for (1.1) requires that $0 \leq m_1, m_2 < 1$.

This investigation mainly explores a tri-trophic food chain model that includes a constant proportion of prey refuge and intra-specific competitions among predator species. The most important curiosity of this investigation is to observe chaotic dynamics and their related dynamics. In this investigation, the following issues are addressed in connection with the analytical and numerical consequences:

- What are the dynamical effects of prey refuge and intra-specific competition among predator species on the tri-trophic food chain system?
- What is the genesis of chaotic dynamics in a tri-trophic food chain model system subject to prey refuge phenomena?

2.1 Non-dimensional design

We make a change of variables and time rescaling: $(x, y, z, t) = (\frac{X}{K}, \frac{Y}{KE_1}, \frac{Z}{KE_1E_2}, rT)$. The non-dimensional form of the system (2.1) is given by

$$\frac{dx}{dt} = x(1-x) - \frac{a_1(1-m_1)xy}{1+b_1(1-m_1)x} \equiv F_1(x, y, z), \quad (2.2a)$$

$$\frac{dy}{dt} = \frac{a_1(1-m_1)xy}{1+b_1(1-m_1)x} - d_1y - \frac{a_2(1-m_2)yz}{1+b_2(1-m_2)y} - h_1y^2 \equiv F_2(x, y, z), \quad (2.2b)$$

$$\frac{dz}{dt} = \frac{a_2(1-m_2)yz}{1+b_2(1-m_2)y} - d_2z - h_2z^2 \equiv F_3(x, y, z), \quad (2.2c)$$

$$x(0) \geq 0, \quad y(0) \geq 0, \quad z(0) \geq 0, \quad (2.2d)$$

where, the dimensionless system parameters are given by

$$a_1 = \frac{M_1KE_1}{rA_1}, \quad b_1 = \frac{K}{A_1}, \quad d_1 = \frac{D_1}{r}, \quad h_1 = \frac{H_1KE_1}{r}, \quad a_2 = \frac{M_2KE_1E_2}{rA_2}, \quad b_2 = \frac{KE_1}{A_2}, \quad d_2 = \frac{D_2}{r}, \quad h_2 = \frac{H_2KE_1E_2}{r}.$$

It is important to note that the functions F_i ($i = 1, 2, 3$) of the system (2.2) are continuous and have continuous partial derivatives on the positive octant $\mathbf{R}_+^3 = \{(x, y, z) \in \mathbf{R}_+^3 : x \geq 0, y \geq 0, z \geq 0\}$. Consequently all these functions F_i ($i = 1, 2, 3$) are Lipschitzian on $\mathbf{R}_+^3$, and therefore the solution of the system (2.2) exists and is unique.

In this current investigation, we focus mainly on the local and global stability of equilibria, dissipativeness of three species, Hopf bifurcation and chaos of the system (2.2). Now, we look over the boundedness of the solutions and dissipativeness of the model system (2.2). Permanence is an important behaviour of the system in the sense that it describes long-term behaviour of the system. Analytically, a system is said to be permanent if it is dissipative, i.e., all trajectories of the model system (2.2) initiating from any point in $\mathbf{R}_+^3$ ultimately lie in the fixed bounded region for sufficiently large time t .

3 Boundedness

Lemma 1: All the solutions of the model system (2.2) which initiate in $\mathbf{R}_+^3$ are uniformly bounded.

Proof: Define a positive definite function $W(t) = x + y + z$. Then for each $\eta > 0$, the time derivative of $W(t)$ along the solution of the system (2.2) is

$$\frac{dW}{dt} + \eta W(t) = x(1-x) + y(\eta - d_1 - h_1) + z(\eta - d_2 - h_2z) \leq (\eta + 1)^2 + (\eta - d_1)^2 + (\eta - d_2)^2.$$

Hence, one may find $\phi > 0$ such that $\frac{dW}{dt} + \eta W(t) \leq \phi$.

$0 \leq W(t) \leq \frac{\phi}{\eta}(1 - e^{-\eta t}) + W_0$, where W_0 is an arbitrary constant.

As $t \rightarrow \infty$, $W(t) \rightarrow \frac{\phi}{\eta}$.

Hence all the solutions of the system that initiate at $(x(0), y(0), z(0))$ lie in $\mathbf{R}_+^3$ and are confined in the compact region $S = [(x, y, z) : x(t) + y(t) + z(t) = \frac{\phi}{\eta} + \epsilon, \forall \epsilon > 0]$.

4 Dissipativeness

Lemma 2: If $a_i > b_i d_i$; $i = 1, 2$, then the flow / dynamical systems associated with model system (2.2) is dissipative.

Proof: By using straight forward arguments, one can show that the solution of the system (2.2) always exists and keeps on positive. Since all the interacting populations and parameters associated with system (2.2) are positive and $m_1, m_2 \in [0, 1]$, hence from the first equation of the model system (2.2), we have $\frac{dx}{dt} \leq x(1-x)$. Consequently, by solving one can observe that $x(t) \leq \frac{x(0)}{x(0) + (1-x(0))e^{-t}}$. Thus, any solution of the system (2.2) must satisfy $\limsup_{t \rightarrow \infty} x(t) \leq 1$.

Now, from the second equation of the model system (2.2) one may examine that

$$\frac{dy}{dt} \leq y \left(\frac{a_1}{b_1} - d_1 - h_1 y \right).$$

Therefore, by using similar arguments we have $\limsup_{t \rightarrow \infty} y(t) \leq \bar{y} = \frac{a_1 - b_1 d_1}{b_1 h_1}$, where $\bar{y}$ denotes an upper bound of $y(t)$ which is positive provided $a_1 - b_1 d_1 > 0$.

Similarly examining the third equation of the system (2.2)

$$\frac{dz}{dt} \leq z \left(\frac{a_2}{b_2} - d_2 - h_2 z \right).$$

Consequently, we have $\limsup_{t \rightarrow \infty} z(t) \leq \bar{z} = \frac{a_2 - b_2 d_2}{b_2 h_2}$, where $\bar{z}$ denotes an upper bound of $z(t)$ which is positive provided $a_2 - b_2 d_2 > 0$.

Hence, the proof.

5 Equilibria and their feasibility

We now analyze the behaviour of the proposed model through the existence of equilibrium of the system (2.2). The corresponding model (2.2) has at most four ecologically meaningful equilibria:

(i) Trivial equilibrium: $E_0(0, 0, 0)$,

(ii) Axial equilibrium: $E_1(1, 0, 0)$,

(iii) Boundary equilibrium: $E_2(x_2, y_2, 0)$,

(iv) Positive interior equilibrium: $E_3(x_3, y_3, z_3)$,

where $y_2 = \frac{(1-x_2)(1+b_1(1-m_1)x_2)}{a_1(1-m_1)}$ and x_2 satisfies the following cubic equation $\sum A_i x^i = 0$ and the coefficients A_i 's are given by

$$\begin{aligned} A_3 &= h_1 b_1^2 (1-m_1)^2 x_3^2 (> 0), \\ A_2 &= h_1 b_1 (1-m_1) [2 - b_1 (1-m_1)], \\ A_1 &= h_1 + a_1^2 (1-m_1)^2 - 2h_1 b_1 (1-m_1) - a_1 b_1 d_1 (1-m_1)^2, \\ A_0 &= -(a_1 d_1 (1-m_1) + h_1) (< 0). \end{aligned}$$

For the positive interior equilibrium $E_3(x_3, y_3, z_3)$, we have $z_3 = \frac{(a_2 - b_2 d_2)(1-m_2)y_3 - d_2}{h_2(1+b_2(1-m_2)y_3)}$, $y_3 = \frac{(1-x_3)(1+b_1(1-m_1)x_3)}{a_1(1-m_1)}$ and x_3 can be obtained by putting the value of y_3 in z_3 .

6 Stability analysis

The system (2.2) can be written in the form $\dot{X} = F(X)$, where $X = (x, y, z)^T$. The Jacobian matrix $J = DF(X)$ of the system (2.2) at any arbitrary point (x, y, z) is given by

$$J = \begin{bmatrix} J_{11} & J_{12} & J_{13} \\ J_{21} & J_{22} & J_{23} \\ J_{31} & J_{32} & J_{33} \end{bmatrix},$$

where $J_{11} = 1 - 2x - \frac{a_1(1-m_1)y}{1+b_1(1-m_1)x} + \frac{a_1 b_1 (1-m_1)^2 x y}{(1+b_1(1-m_1)x)^2}$, $J_{12} = -\frac{a_1(1-m_1)x}{1+b_1(1-m_1)x}$, $J_{13} = 0$, $J_{21} = \frac{a_1(1-m_1)y}{(1+b_1(1-m_1)x)^2}$, $J_{22} = \frac{a_1(1-m_1)x}{1+b_1(1-m_1)x} - \frac{a_2(1-m_2)z}{1+b_2(1-m_2)y} + \frac{a_2 b_2 (1-m_2)^2 y z}{(1+b_2(1-m_2)y)^2} - d_1 - 2h_1 y$, $J_{23} = \frac{a_2(1-m_2)y}{1+b_2(1-m_2)y}$, $J_{31} = 0$, $J_{32} = \frac{a_2(1-m_2)z}{(1+b_2(1-m_2)y)^2}$ and $J_{33} = \frac{a_2(1-m_2)y}{1+b_2(1-m_2)y} - d_2 - 2h_2 z$.

We denote $J_k = J$, the Jacobian evaluated at E_k , $k = 0, 1, 2, 3$ with elements (J_{ij}) , $i = 1, 2, 3$; $j = 1, 2, 3$ and the determinant $J_k = \det(J_k)$, trace $J_k = \text{tr}(J_k)$.

6.1 Stability of the system around the origin $E_0(0, 0, 0)$

The eigenvalues of the Jacobian matrix J_0 of the system (2.2) around E_0 are $1, -d_1, -d_2$. As a result, the hyperbolic equilibrium point is an unstable saddle point.

6.2 Stability of the system around the equilibrium point $E_1(1, 0, 0)$

Lemma 3: The equilibrium point $E_1(1, 0, 0)$ is locally asymptotically stable (LAS) iff $a_1 < d_1[b_1 + \frac{1}{(1-m_1)}]$.

Proof: The eigenvalues of the Jacobian matrix J_1 of the system (2.2) around the axial equilibrium E_1 are $-1, \frac{a_1(1-m_1)}{1+b_1(1-m_1)} - d_1, -d_2$. Therefore E_1 is locally asymptotically stable if $a_1 < d_1[b_1 + \frac{1}{(1-m_1)}]$.

6.3 Stability of the system around the equilibrium point $E_2(x_2, y_2, 0)$

Lemma 4: The equilibrium point $E_2(x_2, y_2, 0)$ is locally asymptotically stable if the conditions $a_1 b_1 (1 - m_1)^2 y_2 < [1 + b_1 (1 - m_1) x_2]^2$ and $y_2 < \frac{d_2}{(1-m_1)(a_2-b_2 d_2)}$ holds.

Proof:

$$J_2 = \begin{bmatrix} -x_2 + \frac{a_1 b_1 (1-m_1)^2 x_2 y_2}{(1+b_1(1-m_1)x_2)^2} & -\frac{a_1 (1-m_1) x_2}{(1+b_1(1-m_1)x_2)} & 0 \\ \frac{a_1 (1-m_1) y_2}{(1+b_1(1-m_1)x_2)^2} & -h_2 y_2 & -\frac{a_2 (1-m_2) y_2}{1+b_2(1-m_2)y_2} \\ 0 & 0 & \frac{a_2 (1-m_2) y_2}{1+b_2(1-m_2)y_2} - d_2 \end{bmatrix} = \begin{bmatrix} J_{11} & J_{12} & J_{13} \\ J_{21} & J_{22} & J_{23} \\ J_{31} & J_{32} & J_{33} \end{bmatrix}.$$

Now $tr(J_2) = J_{11} + J_{22} + J_{33}$ and $det(J_2) = (J_{11} J_{22} - J_{21} J_{12}) J_{33}$, where we notice that $J_{22} J_{12} < 0$ and $J_{21} > 0$. Using Routh-Hurwitz criterion we obtain that the system is locally asymptotically stable around E_2 if $tr(J_2) < 0$ and $det(J_2) > 0$, i.e., if $J_{11} < 0$ and $J_{33} > 0$, i.e., if $a_1 b_1 (1 - m_1)^2 y_2 < (1 + b_1 (1 - m_1) x_2)^2$ and $y_2 < \frac{d_2}{(1-m_1)(a_2-b_2 d_2)}$ holds.

6.4 The stability of the system around the feasible interior equilibrium point

$$E_3(x_3, y_3, z_3)$$

Lemma 5:

(i) The equilibrium point E_3 is locally asymptotically stable if $a_1 b_1 (1 - m_1)^2 < \frac{(1+b_1(1-m_1)x_3)^2}{(1-m_1)^2 y_3}$ and

$a_2 b_2 (1 - m_2)^2 < h_1 \frac{(1+b_2(1-m_2)y_3)^2}{((1-m_2)^2 z_3)}$ holds.

(ii) The global stability of the system (2.2) occurs at the interior equilibrium point E_3 if the conditions $a_1 b_1 (1 - m_1)^2 y_3 < 1 + b_1 (1 - m_1) x_3$ and $a_2 b_2 (1 - m_2)^2 z_3 < 1 + b_2 (1 - m_2) y_3$ holds.

(iii) The system exhibits a Hopf bifurcation at E_3 at $m_1 = m_1^{[HB]}$, where m_1 satisfies the equation (6.1) with $A_1 A_2 = A_3$.

Proof: (i) The Jacobian matrix J_3 of the system (2.2) the interior equilibrium point E_3 is given by $J_3 = (J_{ij})$. Let $\lambda_l; l = 1, 2, 3$, be the roots of the characteristic equation of J_3 is given by

$$\lambda^3 + A_1 \lambda^2 + A_2 \lambda + A_3 = 0, \quad (6.1)$$

where the coefficients of the characteristic equation (6.1) are defined as

$$A_1 = -(J_{11} + J_{22} + J_{33}), \quad (6.2a)$$

$$A_2 = J_{11} J_{22} + J_{22} J_{33} + J_{33} J_{11} - J_{12} J_{21} - J_{32} J_{23}, \quad (6.2b)$$

$$A_3 = J_{12} J_{21} J_{33} + J_{11} J_{23} J_{32} - J_{11} J_{22} J_{33}. \quad (6.2c)$$

Using Routh-Hurwitz criterion, E_3 is LAS if and only if $A_1, A_3 > 0$ and $A_1 A_2 > A_3$.

Now, $A_1 A_2 - A_3 = -(J_{11})^2 (J_{22} - J_{33}) + J_{12} J_{21} (J_{11} + J_{22}) - (J_{22})^2 (J_{11} + J_{33}) - 2 J_{11} J_{22} J_{33} - (J_{33})^2 (J_{11} + J_{22}) + J_{32} J_{23} J_{33}$,

as $J_{12}, J_{23} < 0$; $J_{21}, J_{32} > 0$; it is easy to check that $A_1 A_2 - A_3 > 0$ if, that is, if $a_1 b_1 (1 - m_1)^2 < \frac{(1 + b_1 (1 - m_1) x_3)^2}{(1 - m_1)^2 y_3}$ and $a_2 b_2 (1 - m_2)^2 < h_1 \frac{(1 + b_2 (1 - m_2) y_3)^2}{(1 - m_2)^2 z_3}$.

Hence, the equilibrium point is LAS.

(ii) To show the global stability of the system (2.2), let us construct a Lyapunov function as follows:

$$L(x, y, z) = (x - x_3 - x_3 \log \frac{x}{x_3}) + k_2 (y - y_3 - y_3 \log \frac{y}{y_3}) + k_3 (z - z_3 - z_3 \log \frac{z}{z_3}).$$

One may easily verify that the function $L(x, y, z) = 0$ at $(x, y, z) = (x_3, y_3, z_3)$ and is positive for all other positive values of x, y and z . The time derivative of $L(x, y, z)$ along the solution of the system (2.2) is given by

$$\begin{aligned} \frac{dL}{dt} &= \frac{x - x_3}{x} \frac{dx}{dt} + k_2 \frac{y - y_3}{y} \frac{dy}{dt} + k_3 \frac{z - z_3}{z} \frac{dz}{dt} \\ &= (x - x_3) \left[(1 - x) - \frac{a_1 (1 - m_1) y}{1 + b_1 (1 - m_1) x} \right] + k_2 (y - y_3) \left[\frac{a_1 (1 - m_1) y}{1 + b_1 (1 - m_1) x} \right. \\ &\quad \left. - h_1 y - d_1 - \frac{a_2 (1 - m_2) z}{1 + b_2 (1 - m_2) y} \right] + k_3 (z - z_3) \left[\frac{a_2 (1 - m_2) y}{1 + b_2 (1 - m_2) y} - h_2 z - d_2 \right] \\ &= (x - x_3) \left[-(x - x_3) + \frac{a_1 (1 - m_1) y}{1 + b_1 (1 - m_1) x} + \frac{a_1 (1 - m_1) y_3}{1 + b_1 (1 - m_1) x_3} \right] \\ &\quad + k_2 (y - y_3) \left[\frac{a_1 (1 - m_1) x}{1 + b_1 (1 - m_1) x} - \frac{a_1 (1 - m_1) x_3}{1 + b_1 (1 - m_1) x_3} - \frac{a_2 (1 - m_2) z}{1 + b_2 (1 - m_2) y} + \frac{a_2 (1 - m_2) z_3}{1 + b_2 (1 - m_2) y_3} - h_1 (y - y_3) \right] \\ &\quad + k_3 (z - z_3) \left[\frac{a_2 (1 - m_2) y}{1 + b_2 (1 - m_2) y} - \frac{a_2 (1 - m_2) y_3}{1 + b_2 (1 - m_2) y_3} - h_2 (z - z_3) \right] \\ &= -(x - x_3)^2 + \frac{a_1 b_1 (1 - m_1)^2 y_3 (x - x_3)^2}{1 + b_1 (1 - m_1) x_1 + b_1 (1 - m_1) x_3} \\ &\quad + (1 - m_1) (x - x_3) (y - y_3) \frac{-a_1 - a_1 b_1 x_3 (1 - m_1) + a_1 k_2}{(1 + b_1 (1 - m_1) x) (1 + b_1 (1 - m_1) x_3)} \\ &\quad - k_2 h_1 (y - y_3)^2 + k_2 a_2 b_2 (1 - m_2)^2 \frac{z_3 (y - y_3)^2}{(1 + b_2 (1 - m_2) y) (1 + b_2 (1 - m_2) y_3)} \\ &\quad + (1 - m_2) a_2 (y - y_3) (z - z_3) \frac{-k_2 - k_2 b_2 y_3 (1 - m_2) + k_3}{(1 + b_2 (1 - m_2) y) (1 + b_2 (1 - m_2) y_3)} - k_3 h_2 (z - z_3)^2. \end{aligned}$$

Now we choose $k_2 = 1 + b_1 (1 - m_1 x_3)$ and $k_3 = k_2 (1 + b_2 (1 - m_2 y_3))$ so that

$$\frac{dL}{dt} = \left(-1 + \frac{a_1 b_1 (1 - m_1)^2 y_3}{1 + b_1 (1 - m_1) x_1 + b_1 (1 - m_1) x_3} \right) (x - x_3)^2 + k_2 \left(-h_1 + \frac{a_2 b_2 (1 - m_2)^2 z_3}{1 + b_2 (1 - m_2) y_1 + b_2 (1 - m_2) y_3} \right) (y - y_3)^2 - k_3 h_2 (z - z_3)^2$$

$$\leq \left(-1 + \frac{a_1 b_1 (1 - m_1)^2 y_3}{1 + b_1 (1 - m_1) x_3} \right) (x - x_3)^2 + k_2 \left(-h_1 + \frac{a_2 b_2 (1 - m_2)^2 z_3}{1 + b_2 (1 - m_2) y_3} \right) (y - y_3)^2 - k_3 h_2 (z - z_3)^2$$

≤ 0 , if $a_1 b_1 (1 - m_1)^2 y_3 < 1 + b_1 (1 - m_1) x_3$ and $a_2 b_2 (1 - m_2)^2 z_3 < 1 + b_2 (1 - m_2) y_3$

and $\frac{dL}{dt} = 0$ when $(x, y, z) = (x_3, y_3, z_3)$. Hence by Lyapunov-LaSalle's invariance principle (Hale, 1969), E_3 is globally asymptotically stable (GAS).

(iii) The characteristic equation of J_3 is given by (6.1). By the Routh-Hurwitz criterion for all the roots of the characteristic equation to have negative real parts if $A_1 > 0, A_3 > 0$ and $A_1 A_2 > A_3$. From the expression given in the equations (6.2), we obtain that $A_1 > 0, A_2 > 0, A_3 > 0$ if $n_{11} < 0$ and $n_{22} < 0$. Clearly the characteristic equation has two purely imaginary roots if and only if $A_1 A_2 = A_3$, which provides at $m_1 = m_1^{[HB]}$, there is an interval $(m_1 - \epsilon, m_1 + \epsilon)$ containing m_1 for some $\epsilon > 0$ for which $m_1 - \epsilon > 0$ such that $A_2 > 0$ for m_1 in $(m_1 - \epsilon, m_1 + \epsilon)$. Thus for m_1 in $(m_1 - \epsilon, m_1 + \epsilon)$, the characteristic

equation can not have real positive roots. Now at $m_1 = m_1^{[HB]}$, we have $(\lambda^2 + A_2)(\lambda + A_1) = 0$, which has three roots $\lambda_1 = i\sqrt{A_2}$, $\lambda_2 = -i\sqrt{A_2}$, $\lambda_3 = -A_1$.

For m_1 in $(m_1 - \epsilon, m_1 + \epsilon)$, the roots are in general of the form $\lambda_1 = \alpha(m_1) + i\beta(m_1)$, $\lambda_2 = \alpha(m_1) - i\beta(m_1)$, $\lambda_3 = -A_1(m_1)$. Now we shall verify the transversality condition $Re \left[\frac{d(\lambda_j)}{dm_1} \right]_{m_1=m_1^{[HB]}} \neq 0$, $j = 1, 2$.

Substituting $\lambda_1 = \alpha(m_1) + i\beta(m_1)$ into the characteristic equation and calculating the derivative, we obtain

$$\begin{aligned}\omega(m_1)\alpha(m_1) - \phi(m_1)\beta(m_1) + \eta(m_1) &= 0, \\ \phi(m_1)\alpha(m_1) - \omega(m_1)\beta(m_1) + \mu(m_1) &= 0,\end{aligned}$$

where $\omega(m_1) = 3\alpha(m_1)^2 + 2A_1(m_1)\alpha(m_1) - 3\beta(m_1)^2$,

$\phi(m_1) = 6\alpha(m_1)\beta(m_1) + 2A_1(m_1)\beta(m_1)$,

$\eta(m_1) = A_1(m_1)\alpha(m_1)^2 + A_2(m_1)\alpha(m_1) + A_3(m_1) - A_1(m_1)\beta(m_1)^2$,

$\mu(m_1) = 2\alpha(m_1)\beta(m_1) + A_2(m_1)\beta(m_1)$.

Since $\phi(m_1)\mu(m_1) + \omega(m_1)\eta(m_1) \neq 0$, one may obtain

$$Re \left[\frac{d(\lambda_j)}{dm_1} \right]_{m_1=m_1^{[HB]}} = -\frac{\phi\mu + \omega\eta}{\phi^2 + \omega^2} \neq 0, j = 1, 2,$$

$\lambda_3 = -A_1(m_1) \neq 0$. Hence the proof.

7 Numerical explorations

In this section, we wish to present some numerical simulations to verify our theoretical results proved in the previous sections by using MATLAB R2015a and Maple 2016 software with the fourth order Runge-Kutta method because to support the theoretical findings; we should have the evidence of some numerical results. We have modified the model of Hastings and Powell (Hastings and Powell, 1991) by bringing in prey refuge and intra-specific competition. Hastings and Powell (Hastings and Powell, 1991) observed stable focus, period doubling, and chaotic dynamics of the system.

In dynamical systems, global stability indicates that the attracting basin of trajectories is either the state space or a specific region in the state space, which is the defining region of the state variables of the proposed non-linear system. All the three species will coexist under what conditions; that scenario is shown in the global stability of the system (2.2) around E_3 (cf. Fig. 1). By changing the parameter values, we have shown that the system (2.2) enters into a Hopf bifurcation around E_3 . This happens, if we choose $a_1 = 6.109$, $a_2 = 2.8$, $b_1 = 2.0$, $b_2 = 0.5$, $d_1 = 0.5$, $d_2 = 0.4$, $h_1 = 0.05$, $h_2 = 0.05$, $m_1 = 0.0765$, $m_2 = 0.1$. This circumstance has been illustrated in Fig. 2.

In the absence of intra-specific competitions, that is, for $h_1 = 0 = h_2$, chaotic behaviour of the system (2.2) is shown in Fig. 3 for the set of parameter values mentioned in the caption of that figure. We also draw the spectrum of Lyapunov exponents (Lynch, 2004), (Layek, 2015) concerning time through a particular choice of system parameters; the positive value of the maximum Lyapunov exponent in Fig. 4 points out the chaotic behaviour of the system. Moreover, the outcomes of Fig. 5(a) exhibit the transfer of trajectories of the system (2.2) into the period-doubling phase from chaotic dynamics when the parameters b_1 , h_1 , h_2 are chosen to have their respective values 3.0, 0.0, 0.0002. At the same time, they are found to enter into the limit cycle state from the period doubling phase for a slight change of the parameter h_2 from 0.0002 to 0.00025 by keeping other parameters unchanged, as shown in Fig. 5(b).

7.1 Impact of the refuge parameters m_1 and m_2

In this subsection, we will investigate the effect of the refuge parameters m_1 and m_2 on the system (2.2) in the presence of intra-specific competitions. The limit cycle state of the model system (2.2) is obtained

for $a_1 = 5.0$, $a_2 = 0.1$, $b_1 = 3.0$, $b_2 = 2.0$, $d_1 = 0.4$, $d_2 = 0.01$, $h_1 = 0.0$, $h_2 = 0.00025$, $m_1 = 0.01$, $m_2 = 0.04$. From this, one may obtain the following results due to the change of m_1 and m_2 . If there is a variation of m_1 from 0.01 to 0.3, then the system displays a chaotic regime with a stable limit cycle as shown in Fig. 6(b). On the other hand, by changing the value of m_2 from 0.04 to 0.065, we find that the system becomes globally stable (cf. Fig. 6(a)).

In recent times the concept of chaotic dynamics has been one of the significant discoveries. One may verify the mathematical theory to explore chaotic dynamics with the help of the Lyapunov exponent. Lyapunov exponent and bifurcation diagrams are exercised to disclose the system's dynamics. Fig. 7(a) reveals the spectrum of maximal Lyapunov exponent of system (2.2) with the parameters $a_1 = 5.0$, $a_2 = 0.1$, $b_1 = 4.0$, $b_2 = 2.0$, $d_1 = 0.4$, $d_2 = 0.01$, $h_1 = 0.0$, $h_2 = 0.0$, $m_2 = 0.1$ when $0.02 \leq m_1 \leq 0.2$. With the help of this specific parameter set, one may be able to design bifurcation diagrams of the system concerning m_1 (cf. Fig. 9). Fig. 7(b) discloses the spectrum of maximal Lyapunov exponent of system (2.2) with the system parameters $a_1 = 5.0$, $a_2 = 0.1$, $b_1 = 4.0$, $b_2 = 2.0$, $d_1 = 0.4$, $d_2 = 0.01$, $h_1 = 0.0$, $h_2 = 0.0$, $m_1 = 0.1$ when $0.02 \leq m_2 \leq 0.1$. With the help of this specific parameter set, one may be able to design bifurcation diagrams of the system concerning m_2 (cf. Fig. 10). Similarly, Fig. 8(a) demonstrates the spectrum of maximal Lyapunov exponent of system (2.2) with the parameters $a_1 = 5.0$, $a_2 = 0.1$, $b_1 = 3.05$, $b_2 = 2.0$, $d_1 = 0.4$, $d_2 = 0.01$, $h_1 = 0.0$, $h_2 = 0.0$, $m_2 = 0.1$ when $0.02 \leq m_1 \leq 0.2$; consequently, bifurcation diagrams of the system (2.2) with respect to m_1 has been plotted in Fig. 11. Fig. 8(b) signifies the spectrum of maximal Lyapunov exponent of system (2.2) with the parameters $a_1 = 5.0$, $a_2 = 0.1$, $b_1 = 3.05$, $b_2 = 2.0$, $d_1 = 0.4$, $d_2 = 0.01$, $h_1 = 0.0$, $h_2 = 0.0$, $m_1 = 0.1$ when $0.02 \leq m_2 \leq 0.1$. With the help of this specific parameter set, one may be able to propose bifurcation diagrams of the system concerning m_2 (cf. Fig. 12). In addition, dynamical chaos has been observed for other numeric support of b_1 throughout the precise assumption, namely with prey refuge and without prey refuge (cf. Figs. 13-15).

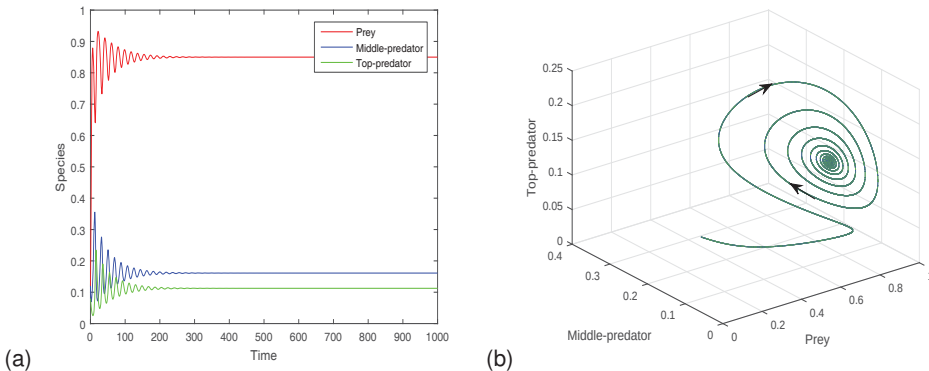


Figure 1: Global stability of the model system (2.2) around the positive interior equilibrium point $E_3(x_3, y_3, z_3)$ corresponding to the values of parameters $a_1 = 3.39$, $a_2 = 3.0$, $b_1 = 2.9$, $b_2 = 0.5$, $d_1 = 0.5$, $d_2 = 0.4$, $h_1 = 0.05$, $h_2 = 0.05$, $m_1 = 0.15$, $m_2 = 0.1$. (a) Time series evolution of the species, (b) Phase portrait of the system.

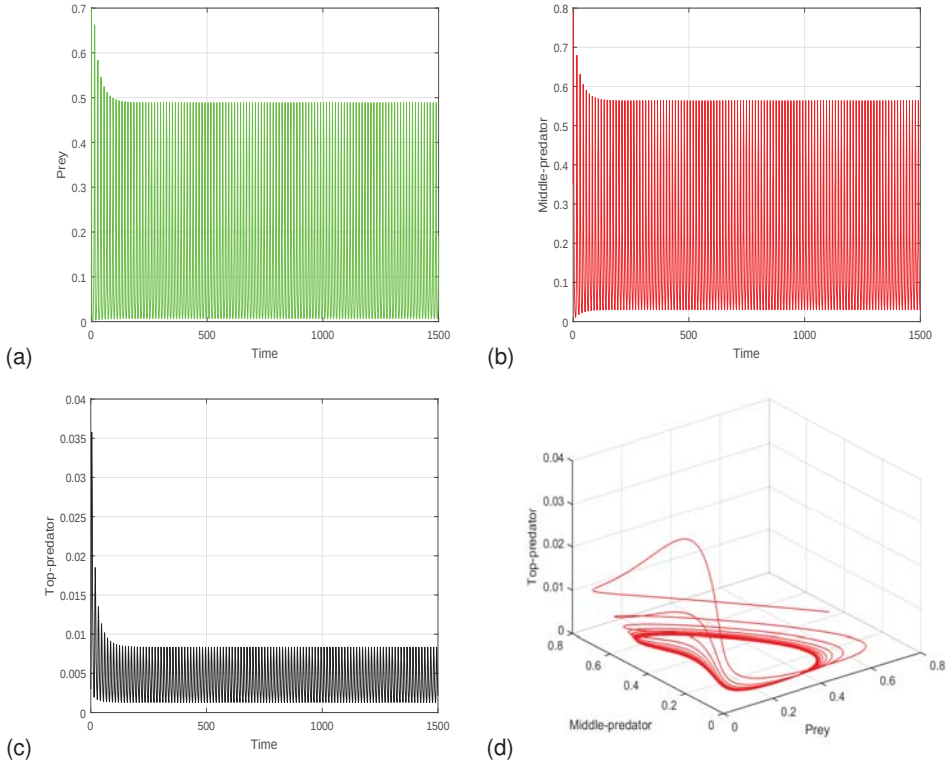


Figure 2: Unstable scenario of the model system (2.2) surrounding the positive interior equilibrium point $E_3(x_3, y_3, z_3)$ corresponding to the values of parameters $a_1 = 6.109$, $a_2 = 2.8$, $b_1 = 2.0$, $b_2 = 0.5$, $d_1 = 0.5$, $d_2 = 0.4$, $h_1 = 0.05$, $h_2 = 0.05$, $m_1 = 0.0765$, $m_2 = 0.1$. (a)-(c) Time series for $x(t)$, $y(t)$ and $z(t)$ respectively, (d) Limit cycle of the system.

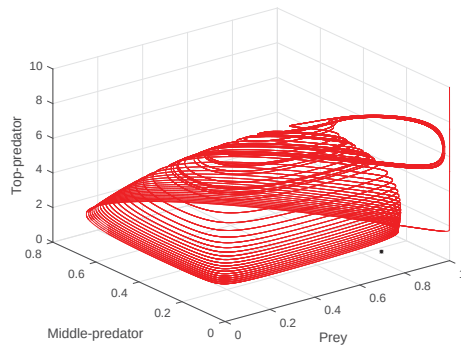


Figure 3: Chaotic behaviour of the model system (2.2) for $b_1 = 4.0$ and the other parameter values are $a_1 = 5.0$, $a_2 = 0.1$, $b_2 = 2.0$, $d_1 = 0.4$, $d_2 = 0.01$, $h_1 = 0.0$, $h_2 = 0.0$, $m_1 = 0.15$ and $m_2 = 0.1$.

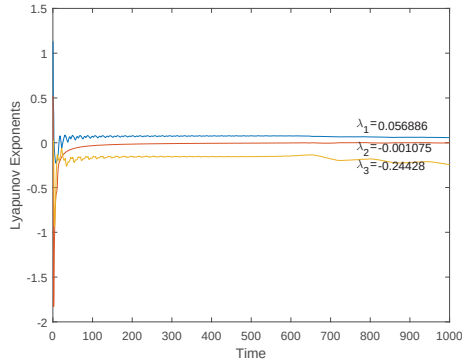


Figure 4: The spectrum of Lyapunov exponent of the model system (2.2) corresponding to the values of parameters $a_1 = 5.0$, $a_2 = 0.1$, $b_2 = 2.0$, $d_1 = 0.4$, $d_2 = 0.01$, $h_1 = 0.0$, $h_2 = 0.0$, $m_1 = 0.15$ and $m_2 = 0.1$.

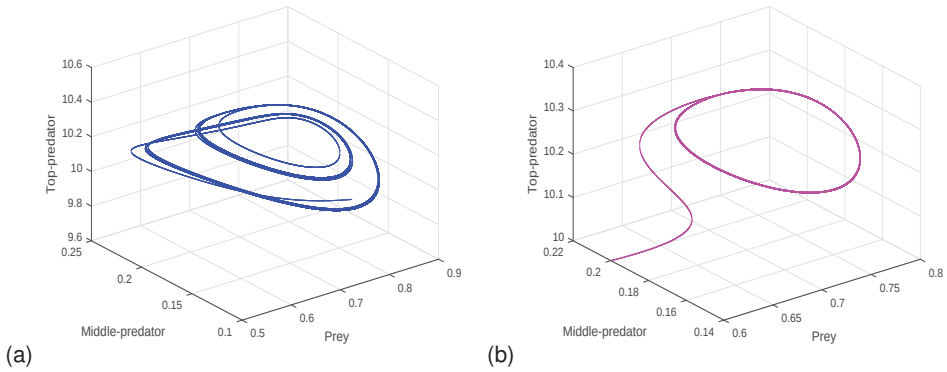


Figure 5: (a) Period doubling behaviour of the model system (2.2) for $h_2 = 0.0002$. (b) Limit cycle behaviour of the model system (2.2) for $h_2 = 0.00025$. Other system parameter values are $a_1 = 5.0$, $a_2 = 0.1$, $b_1 = 3.0$, $b_2 = 2.0$, $d_1 = 0.4$, $d_2 = 0.01$, $h_1 = 0.0$, $m_1 = 0.01$ and $m_2 = 0.04$.

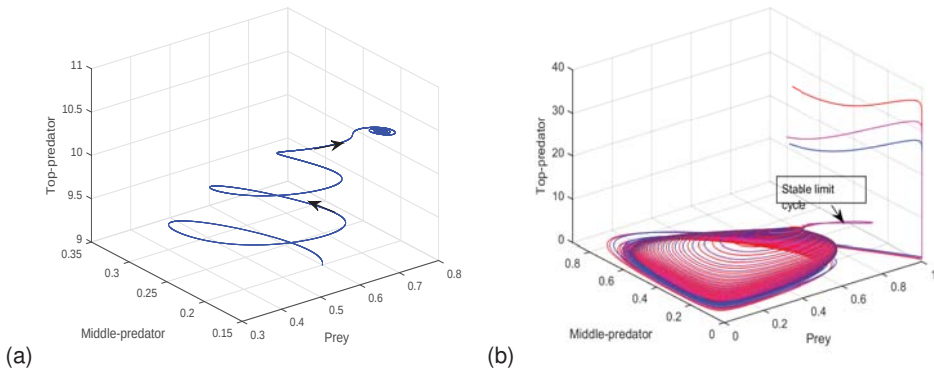


Figure 6: (a) Global stability of the model system (2.2) for $m_2 = 0.065$. Other system parameter values are $a_1 = 5.0$, $a_2 = 0.1$, $b_1 = 3.0$, $b_2 = 2.0$, $d_1 = 0.4$, $d_2 = 0.01$, $h_1 = 0.0$, $h_2 = 0.00025$, $m_1 = 0.01$. (b) Existence of stable limit cycle in chaotic state of the model system (2.2) for $m_1 = 0.3$. Other system parameter values are $a_1 = 5.0$, $a_2 = 0.1$, $b_1 = 3.0$, $b_2 = 2.0$, $d_1 = 0.4$, $d_2 = 0.01$, $h_1 = 0.0$, $h_2 = 0.00025$ and $m_2 = 0.04$.

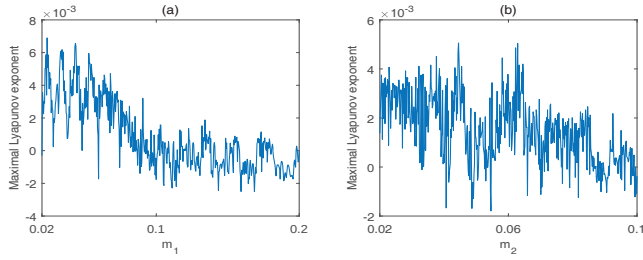


Figure 7: (a) The spectrum of maximal Lyapunov exponent of system (2.2) with the parameters $a_1 = 5.0$, $a_2 = 0.1$, $b_1 = 4.0$, $b_2 = 2.0$, $d_1 = 0.4$, $d_2 = 0.01$, $h_1 = 0.0$, $h_2 = 0.0$, $m_2 = 0.1$ when $0.02 \leq m_1 \leq 0.2$. (b) The spectrum of maximal Lyapunov exponent of system (2.2) with the parameters $a_1 = 5.0$, $a_2 = 0.1$, $b_1 = 4.0$, $b_2 = 2.0$, $d_1 = 0.4$, $d_2 = 0.01$, $h_1 = 0.0$, $h_2 = 0.0$, $m_1 = 0.1$ when $0.02 \leq m_2 \leq 0.1$.

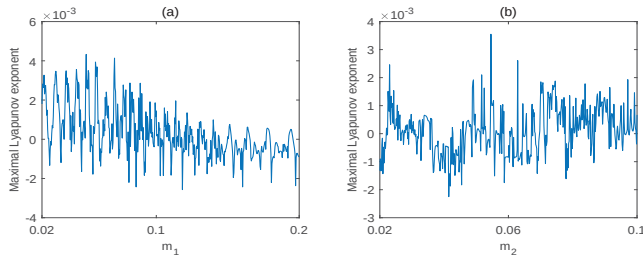


Figure 8: (a) The spectrum of maximal Lyapunov exponent of system (2.2) with the parameters $a_1 = 5.0$, $a_2 = 0.1$, $b_1 = 3.05$, $b_2 = 2.0$, $d_1 = 0.4$, $d_2 = 0.01$, $h_1 = 0.0$, $h_2 = 0.0$, $m_2 = 0.1$ when $0.02 \leq m_1 \leq 0.2$. (b) The spectrum of maximal Lyapunov exponent of system (2.2) with the parameters $a_1 = 5.0$, $a_2 = 0.1$, $b_1 = 3.05$, $b_2 = 2.0$, $d_1 = 0.4$, $d_2 = 0.01$, $h_1 = 0.0$, $h_2 = 0.0$, $m_1 = 0.1$ when $0.02 \leq m_2 \leq 0.1$.

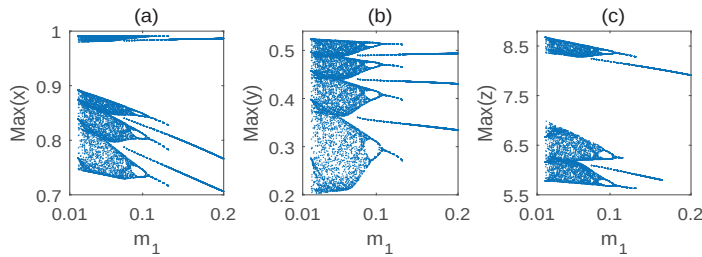


Figure 9: (a) Bifurcation diagrams of system (2.2) with respect to m_1 for the same parameter values used in Figure 7(a): (a) Maxima for the prey, (b) Maxima for the middle-predator, (c) Maxima for the top-predator.

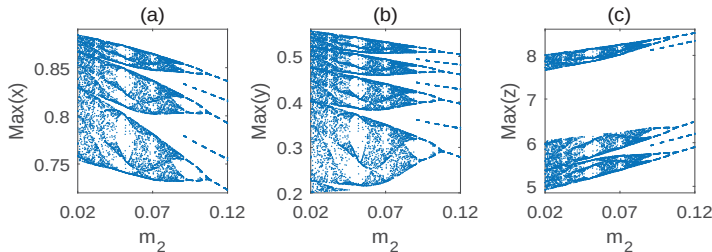


Figure 10: (a) Bifurcation diagrams of system (2.2) with respect to m_2 for the same parameter values used in Figure 7(b): (a) Maxima for the prey, (b) Maxima for the middle-predator, (c) Maxima for the top-predator.

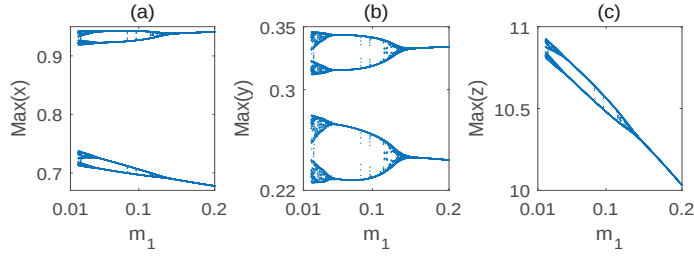


Figure 11: (a) Bifurcation diagrams of system (2.2) with respect to m_1 for the same parameter values used in Figure 8(a): (a) Maxima for the prey, (b) Maxima for the middle-predator, (c) Maxima for the top-predator.

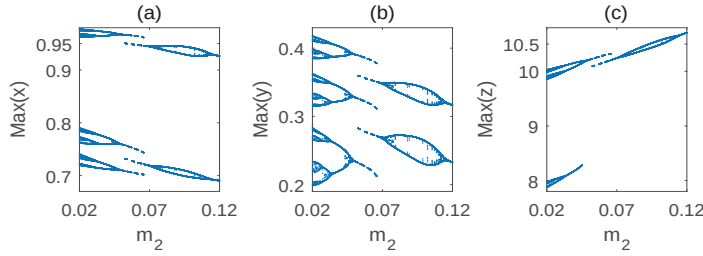


Figure 12: (a) Bifurcation diagrams of system (2.2) with respect to m_2 for the same parameter values used in Figure 8(b): (a) Maxima for the prey, (b) Maxima for the middle-predator, (c) Maxima for the top-predator.

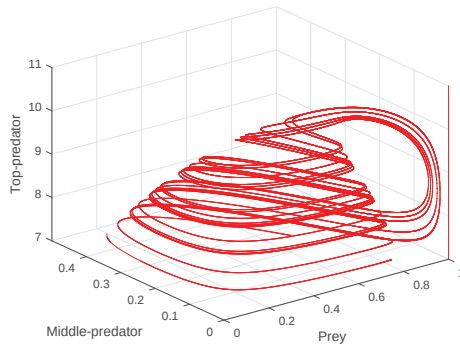


Figure 13: Chaotic behaviour of the model system (2.2) for $b_1 = 3.05$ and the other parameter values are $a_1 = 5.0$, $a_2 = 0.1$, $b_2 = 2.0$, $d_1 = 0.4$, $d_2 = 0.01$, $h_1 = 0.0$, $h_2 = 0.0$, $m_1 = 0.0$, $m_2 = 0.0$.

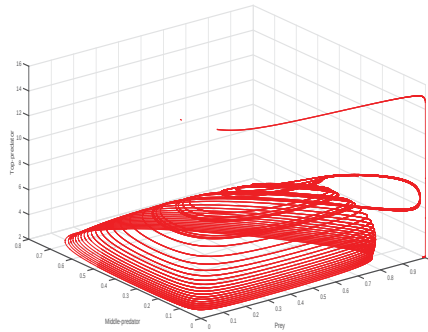


Figure 14: Chaotic behaviour of the model system (2.2) for $b_1 = 3.5$ and the other parameter values are $a_1 = 5.0$, $a_2 = 0.1$, $b_2 = 2.0$, $d_1 = 0.4$, $d_2 = 0.01$, $h_1 = 0.0$, $h_2 = 0.0$, $m_1 = 0.15$, $m_2 = 0.0$.

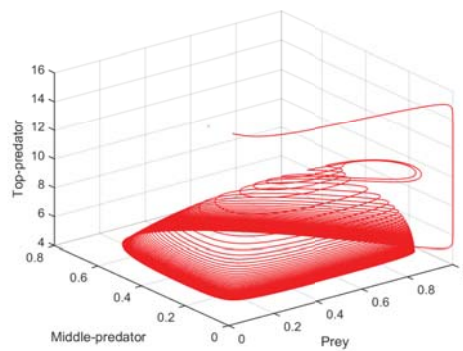


Figure 15: Chaotic behaviour of the model system (2.2) for $b_1 = 3.0$ and the other parameter values are $a_1 = 5.0$, $a_2 = 0.1$, $b_2 = 2.0$, $d_1 = 0.4$, $d_2 = 0.01$, $h_1 = 0.0$, $h_2 = 0.0$, $m_1 = 0.0$, $m_2 = 0.1$.

8 Concluding comments and discussion

A considerable number of investigations have been carried out for better understanding the classical prey-dependent food chain model (1.1) of which one may particularly mention the work (May, 2019), (Freedman, 1980) for a more desirable perception of the model outcomes. With the inclusion of both the prey refuge policy and the intra-specific competition among predator species into the existing model (May, 2019), an attempt has been made to update the food chain model having the hyperbolic functional response to discuss the behavior of the model trajectories in a more suitable natural situation. That means the upswing of the concerned model makes the system relatively closer to the actual situation based on their sources of food, shelter, and imperative biological needs. The colour, smell, size, skin, body cover, habitat complexity, predator density, injection of poisonous agents, etc., may be used as diverse forms of refuge to save prey species from predators. The proportional density-dependent prey refuge in a system of interacting species is an ecological feature indispensable for the continued existence of species, and consequently it deserves a factor of great interest as self-protective properties of prey in opposition to predation policy.

Right now to sum up the analytical findings together with their validation through numerical simulations, the preliminary measures for the boundedness, the existence and also the dissipativeness of the system are taken into account. The present model admits of four biologically feasible equilibria of which the trivial equilibrium position E_0 where all the species disappear is always unstable. The remaining non-trivial equilibria are asymptotically stable depending on their respective parametric constraints. The first non-trivial equilibrium point E_1 is manifested where the prey species is at its highest density in the absence of both the predators. The equilibrium position E_2 corresponds to the presence of both prey and middle-predator species in the absence of top-predator only. Unlike these equilibria, the fourth one E_3 admits the existence of all the species is termed as interior equilibrium position. The global asymptotic stability of the projected system about the interior equilibrium position is carried out by making use of suitable Lyapunov function and Lyapunov-LaSalle invariance principle. The system also experiences a Hopf-bifurcation around the coexistence equilibrium E_3 at $m_1 = m_1^{[HB]}$, where $m_1^{[HB]}$ is provided in subsection 6.4 with $A_1 A_2 = A_3$.

The existence of all possible biologically feasible equilibrium positions of the system under consideration is explored along with the stability criteria at these points. The mathematical analysis illustrates the existence of stability switching, Hopf-bifurcation and chaos as well around the coexistence equilibrium position E_3 . Numerical simulations based on the set of model parameter values are conducted by employing suitable codes for MATLAB R2015a and Maple 2016 software. Numerical results agree well with the theoretical findings regarding the stability response of the model system in the event of lesser value of prey refuge parameter than its critical one. However, the system becomes unstable at E_3 when the prey refuge parameter exceeds its critical value, but the chaotic response of the interacting species prevails for different variations of momentous system parameters.

Furthermore, if one disregards the existence of intra-specific competition among the species, the present model system exhibits a chaotic response in the long run. But the system bears the potential to control chaotic dynamical situations treating non-zero values of h_1 and h_2 as controlling parameters. It appears that self-interactions are primarily responsible for transforming the system dynamics from chaos to limit cycle and then to a stable steady-state environment. One may note that apart from the system parameters, initial conditions do play a significant role in the progression of all the species concerned, although it is critical to estimate the specific effect on progression. The ecological implication of these results reveals that the model system is either in a steady-state situation or in an oscillatory mode. Essentially, the outcomes of the model support one to make out better understanding of the concept of theoretical ecology in the realm of food chain system. It is believed that the remarks made in the present investigation would certainly be of some help to the experimental ecologists working in the relevant domain

of food chain dynamics. The work undertaken may further be extended with the inclusion of specific important ecological components (Guin, 2014), (Guin, 2015), (Guin and Acharya, 2017), (Guin and Baek, 2018), (Guin, Das and Sambath, 2020), (Han, Guin and Dai, 2020), (Guin, Pal, Chakravarty and Djilali, 2021), (Guin, Mapa and Chakravarty, 2021) in future.

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