

A Time Delay Viral Micro Parasite Model

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ABSTRACT

This paper deals with the dynamics of a micro parasite viral model which includes three compartments (susceptible(x), infective (y) and removable (z) populations). The total population is given by $N = x + y + z$. A time delay is incorporated in the interaction of susceptible and infective populations. The system is described by the couple of delay differential equations. The stability analysis is described at endemic equilibrium point. The existence of Hopf bifurcation is analysed using the time delay argument (τ).

Keywords: Equilibrium points, Local stability, Hopf bifurcation, Delay argument.

AMS classification: 34D, 92B, 92D.

JEL classification : C62,C63.

1. INTRODUCTION

Epidemiology deals with epidemics in human population which determines the disease conditions and its analysis. In recent decades it gains a lot of importance. Mathematical models are well known to describe the nature of the diseases (epidemics). Differential equations plays a vital role in illumination the epidemic models. Murray [1], R.M.Anderson [2] studied the different various epidemic models and their stability analysis. Epidemiology studies the diseases causation, transmission of diseases, outbreak of disease etc. Epidemiology deals with the dynamics of the parasite's interaction. Heathcote [3], Anderson et.[4] studied the parasites and its impact on the dynamics of host population. N.J.T Bayley [5] studied the dynamics of transmission of diseases. Parasitism is kind of symbiosis. It is biological interaction between the host and parasite. Different kinds of parasites are present in nature. These parasites are classified as micro and macro. A micro parasite can reproduce and complete their life cycle within the host, whereas macro parasite reproduces and complete their life cycle outside the host. These parasites transmit the diseases directly by the close contact or through environment contact

and those that require intermediate host or a vector for transmission. The dynamics of vector disease models is studied by K. L. Cooke [8]. Heathcote [7] discussed the stability analysis communicable diseases. These diseases are controlled by vaccination. Dynamics of the Models which deals with pulse vaccination and control of viral infection is studied by D. Nokes and J. Swinton [6]. The communicable disease like HIV with time delay was studied by Karuna et al [9]. Ranjith Kumar [10, 11] dealt with SIR models with time delays, saturated incidence rate. Recently apparao [12] study stability analysis of micro viral parasite model. In spite of that, we study the delay dynamics of micro viral parasite model. We introduce the delay parameter 'τ' in the interaction of susceptible and infective population. The stability analysis is discussed at endemic equilibrium point and derived the condition for existence of stability. The Hopf bifurcation analysis is carried out with delay parameter 'τ' using numerical simulation with aid of MATLAB simulation.

2. MATHEMATICAL MODEL

The governing equations of the system are as follows for the proposed model dealt with delay arguments are as follows

$$\begin{aligned}\frac{dx}{dt} &= bN - \beta x(t-\tau)y(t-\tau) - bx \\ \frac{dy}{dt} &= \beta x(t-\tau)y(t-\tau) - (b+r)y \\ \frac{dz}{dt} &= ry - bz\end{aligned}\tag{2.1}$$

N: Total population ($N = x + y + z$), x susceptible population, y infective population, z , removable population, β disease transmission rate, r recovery rate, b proportion rate

3. EQUILIBRIUM POINTS

The system (2.1) has two equilibrium points namely given by

- (i) Diseases free equilibrium point: $E_1 (N, 0, 0)$ (3.1)
- (ii) Endemic equilibrium point (E_2) is given by

$$E_2(\bar{x}, \bar{y}, \bar{z}) = \left[\frac{b+r}{\beta}, b \left(\frac{N\beta - b - r}{\beta(b+r)} \right), r \left(\frac{N\beta - b - r}{\beta(b+r)} \right) \right]\tag{3.2}$$

With a basic reproduction number (R_0) is given by $R_0 = \frac{N\beta}{b+r}$

If $R_0 < 1$, there is no positive solution.

If $R_0 > 1$, the infection spread out

4. STABILITY ANALYSIS AT ENDEMIC EQUILIBRIUM POINT $E_2(\bar{x}, \bar{y}, \bar{z})$

The Jacobean matrix for the system is at endemic equilibrium point $E_2(\bar{x}, \bar{y}, \bar{z})$ is given by

$$J = \begin{vmatrix} -\beta\bar{y}e^{-\lambda\tau} - b & -\beta\bar{x}e^{-\lambda\tau} & 0 \\ \beta\bar{y}e^{-\lambda\tau} & \beta\bar{x}e^{-\lambda\tau} - (b+r) & 0 \\ 0 & r & -b-\lambda \end{vmatrix} \quad (4.1)$$

The characteristic equation of (4.1) is given by

$$|J - \lambda I| = \begin{vmatrix} -\beta\bar{y}e^{-\lambda\tau} - b - \lambda & -\beta\bar{x}e^{-\lambda\tau} & 0 \\ \beta\bar{y}e^{-\lambda\tau} & \beta\bar{x}e^{-\lambda\tau} - (b+r) - \lambda & 0 \\ 0 & r & -b-\lambda \end{vmatrix} = 0$$

$$\psi(\lambda, \tau) = \lambda^3 + p_1\lambda^2 + p_2\lambda + p_3 + e^{-\lambda\tau}(q_1\lambda^2 + q_2\lambda + q_3) = 0 \quad (4.2)$$

Where

$$p_1 = 3b + r$$

$$p_2 = 3b^2 + 2rb$$

$$p_3 = b^3 + rb^2$$

$$q_1 = \beta(\bar{y} - \bar{x})$$

$$q_2 = 2b\beta(\bar{y} - \bar{x}) + r\beta\bar{y}$$

$$q_3 = \beta b^2(\bar{y} - \bar{x}) + r\beta b\bar{y}$$

We need find the condition for existence of negative real roots

Case (i) : For $\tau = 0$ equation (4.2) becomes

$$\psi(\lambda, 0) = -(b + \lambda) \left[\lambda^2 + \lambda(\beta(\bar{y} - \bar{x}) + 2b + r) + b\beta(\bar{y} - \bar{x}) + r\beta\bar{y} + b^2 + rb \right] = 0$$

$$(b + \lambda) = 0, (or) \left[\lambda^2 + \lambda(\beta(\bar{y} - \bar{x}) + 2b + r) + b\beta(\bar{y} - \bar{x}) + r\beta\bar{y} + b^2 + rb \right] = 0$$

$$\lambda = -b \quad (4.3)$$

$$\text{and} \quad \left[\lambda^2 + \lambda(\beta(\bar{y} - \bar{x}) + 2b + r) + b\beta(\bar{y} - \bar{x}) + r\beta\bar{y} + b^2 + rb \right] = 0 \quad (4.4)$$

The system becomes asymptotically stable if (4.4) possess negative real roots.

From the equation (4.3) one of the roots is negative and the equation (4.4) has negative real roots if the

trace of the equation $\left(\frac{-b}{a} \right)$ is negative and the determinant $\left(\frac{c}{a} \right)$ is positive

$$\text{Here the trace is } = \frac{-b}{a} = \frac{-(\beta(\bar{y} - \bar{x}) + 2b + r)}{1} = -(\beta(\bar{y} - \bar{x}) + 2b + r) < 0 \text{ if } \bar{y} > \bar{x}$$

$$\text{Determinant} = \frac{c}{a} = \frac{b\beta(\bar{y} - \bar{x}) + r\beta\bar{y} + b^2 + rb}{1} = b\beta(\bar{y} - \bar{x}) + r\beta\bar{y} + b^2 + rb > 0 \text{ if } \bar{y} > \bar{x}$$

Hence the endemic point $E_2(\bar{x}, \bar{y}, \bar{z})$ is locally asymptotically stable if $\bar{y} > \bar{x}$

Case (ii) for $\tau > 0$: suppose there is positive τ_0 such that the equation (4.1) has pair of purely imaginary root, can be taken as $\pm i\omega, \omega > 0$ then ω satisfies the equation (4.1)

$$(i\omega)^3 + p_1(i\omega)^2 + p_2(i\omega) + p_3 + e^{-i\omega\tau}(q_1(i\omega)^2 + q_2(i\omega) + q_3) = 0$$

$$-\omega^2 p_1 + p_3 - q_1 \omega^2 \cos \omega\tau + q_3 \cos \omega\tau + q_2 \omega \sin \omega\tau + i[-\omega^3 + \omega p_2 + q_2 \omega \cos \omega\tau + q_1 \omega^2 \sin \omega\tau - q_3 \sin \omega\tau] = 0$$

Separating real and imaginary parts we get the following two equations

$$(q_3 - q_1 \omega^2) \cos \omega\tau + q_2 \omega \sin \omega\tau = \omega^2 p_1 - p_3$$

$$q_2 \omega \cos \omega\tau - (q_3 - q_1 \omega^2) \sin \omega\tau = \omega^3 - \omega p_2 \quad (4.5)$$

from the above equations we get the following equation (by square and add the two results)

$$(q_3 - q_1 \omega^2)^2 + (q_2 \omega)^2 = (\omega^2 p_1 - p_3)^2 + (\omega^3 - \omega p_2)^2$$

$$\omega^6 + \omega^4(p_1^2 - 2p_2 - q_1^2) + \omega^2(p_2^2 - 2p_1 p_3 - q_2^2 + 2q_1 q_3) + q_3^2 + p_3^2 = 0$$

$$\text{Let } \psi(p) = p^3 + p^2 N_1 + p N_2 + N_3 = 0 \quad (4.6)$$

Where

$$N_1 = p_1^2 - 2p_1 - q_1^2$$

$$N_2 = p_2^2 - 2p_1p_2 - q_2^2 + 2q_1q_3$$

$$N_3 = q_3^3 + p_3^3$$

$$p = \omega^2$$

$$\therefore \psi(p) = 0$$

If we assume that $N_1 > 0, N_2 > 0, N_3 > 0$ then equation (4.6) has no positive real roots

Therefore, the equation (4.6) admits negative real roots. Hence, we can derive the conditions for existences of stability at endemic equilibrium point.

Theorem 4. 1: The endemic equilibrium E_2 is locally asymptotically stable for all τ , if following conditions hold.

$$(i). R_0 > 1$$

$$(ii). (p_1 + q_1) > 0, (p_2 + q_2) > 0, (p_3 + q_3) > 0$$

$$(ii). N_1 > 0, N_2 > 0, N_3 > 0$$

Now if any one of N_1, N_2, N_3 is negative then equation (4.6) has a positive root ω_0

Eliminating $\cos \omega \tau$ from equation (4.5)

$$\cos \omega \tau = \frac{\begin{vmatrix} \omega^2 p_1 - p_3 & q_2 \omega \\ \omega^3 - \omega p_2 & -(q_3 - q_1 \omega^2) \end{vmatrix}}{\begin{vmatrix} q_3 - q_1 \omega^2 & q_2 \omega \\ q_2 \omega & -(q_3 - q_1 \omega^2) \end{vmatrix}}$$

$$\cos \omega \tau = \frac{\omega^2 p_1 q_3 - p_3 q_3 - \omega^4 p_1 q_1 + \omega^2 q_1 p_3 + q_3 \omega^3 - p_2 q_3 \omega^2}{q_3^2 + q_1^2 \omega^4 - 2q_1 q_3 \omega^2 + q_2^2 \omega^2}$$

$$\tau_k = \frac{1}{\omega_0} \cos^{-1} \left[\frac{\omega_0^4 (q_3 - p_1 q_1) + \omega_0^2 (p_1 q_3 + q_1 p_3 - p_2 q_3) - p_3 q_3}{q_1^2 \omega_0^4 + \omega_0^2 (q_2^2 - 2q_1 q_3) + q_3^2} \right] + \frac{2k\pi}{\omega_0}$$

where $k = 0, 1, 2, 3, \dots$

4.1 HOPF BIFURACTION

Based on the above results the following holds good

Theorem 4.1.2: If $R_0 > 1$ there exist a positive τ_0 such that the following results hold

- (i) If $0 < \tau < \tau_0$, the system of equations (2.1) has an endemic equilibrium point E_2 is locally asymptotically stable.
- (ii) The system (2.1) undergoes a Hopf bifurcation if $\tau > \tau_0$

Proof: To obtain Hopf bifurcation we need to check the transversal condition for the existence of complex eigen root of the endemic equilibrium point (E_2) at $\tau = \tau_0$

i.e. the real part of $\lambda(t)$ become positive when $\tau > \tau_0$ and the steady state become unstable moreover, a Hopf bifurcations when τ passes through the critical value τ_0 .

Differentiating (4.2) W.r.t. τ , we get

$$= 3\lambda^2 \frac{d\lambda}{d\tau} + 2p_1\lambda \frac{d\lambda}{d\tau} + p_2 \frac{d\lambda}{d\tau} + e^{-\lambda\tau} (2q_1\lambda \frac{d\lambda}{d\tau} + q_2 \frac{d\lambda}{d\tau}) + (q_1\lambda^2 + q_2\lambda + q_3)(-\lambda - \lambda \frac{d\lambda}{d\tau})e^{-\lambda\tau} = 0$$

$$\frac{d\lambda}{d\tau} [3\lambda^2 + 2p_1\lambda + p_2 + e^{-\lambda\tau} (2q_1\lambda + q_2) - (q_1\lambda^2 + q_2\lambda + q_3)\lambda e^{-\lambda\tau}] = (q_1\lambda^2 + q_2\lambda + q_3)\lambda e^{-\lambda\tau}$$

$$\left[\frac{d\lambda}{d\tau} \right]^{-1} = \frac{[3\lambda^2 + 2p_1\lambda + p_2 + e^{-\lambda\tau} (2q_1\lambda + q_2) - (q_1\lambda^2 + q_2\lambda + q_3)\lambda e^{-\lambda\tau}]}{(q_1\lambda^2 + q_2\lambda + q_3)\lambda e^{-\lambda\tau}}$$

$$\left[\frac{d\lambda}{d\tau} \right]^{-1} = \frac{3\lambda^2 + 2p_1\lambda + p_2}{(q_1\lambda^2 + q_2\lambda + q_3)\lambda e^{-\lambda\tau}} + \frac{(2q_1\lambda + q_2)}{(q_1\lambda^2 + q_2\lambda + q_3)\lambda} - \frac{\tau}{\lambda}$$

$$\left[\frac{d\lambda}{d\tau} \right]^{-1} = \frac{3\lambda^2 + 2p_1\lambda + p_2}{-\lambda(\lambda^3 + p_1\lambda^2 + p_2\lambda + p_3)} + \frac{(2q_1\lambda + q_2)}{(q_1\lambda^2 + q_2\lambda + q_3)\lambda} - \frac{\tau}{\lambda}$$

$$\lambda = i\omega_0$$

$$\left[\frac{d\lambda}{d\tau} \right]^{-1} = \frac{1}{\omega_0} \left[\frac{-3\omega_0^2 + 2ip_1\omega_0 + p_2}{(-\omega_0^3 + p_1\omega_0^2 + ip_1\omega_0^2 - p_3)^2} + \frac{(2iq_1\omega_0 + q_2)}{-q_2\omega_0 + i(q_3 - q_1\omega_0^2)} + \tau i \right]$$

$$\left[\frac{d\lambda}{d\tau} \right]^{-1} = \frac{1}{\omega_0} \left[\frac{(-3\omega_0^2 + 2ip_1\omega_0 + p_2)((-\omega_0^3 + p_2\omega_0 - i(p_1\omega_0^2 - p_3)) + \frac{(2iq_1\omega_0 + q_2)(-q_2\omega_0 - i(q_3 - q_1\omega_0^2))}{(q_2\omega_0)^2 + (q_3 - q_1\omega_0^2)^2} + \tau i \right]$$

$$\text{Real part of } \left[\frac{d\lambda}{d\tau} \right]^{-1}$$

$$= \frac{1}{\omega_0} \left[\frac{(-3\omega_0^2 + p_2)(-\omega_0^3 + p_2\omega_0) + 2p_1\omega_0(p_1\omega_0^2 - p_3)}{(-\omega_0^3 + p_2\omega_0)^2 + (p_1\omega_0^2 - p_3)^2} + \frac{-q_2^2\omega_0 + 2q_1\omega_0(q_3 - q_1\omega_0^2)}{(q_2\omega_0)^2 + (q_3 - q_1\omega_0^2)^2} \right]$$

$$(-\omega_0^3 + p_2\omega_0)^2 + (p_1\omega_0^2 - p_3)^2 = (q_2\omega_0)^2 + (q_3 - q_1\omega_0^2)^2$$

$$= \frac{1}{\omega_0} \left[\frac{3\omega_0^5 + \omega_0^3(2p_1^2 - p_2 - 3p_2 - 2q_1^2) + (p_2^2 - 2p_1p_3 + 2q_1q_2 - q_2^2)\omega_0}{(q_2\omega_0)^2 + (q_3 - q_1\omega_0^2)^2} \right]$$

$$\text{Re} \left[\frac{d\lambda}{d\tau} \right]^{-1} = \left[\frac{3\omega_0^4 + \omega_0^2(2p_1^2 - 4p_2 - 2q_1^2) + p_2^2 - 2p_1p_3 + 2q_1q_2 - q_2^2}{(q_2\omega_0)^2 + (q_3 - q_1\omega_0^2)^2} \right]$$

$$\left[\frac{d}{d\tau} \text{Re}(\lambda) \right] = \left[\text{Re} \left(\frac{d\lambda}{d\tau} \right)^{-1} \right]_{\lambda=i\omega_0} = \left[\frac{3\omega_0^4 + \omega_0^2(2p_1^2 - 4p_2 - 2q_1^2) + p_2^2 - 2p_1p_3 + 2q_1q_2 - q_2^2}{(q_2\omega_0)^2 + (q_3 - q_1\omega_0^2)^2} \right]$$

Under the condition that $p_1^2 - 2p_1 - q_1^2 > 0$ & $p_2^2 - 2p_1p_2 - q_2^2 + 2q_1q_3 > 0$

$$\text{We have } \left[\frac{d}{d\tau} \text{Re}(\lambda) \right]_{\lambda=i\omega_0} > 0$$

Therefore, the transversality condition holds and Hopf bifurcation occurs at $\tau = \tau_0$

5. NUMERICAL EXAMPLE

In each graph, figure (A) represents Time series responses and (B) represents Phase portraits

Example 5.1: let us take $N=15$, $b=0.7$, $r = 0.1$, $\beta = 0.5$, $x = 7$, $y = 5$, $z = 3$

The system of equations (2.1) has an equilibrium point $E(2, 12, 2)$ and $R_0 = 9.375 > 1$. We identified the critical positive time delay parameter $\tau_0 = 0.3$, for $0 < \tau < \tau_0$, E_2 is asymptotically stable. When $\tau_0 \geq 0.34$, the endemic equilibrium point loses its stability and exhibit periodic solutions bifurcate from E_2 is shown in the graphs (5.1.1 -5.1.3)

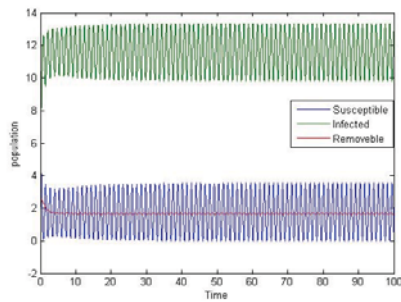


Fig 5.1.1(A)

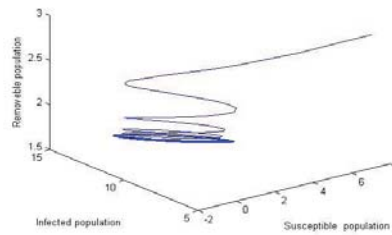


Fig 5.1.1(B)

Unstable variations of system (2.1) when $\tau=0.35$

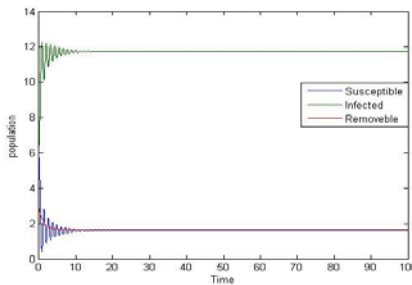


Fig 5.1. 2(A)

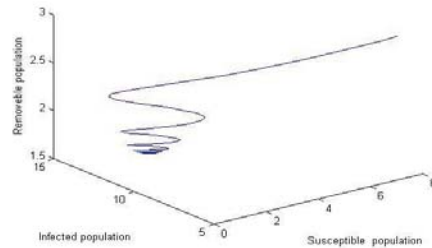


Fig 5.1.2(B)

Stable variation of system (2.1) when $\tau=0.3$

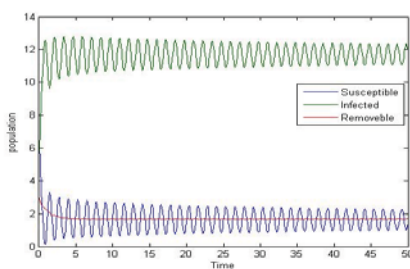


Fig 5.1. 3(A)

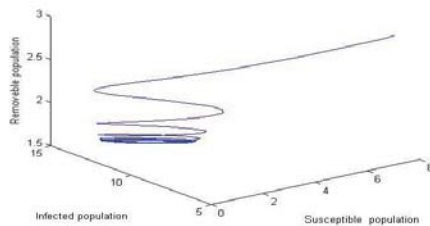


Fig 5.1.3(B)

Stable variations of system (2.1) when $\tau=0.34$

similar kind of analysis is done by taking different examples shown below and identified the critical values τ_0 which bifurcation took place.

Example 5. 2: $N=18$, $b=1.5$, $r = 0.1$, $\beta = 0.5$, $x = 8$, $y = 7$, $z = 3$.

The system is has a unique positive equilibrium point $E(4, 14, 1)$ and $R_0 = 5.625 > 1$. The critical positive time delay parameter $\tau_0 = 0.36$, for $0 < \tau < \tau_0$, E_2 is asymptotically stable. When $\tau_0 \geq 0.36$, the endemic equilibrium point losses its stability and exhibit periodic solutions bifurcate from E_2 is shown in the graphs (5.2.1 -5.2.4)

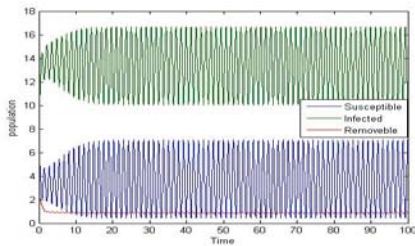


Fig 5.2.1.(A)

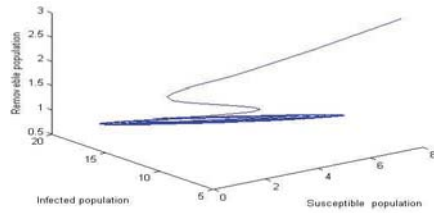


Fig 5.2.1(B)

Unstable variation of system (2.1) when $\tau=0.4$

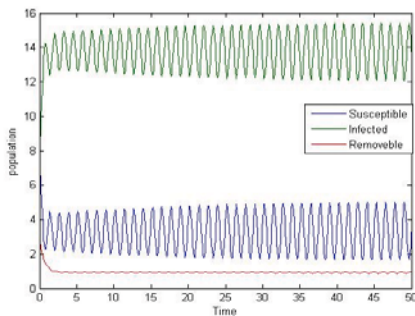


Fig 5.2.2.(A)

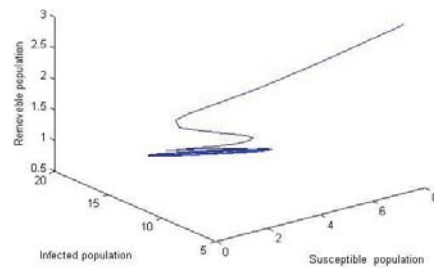


Fig 5.2.2(B)

Unstable variation of system (2.1) when $\tau=0.37$

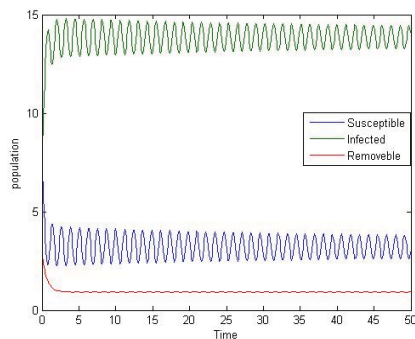


Fig 5.2.3(A)

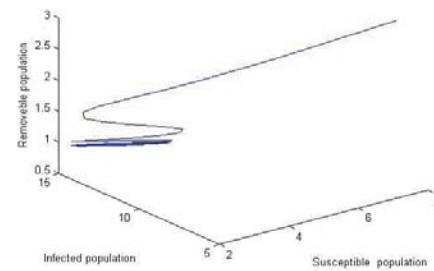


Fig 5.2.3(B)

Stable variation of populations when $\tau=0.36$

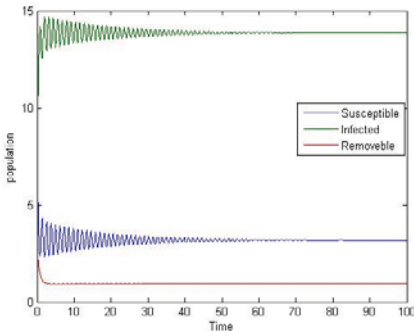


Fig 5.2.4(A)

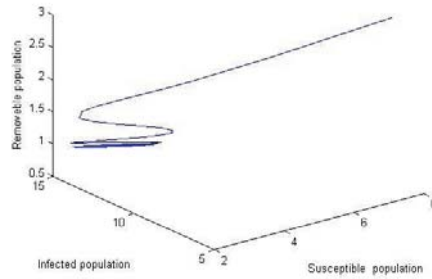


Fig 5.2.4(B)

Stable variation of populations when $\tau=0.35$

Example 5. 3: $N=10$, $b=0.5$, $r = 0.05$, $\beta = 0.5$, $x = 4$, $y = 3$, $z = 3$

The critical positive time delay parameter $\tau_0 = 0.49$, for $0 < \tau < \tau_0$, E_2 is asymptotically stable. when $\tau_0 \geq 0.49$, the system bifurcate from E_2 is shown in the graphs (5.3.1 -5.3.4)

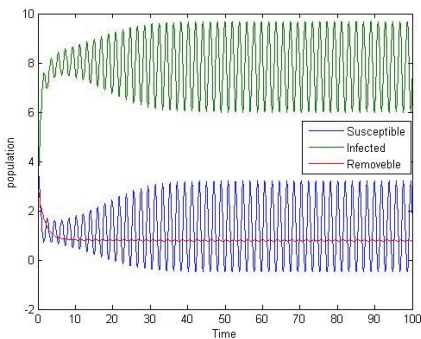


Fig 5.3.1(A)

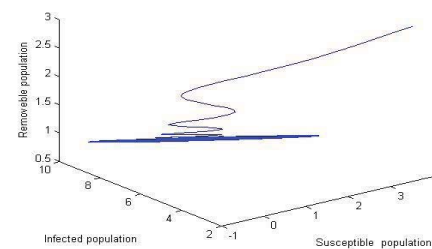


Fig 5.3.1(B)

Unstable variation of system (2.1) when $\tau=0.53$

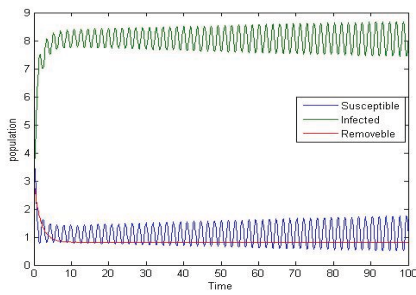


Fig 5.3.2(A)

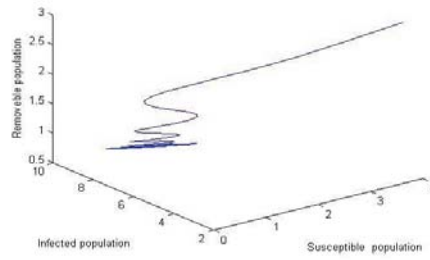


Fig 5.3.2(B)

Unstable variation of system (2.1) when $\tau=0.5$

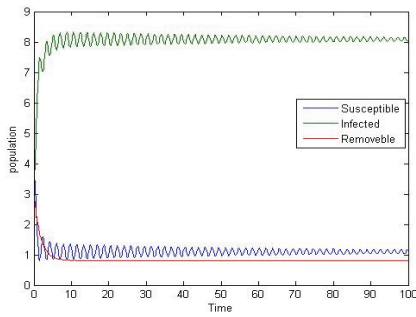


Fig 5.3.3(A)

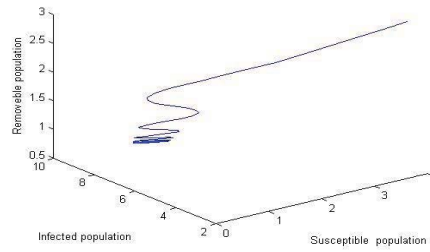


Fig 5.3.3(B)

Stable variation of system (2.1) when $\tau=4.9$

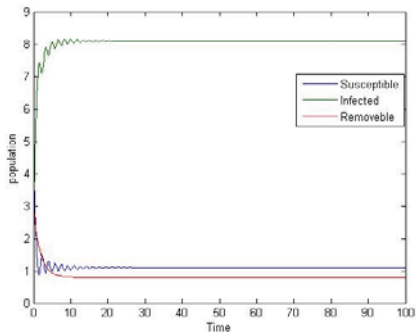


Fig 5.3.4(A)

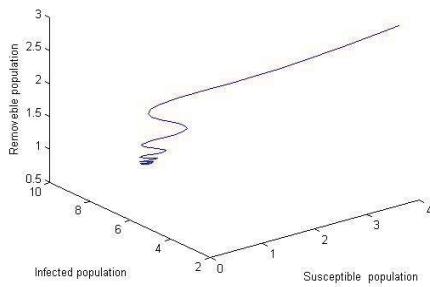


Fig 5.3.4(B)

Stable variation of system (2.1) when $\tau=4.5$

Example4: $N=20$, $b=1.5$, $r = 0.05$, $\beta = 0.5$, $x = 10$, $y = 5$, $z = 5$

The system has an equilibrium point $E(3, 17, 0)$ and $R_0 = 6.45 > 1$. The critical positive time delay parameter $\tau_0 = 0.28$, for $0 < \tau < \tau_0$, E_2 is asymptotically stable. when $\tau_0 \geq 0.28$, the endemic equilibrium point loses its stability and exhibit periodic solutions bifurcate from E_2 is shown in the graphs (5.4.1 -5.4.3)

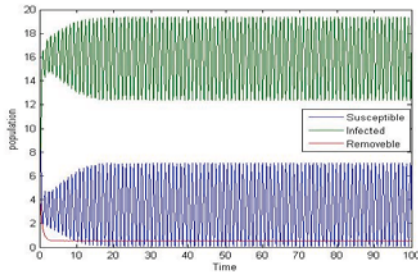


Fig 5.4.1(A)

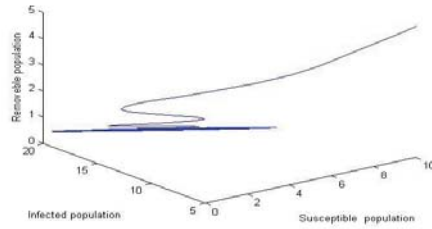


Fig 5.4.1(B)

Unstable variation of system (2.1) when $\tau=0.3$

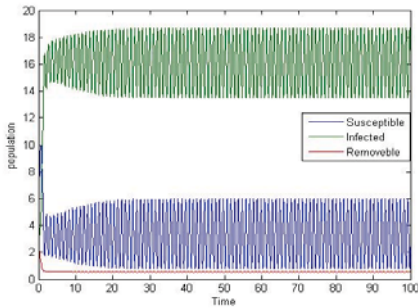


Fig 5.4.2(A)

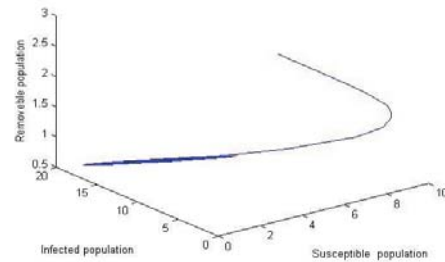


Fig 5.4.2(B)

Unstable variation of system (2.1) when $\tau=0.29$

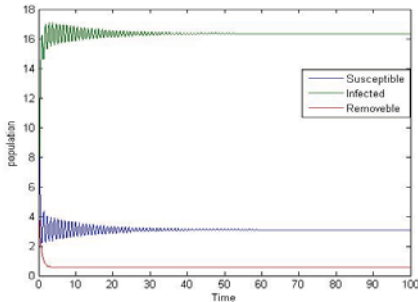


Fig 5.4.3(A)

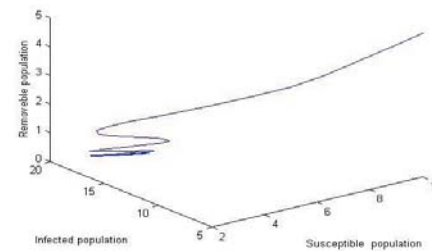


Fig 5.4.3(B)

Stable variation of system (2.1) when $\tau=0.27$

Example 5.5: $N=40$, $b=2.0$, $r = 0.05$, $\beta = 0.25$, $x = 20$, $y = 10$, $z = 10$

The system has an equilibrium point $E(10, 29, 1)$ and $R_0 = 4.87 > 1$. The critical positive time delay parameter $\tau_0 = 0.36$, for $0 < \tau < \tau_0$, E_2 is asymptotically stable. From theorem (2) when $\tau \geq 0.36$, the endemic equilibrium point loses its stability and exhibit periodic solutions bifurcate from E_2 is shown in the graphs (5.5.1 -5.5.3)

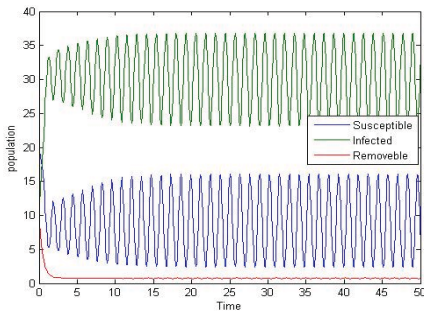


Fig 5.5.1(A)

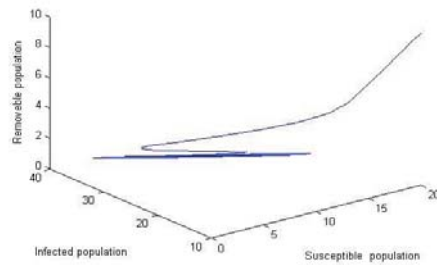


Fig 5.5.1(B)

Unstable variation of system (2.1) when $\tau=0.4$

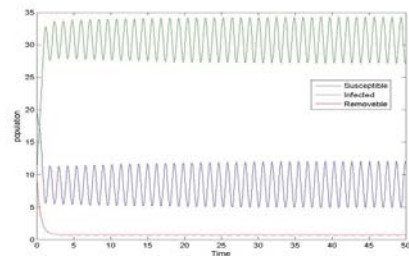


Fig 5.5.2(A)

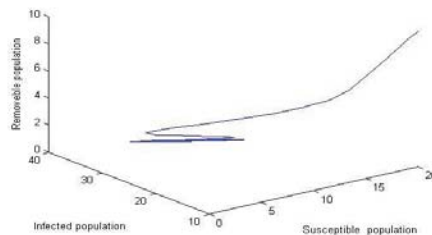


Fig 5.5.2(B)

Unstable variation of system (2.1) when $\tau=0.37$

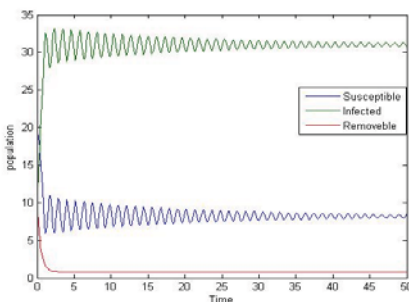


Fig 5.5.3(A)

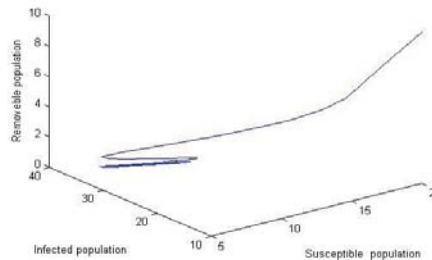


Fig 5.5.3(B)

Stable variation of system (2.1) when $\tau=0.35$

6. DISCUSSION AND CONCLUSIONS

A micro parasite model is considered for investigation with three compartments (susceptible, infective, and removable). The total population (N) is constant in view of births and deaths. The delay parameter is incorporated in the interaction of susceptible (x), infective (y) populations. From the above results the system is asymptotically at endemic equilibrium point if $\bar{y} > \bar{x}$.

The local stability of endemic equilibrium point depends on time delay parameter (τ). The system exhibits Hopf bifurcation when the τ crosses the critical value τ_0 . In support of bifurcation analysis, we choose five set of parametric values and identified the critical bifurcation parameter (τ_0). The critical parameter (τ_0) for each example is given as follows

S.No	Example	Critical positive value (τ_0)
1	1	$\tau_0 = 0.3$
2	2	$\tau_0 = 0.36$
3	3	$\tau_0 = 0.49$
4	4	$\tau_0 = 0.28$
5	5	$\tau_0 = 0.36$

Hence the system exhibits Hopf bifurcation, and the stability analysis is dependent on critical positive value (τ_0).

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