

A COVID-19 time-dependent SIRD model using functional differential equations

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ABSTRACT

In this paper we develop a mathematical model for the spread of the coronavirus disease 2019 (COVID-19) with the goal to do forecasts for the number of infected, recovered and deceased individuals. To do this, we propose a SIRD model, which takes into account the fact that susceptible individuals that meet infected individuals are not immediately infected, they become infected after a period which is denoted by τ . Similarly, infected individuals recovered or deceased not immediately but after a period which is denoted by δ . These considerations produce discrete delays in the classic SIRD model which is a particular case of functional differential equations. Furthermore, to take into account the effect of the social distancing, mask use or other actions to reduce the risk of infectivity, we consider that the transmission, the recovery and the death rates are not constants but time-dependent. To apply the model proposed, we developed an algorithm which uses ridge regression to do forecasts for the Peruvian regions of Moquegua and Tacna.

Keywords: Covid-19, SIRD model, Functional differential equations, Ridge regression.

2000 Mathematics Subject Classification: 92B05, 34K99, 62J07.

1 Introduction

The SARS-CoV-2 (or Coronavirus) is a new respiratory virus that not has not been previously identified. The disease it causes has been called “coronavirus disease 2019” (abbreviated “COVID-19”). The first COVID-19 case was reported in Wuhan, Hubei province, China, in December 2019. In the weeks that followed, the disease spread around the globe at a frightening pace. The first case of COVID-19 in Perú was confirmed on March 06 of 2020. After this date more infected people were reported in different Peruvian regions. In the region of Moquegua the first infected person was reported on April 04 of 2020, while in the region of Tacna the first case was reported on March 29 of 2020. To contain the spread, several sanitary authorities have adopted strategies such as social distancing. This produce effects in the local economy. For this reason we believe that the use of Mathematical models can help to observe the trend of the epidemic.

Since the fundamental epidemic model of (Kermack and McKendrick, 1927), several models were proposed to improve that work. Particularly in the COVID-19 case, several works attempted to adapt that work to the case in China (Ivorra, Ferrández, Vela-Pérez, and Ramos,

2020) for example, the authors proposed a model taking into account the undetected infections. The work (Chen, Lu, Chang and Liu, 2020), considers a model taking into account the undetectable infected individuals and with the coefficients time-dependent. The work (Maleewong, 2020) considers a model with delay to take into account the observation that a susceptible person will become infected due to the contact with an infected person some days before, but the model considers the case when the coefficients are constant. The novelty of the present work is that it proposes an epidemic model that takes into account that the present number of infected individuals was produced by the interactions between susceptible and infected people in the past. Analogously, the present number of recovered or deceased individuals are the result of infected individuals in the past. Furthermore, we consider that the coefficients are time-dependent.

The present article is organized in the following way: Section 2 is devoted to notation, definitions and the formulation of the epidemic model used throughout the text. In Section 3 we develop the algorithm used to do the forecasts using ridge regression. Section 4 is devoted to the use of the algorithm developed in the previous section which will do some forecasts for the Peruvian regions of Moquegua and Tacna. Finally, section 5 is dedicated to conclusions.

2 Epidemic models using functional differential equations

The SIR (Susceptible/Infectious/Removed) epidemic model is a system of differential equations that describe the spread of a disease in a population (see Kermack and McKendrick (Kermack and McKendrick, 1927)). An alternative to the SIR model is given by the SIRD mode which takes into account possible deceased patients:

$$\left\{ \begin{array}{l} \frac{dS}{dt} = -\beta S(t)I(t) \\ \frac{dI}{dt} = \beta S(t)I(t) - \eta I(t) - \mu I(t) \\ \frac{dR}{dt} = \eta I(t) \\ \frac{dD}{dt} = \mu I(t). \end{array} \right. \quad (2.1)$$

The variables S and I are as in the SIR model. The variable R gets a new meaning, Recovered, because now the variable D stands for (number of) deceased individuals. The constants β , η , and μ are the mean infection rate, recovery rate and death rate, respectively. The total number of individuals in the system is $N = S + I + R + D$, which is supposed to be constant. The number of individuals in each group is divided by N . In this model the following assumptions are clearly implicit:

- The number of susceptible (S) and contagious infective (I) individuals is very high. So, the random differences between individuals can be neglected.
- The disease spreads through contact from individual to individual.
- Some infected individuals can decrease (denoted by D).
- Recovered individuals (R) become immune.

- The population is closed with no migration.

As we mentioned before β , η , and μ in (2.1) are constant over time. However, in real situations these are time-varying coefficients because generally every local government adopts policies to mitigate the spread. Besides that, one could argue that susceptible individuals that meet an infected person is not immediately infected. They become infected after some time, which implies an incubation period, let's say, $\tau > 0$. Similarly, recovering would demand a convalescence period, let's say, $\delta > 0$. Analogously, infected individuals can decrease after a period, which for simplicity we will adopt the same value as the convalescence period. Taking into account these observations we consider the following SIRD functional differential equation model:

$$\begin{cases} \frac{dS}{dt} = -\beta(t)S(t-\tau)I(t-\tau), \\ \frac{dI}{dt} = \beta(t)S(t-\tau)I(t-\tau) - \eta(t)I(t-\delta) - \mu(t)I(t-\delta), \\ \frac{dR}{dt} = \eta(t)I(t-\delta), \\ \frac{dD}{dt} = \mu(t)I(t-\delta). \end{cases} \quad (2.2)$$

This system is supplemented by initial data $S(t_0 + \theta) = S_0(\theta) > 0$, $I(t_0 + \theta) = I_0(\theta) > 0$, $R(t_0 + \theta) = 0$ and $D(t_0 + \theta) = 0$ with $\theta \in [-\delta, 0]$, $\tau < \delta$ and t_0 the time when the epidemic started. Furthermore, we will suppose that the functions β , η and μ are measurable.

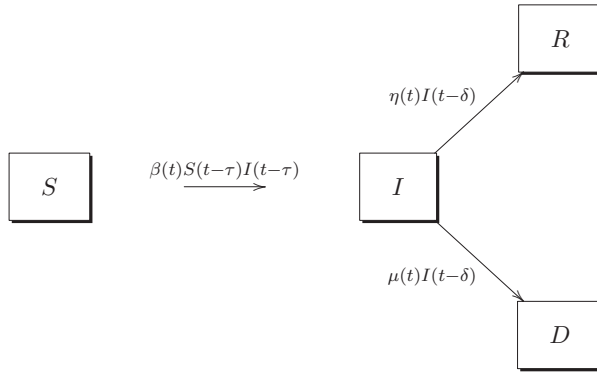


Figure 1: Epidemiological diagram for the Coronavirus model (2.2).

The system (2.2) contains discrete delays and is a particular case of Functional Differential Equations (FDEs). The names Hereditary Differential Equations (HDEs), Retarded Functional Differential Equations (RFDEs), Delay Differential Equations (DDEs), differential equations with aftereffects are used frequently in the FDE literature.

In order to ensure that there is a solution for the system (2.2), we need to provide some terminology. So, consider r and T two positive and finite numbers, we will denote by $\mathbf{C} = C([-r, 0]; \mathbb{R}^n)$ the space of continuous functions $\phi : [-r, 0] \rightarrow \mathbb{R}^n$ equipped with the norm

$$\|\phi\|_{\mathbf{C}} = \sup_{\theta \in [-r, 0]} |\phi(\theta)|, \quad \phi \in \mathbf{C}.$$

If $\psi \in C([-r, T]; \mathbb{R}^n)$, we define, for each $s \in [0, T]$, the function $\psi_s : [-r, 0] \rightarrow \mathbb{R}^n$ by

$$\psi_s(\theta) = \psi(s + \theta), \quad \theta \in [-r, 0].$$

A n -dimensional functional differential equation with bounded memory r and initial datum $(t, x_t) = (t, \phi) \in [0, T] \times \mathbf{C}$ is normally described as follows:

$$\begin{aligned} \frac{dx(s)}{ds} &= f(s, x_s), \quad s \in [t, T], \\ (t, x_t) &= (t, \phi), \end{aligned} \quad (2.3)$$

with $f : [0, T] \times \mathbf{C} \rightarrow \mathbb{R}^n$ satisfying the following conditions to guarantee the existence of a solution to the problem (2.3).

To demonstrate the existence of a solution we will use the Carathéodory's existence theorem. So, suppose Ω is an open subset of $[0, T] \times \mathbf{C}$. A function $f : \Omega \rightarrow \mathbb{R}^n$ is said to satisfy the *Carathéodory conditions* on $\mathbf{C}$ if:

C1 $f(s, \phi)$ is measurable in s for each fixed ϕ , continuous in ϕ for each fixed s .

C2 for any fixed $(t, \phi) \in \Omega$, there is a neighborhood $V(t, \phi)$ and a Lebesgue integrable function m such that

$$|f(s, \psi)| \leq m(s), \quad (s, \psi) \in V(t, \phi).$$

Theorem 2.1. Suppose that the conditions **C1** and **C2** are satisfied. Let $(t, \phi) \in \Omega$, then there is an $A > 0$ such that (2.3) has a solution $\{x = x(s; t, \phi), s \in [t - r, t + A]\}$. Furthermore $x \in C([t - r, t + A], \mathbb{R}^n)$, $x_t = \phi$, and $x(s)$ is absolutely continuous on $[t, t + A]$ and satisfies $\frac{dx(s)}{ds} = f(s, x_s)$ almost everywhere on $[t, t + A]$.

Proof. See (Hale, 1977) p.55. □

We will use the last theorem with

$$\begin{aligned} f(s, \phi) &= (\beta(s) \phi_1(-\tau) \phi_2(-\tau), \beta(s) \phi_1(-\tau) \phi_2(-\tau) - \eta(s) \phi_2(-\delta) - \mu(s) \phi_2(-\delta), \\ &\quad \eta(s) \phi_2(-\delta), \mu(s) \phi_2(-\delta)) \end{aligned}$$

for $\phi = (\phi_1, \phi_2, \phi_3, \phi_4) = (S, I, R, D)$. It is not hard to demonstrate that f satisfies the Carathéodory condition on $\mathbf{C}$. Using the standard continuation method, the local solution obtained in the last theorem can be extended on $[t, T]$ for any T with $0 < t + A \leq T$. Standard arguments are used to demonstrate the uniqueness of the solution to (2.2).

Thus, let (S, I, R, D) be the solution to (2.2) with specified initial conditions.

3 The algorithm used for forecasting

Due to the fact the COVID-19 data are updated daily, we can consider the following discrete time difference system corresponding to (2.2):

$$\begin{aligned} S(t+1) - S(t) &= -\beta(t)S(t-\tau)I(t-\tau), \\ I(t+1) - I(t) &= \beta(t)S(t-\tau)I(t-\tau) - \eta(t)I(t-\delta) - \mu(t)I(t-\delta), \\ R(t+1) - R(t) &= \eta(t)I(t-\delta), \\ D(t+1) - D(t) &= \mu(t)I(t-\delta). \end{aligned} \quad (3.1)$$

From the first, third and fourth equation above we obtain

$$\begin{aligned}\beta(t) &= -\frac{S(t+1) - S(t)}{S(t-\tau)I(t-\tau)}, \\ \eta(t) &= \frac{R(t+1) - R(t)}{I(t-\delta)}, \\ \mu(t) &= \frac{D(t+1) - D(t)}{I(t-\delta)},\end{aligned}\tag{3.2}$$

respectively. Hence, we will estimate $\hat{\beta}$, $\hat{\eta}$ and $\hat{\mu}$ with the purpose of forecasting S , I , R , and D . In order to do this we need historical data from a certain period $\{(S(t), I(t), R(t), D(t)), 1 \leq t \leq T-1\}$, we can measure the corresponding $\{(\beta(t)), \tau+1 \leq t \leq T-2\}$, $\{(\eta(t)), \delta+1 \leq t \leq T-2\}$ and $\{(\mu(t)), \tau+1 \leq t \leq T-2\}$ by means of (3.2).

3.1 Estimation of rates of Transmission β , Recovering η and Death μ by Ridge Regression

In this section we adapt the arguments from (Chen et al, 2020). Thus, we denote by $\hat{\beta}$, $\hat{\eta}$ and $\hat{\mu}$ the forecasted rates. From the **FIR** filters, they are forecasted as follows:

$$\begin{aligned}\hat{\beta}(t) &= b_1\beta(t-1) + b_2\beta(t-2) + \dots + b_J\beta(t-J) + b_0, \\ &= \sum_{j=1}^J b_j\beta(t-j) + b_0, \\ \hat{\eta}(t) &= e_1\eta(t-1) + e_2\eta(t-2) + \dots + e_K\eta(t-K) + e_0, \\ &= \sum_{j=1}^K e_j\eta(t-j) + e_0, \\ \hat{\mu}(t) &= u_1\mu(t-1) + u_2\mu(t-2) + \dots + u_L\mu(t-L) + u_0, \\ &= \sum_{j=1}^L u_j\mu(t-j) + u_0\end{aligned}\tag{3.3}$$

where J , K and L are the orders of the **FIR** filters with $0 < J, K, L < T-2$, and b_i , $i = 1, \dots, J$, e_j , $j = 1, \dots, K$, u_k , $k = 1, \dots, L$ are the coefficients of the impulse response of an **FIR** filter which we estimate by using ridge regression that solves the following optimization problem:

$$\begin{aligned}\min_{b_j} \quad & \sum_{t=J+1}^{T-2} (\beta(t) - \hat{\beta}(t))^2 + \alpha_1 \sum_{j=0}^J b_j^2, \\ \min_{e_j} \quad & \sum_{t=K+1}^{T-2} (\eta(t) - \hat{\eta}(t))^2 + \alpha_2 \sum_{j=0}^K e_j^2, \\ \min_{u_j} \quad & \sum_{t=L+1}^{T-2} (\mu(t) - \hat{\mu}(t))^2 + \alpha_3 \sum_{j=0}^L u_j^2\end{aligned}\tag{3.4}$$

To resolve the optimization problem (3.4) we are following (Hastie, Tibshirani, and Friedman, 2009) p. 64. With this solution we can forecast $\hat{\beta}(t)$, $\hat{\eta}(t)$ and $\hat{\mu}(t)$ for $t = T-1$ using (3.3). To predict $\hat{S}(t)$, $\hat{I}(t)$, $\hat{R}(t)$ and $\hat{D}(t)$ at time T , we will replace β , η and μ by $\hat{\beta}(t)$, $\hat{\eta}(t)$ and $\hat{\mu}(t)$ in (3.1) to get

$$\begin{aligned}\hat{S}(T) &= \hat{S}(T-1) - \hat{\beta}(T-1)\hat{S}(T-1-\tau)\hat{I}(T-1-\tau), \\ \hat{I}(T) &= \hat{I}(T-1) + \hat{\beta}(T-1)\hat{S}(T-1-\tau)\hat{I}(T-1-\tau) - \\ &\quad \hat{\eta}(T-1)\hat{I}(T-1-\delta) - \hat{\mu}(T-1)\hat{I}(T-1-\delta), \\ \hat{R}(T) &= \hat{R}(T-1) + \hat{\eta}(T-1)\hat{I}(T-1-\delta), \\ \hat{D}(T) &= \hat{D}(T-1) + \hat{\mu}(T-1)\hat{I}(T-1-\delta).\end{aligned}\tag{3.5}$$

Instead of taking the forecast $\hat{S}(T)$ obtained above we will replace by

$$\hat{S}(T) = 1 - \hat{I}(T) - \hat{R}(T) - \hat{D}(T). \quad (3.6)$$

To forecast $\hat{I}(t)$, $\hat{R}(t)$ and $\hat{D}(t)$ for $t > T$, first we estimate $\hat{\beta}(t)$, $\hat{\eta}(t)$ and $\hat{\mu}(t)$ by using (3.3), then we use

$$\begin{aligned} \hat{S}(t+1) &= \hat{S}(t) - \hat{\beta}(t)\hat{S}(t-\tau)\hat{I}(t-\tau), \\ \hat{I}(t+1) &= \hat{I}(t) + \hat{\beta}(t-\tau)\hat{S}(t)\hat{I}(t-\tau) - \\ &\quad \hat{\mu}(t)\hat{I}(t-\delta), \\ \hat{R}(t+1) &= \hat{R}(t) + \hat{\eta}(t)\hat{I}(t-\delta), \\ \hat{D}(t+1) &= \hat{D}(t) + \hat{\mu}(t)\hat{I}(t-\delta) \quad t \geq T. \end{aligned} \quad (3.7)$$

Update

$$\hat{S}(t+1) = 1 - \hat{I}(t+1) - \hat{R}(t+1) - \hat{D}(t+1), \quad t \geq T. \quad (3.8)$$

Algorithm: Forecast method

Input: $\{S(t), I(t), R(t), D(t), 1 \leq t \leq T-1\}$. Regularization parameters α_1 , α_2 and α_3 . Order of FIR filters J , K and L . Window forecast P .

Output:

$$\begin{aligned} &\{\beta(t), \tau+1 \leq t \leq T-2\}, \{\eta(t), \mu(t), \delta+1 \leq t \leq T-2\}, \\ &\{\hat{\beta}(t), \hat{\eta}(t), \hat{\mu}(t), t \geq T-1\}, \text{ and} \\ &\{\hat{S}(t), \hat{I}(t), \hat{R}(t), \hat{D}(t) \mid t \geq T\} \end{aligned}$$

1. Measure $\{\beta(t), \tau+1 \leq t \leq T-2\}$ and $\{\eta(t), \mu(t), \delta+1 \leq t \leq T-2\}$ using (3.2).
 2. Train the ridge regression using (3.4).
 3. Estimate $\hat{\beta}(T-1)$, $\hat{\eta}(T-1)$ and $\hat{\mu}(T-1)$ from (3.3).
 4. Calculate the forecast of the number of susceptible individuals $\hat{S}(T)$, infected individuals $\hat{I}(T)$, recovered individuals $\hat{R}(T)$ and deceased individuals $\hat{D}(T)$ to next day T from (3.5). Replace $\hat{S}(T) = 1 - \hat{I}(T) - \hat{R}(T) - \hat{D}(T)$.
 5. **while** $T \leq t \leq T+P$ **do**
 - Estimate $\hat{\beta}(t)$, $\hat{\eta}(t)$ and $\hat{\mu}(t)$ from (3.3).
 - Predict $\hat{S}(t)$, $\hat{I}(t)$, $\hat{R}(t)$ and $\hat{D}(t)$ from (3.7).
 - Replace $\hat{S}(t) = 1 - \hat{I}(t) - \hat{R}(t) - \hat{D}(t)$.
- end while**

4 Forecasts

4.1 Dataset

In this subsection we do some forecasts to estimate the trend of COVID-19 by using the SIRD (2.2) model in Section 3. We collected the dataset from the institutional facebook page of the Health Regional Management for Moquegua (GERESA, 2020) and Health Regional Direction for Tacna (DIRESA, 2020). In order to do the forecasts we considered the dataset from May 25, 2020 for Moquegua and from April 24, 2020 for Tacna. The forecasts are for the total of infected, recovered and deceased individuals for a period of seven days. In the figures we adopt the notation Infect, Predinfect, Recov, PredRecov, Dece and PredDece for infected,

forecast for infected, recovered, forecast for recovered, deceased and forecast for deceased individuals, respectively.

4.2 Parameter Setup

A. Moquegua

We use $N = 192740$ for population size, which is a estimation for June 30 of 2020 made for the National Institute of Statistics and Computing of Perú (INEI, 2020), $J = 6$, $K = 5$ and $L = 7$ for the orders of the **FIR** filters $\beta(t)$, $\eta(t)$ and $\mu(t)$. For the delays we take $\tau = 7$ and $\delta = 15$. We use the software Scilab 6.0.0 to compute the ridge regression. With respect to the regularization parameters we take the following values:

- 300, 300 and 3000 for $\beta(t)$, $\eta(t)$ and $\mu(t)$, respectively. Forecasts made from August 06.
- 3000000, 1 and 3000000 for $\beta(t)$, $\eta(t)$ and $\mu(t)$, respectively. Forecasts made from August 19.
- 300, 300 and 100 for $\beta(t)$, $\eta(t)$ and $\mu(t)$, respectively. Forecasts made from September 19.
- 400, 1, 1/10000 for $\beta(t)$, $\eta(t)$ and $\mu(t)$, respectively. Forecasts made from October 13.

We will explain the reasons why we made these forecasts in subsection 4.3.

B. Tacna

We use $N = 370974$ for population size, which is an estimate for June 30 of 2020 made for the National Institute of Statistics and Computing of Perú (INEI, 2020), $J = 6$, $K = 5$ and $L = 7$ for the orders of the **FIR** filters $\beta(t)$, $\eta(t)$ and $\mu(t)$. For the delays we take $\tau = 7$ and $\delta = 15$. We use the software Scilab 6.0.0 to compute the ridge regression. With respect to the regularization parameters we selected the values:

- 100, 1/10000 and 1 for $\beta(t)$, $\eta(t)$ and $\mu(t)$, respectively. Forecasts made from July 03.
- 10000, 1/100000, 1 for $\beta(t)$, $\eta(t)$ and $\mu(t)$, respectively. Forecasts made from August 09.

We will explain the reasons why we made these forecasts in subsection 4.4.

4.3 Forecasts for Moquegua

Forecasts to find unpublished data for GERESA

- Forecasts made from August 06 for 7 days. We did this to complete the data because GERESA did not published that day. For August 06, we obtain the values 8130, 3096 and 241 for infected, recovered and deceased individuals, respectively. Those values were used to do the other forecasts. The figure (2) shows the forecasts.
- Forecasts made from August 19. Just as in the previous forecast, we needed to do the forecast to complete the lack of data for the days 19 and 22 of August. This forecast produced for August 19 the values 14172, 4058 and 471 for infected, recovered and

deceased individuals, respectively. For August 22 are obtained the values 16052, 4379 and 512 for infected, recovered and deceased individuals, respectively. It is worth saying that we replaced the data published for GERESA for deceased people for the days 23, 24 and 25 of August for the data produced for the forecast: 483, 497 and 512. The figure (3) shows the forecasts. We will use those values for future forecasts.

- Forecast from September 19. Again, we needed this forecast to complete the data. This forecast produced for September 19 the values 24248, 11826 and 645 for infected, recovered and deceased people, respectively. It is important to mention that the value 645 was adopted for future forecasts, it was rounded down because the algorithm produced values greater than 645.5 but the value 645 is compatible with the value published for GERESA for the next day, which is 645. The figure (4) shows the forecast. It is important to mention that we use those values for future forecasts.

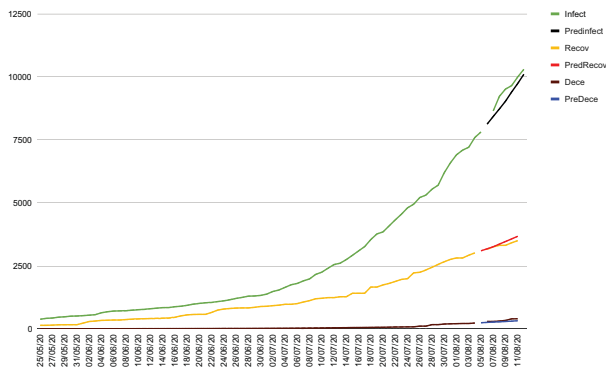


Figure 2: Forecast for Moquegua from August 06

Forecast for seven days

- The forecast made from October 20. As shown Fig. (5) the forecasts follow the trend.

4.4 Forecasts for Tacna

Forecasts to find unpublished data for DIRESA

- The forecast made from July 03. We made this forecast to complete the unpublished data for that day. It produced for July 03 the values 1109, 565 and 18 for infected, recovered and deceased people, respectively. Being that the value 565 was rounded down (see Fig. (6)). It is important to mention that we use those values for future forecasts.

Forecasts for seven days We made the forecast from August 09. It can be seen in Fig. (7) and we can observe that the forecasts follow the trend.

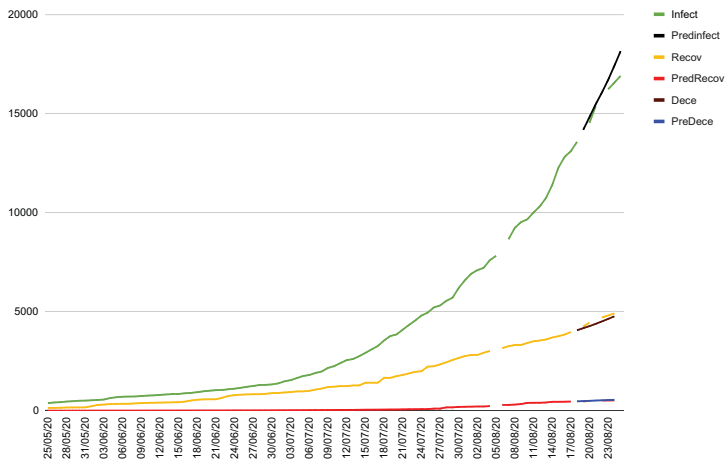


Figure 3: Forecast for Moquegua from August 19

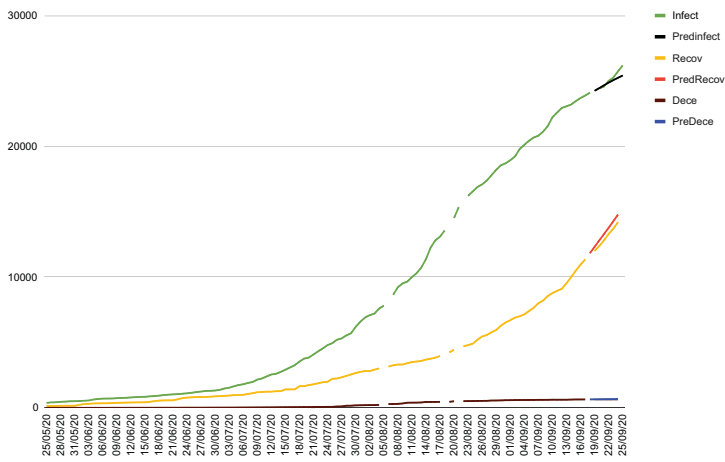


Figure 4: Forecast for Moquegua from September 19

5 Conclusions

The present paper contributes to the Epidemiological Models, more specifically to the COVID-19 epidemic model.

From the graphics exhibited in the present paper we can see that the forecasts follow the trend of published data for infected and deceased individuals for a short interval of seven days.

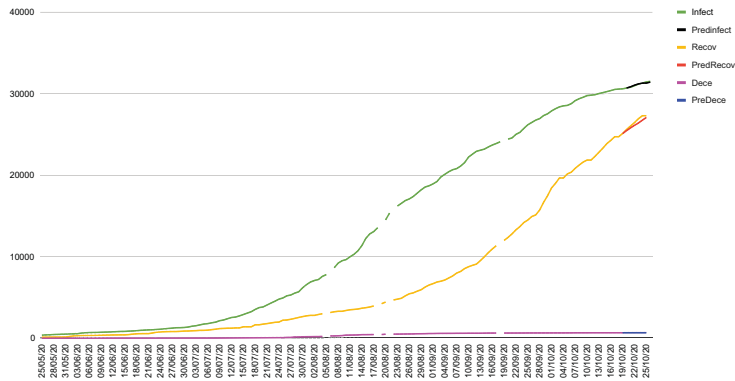


Figure 5: Forecast for Moquegua from October 20

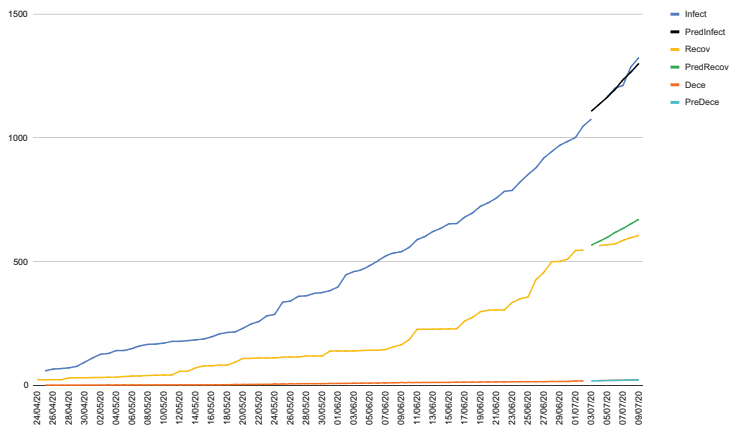


Figure 6: Forecast for Tacna from July 03

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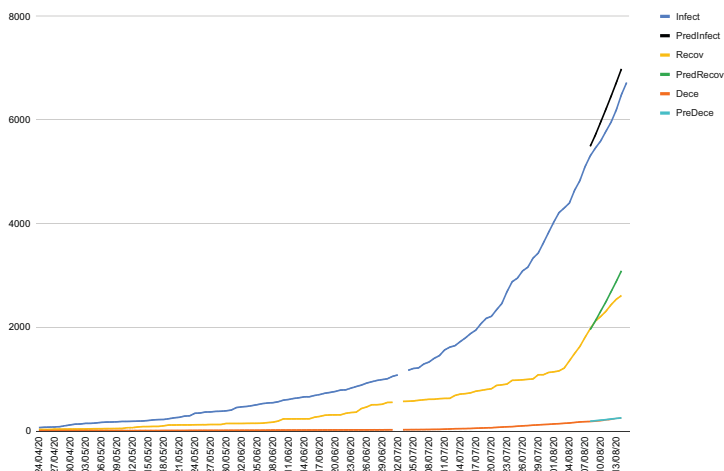


Figure 7: Forecast for Tacna from August 09

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