

Dynamics of an Eco-Epidemic Model with Saturated Incidence Rate

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ABSTRACT

This paper explores the dynamical behaviour of an eco-epidemiological model with communicable disease in prey. Existence of equilibrium points and its stability conditions is derived. Furthermore, stochastic modelling is introduced for the proposed model. The exponential and mean square stability is analysed. Numerical simulations are supported to the deterministic and stochastic models.

Keywords: Prey-Predator model, stochastic stability, Lyapunov function, Eco-epidemiology.

Mathematics Subject Classification: 34D20, 92D30, 60H35, 70K20, 93E15

Journal of Economic literature classification: C60, C62

1. INTRODUCTION

Ecological models are developed after the fundamental work of Lotka and Volterra in 1927 for interactions between prey-predator. These models are further elaborated profusely by [1, 2, 3]. In natural world, the scattering of diseases may arise due to interaction of several species [4]. In a mathematical perspective, contagious diseases in an ecological environment have become predominant and consequently ecological and epidemic aspects emerged as a new branch of science [5]. Eco-epidemiological models illustrate the prey and predator interactions with disease in one population or both the populations [6]. These models play a considerable role to investigate the spread and control of communicable diseases among the populations. The first deterministic epidemic model was initiated by Kermac-Mckendric [7], which presumes that the population is divided into Susceptible, Infectious and Recovered compartments. The preliminary research on Eco-epidemic model was coined by Anderson and May [8]. To detect the mechanism involved in spreading of diseases, various mathematical models are developed and employed for predicting and controlling the contagious diseases [9]. In nature, species do not survive alone, the interaction of species subjected to the parasitism have biological impact where the persistence extinction threshold of each population can be examined [10]. The spread of contagious diseases is probabilistic in environment and fluctuations arise in each compartment. The intrinsic fluctuations play a critical role in

transmission of diseases and whenever the population densities are low the variations in population strengths can be achieved through the stochastic approach compared to deterministic study [11, 12]. Research indicates that parasites can change their behaviour of prey and can be complex so that infected prey is more vulnerable to predation compare to uninfected prey [13, 14].

In this paper, we considered a model with uninfected prey (Susceptible), infected prey (infected by some virus) and predator population. Saturated incidence rate $\frac{\beta SI}{1 + \alpha I}$ is considered, which is specified by Capasso and Serio[15]. We assume predator grows logistically and eats only infected prey with the Holling type-II functional response. We also proposed stochastic model to examine the noise in the system.

2. MATHEMATICAL MODEL

Consider a model, which is formulated by non linear ordinary differential equations as follows

$$\begin{aligned}\frac{dS}{dt} &= A - \frac{\beta SI}{1 + \alpha_1 I} - dS \\ \frac{dI}{dt} &= \frac{\beta SI}{1 + \alpha_1 I} - \mu I - \frac{k_1 IP}{c_1 + I} \\ \frac{dP}{dt} &= \left(\alpha_2 - \frac{k_2 P}{c_2 + I} \right) P\end{aligned}\tag{1}$$

Where $I(t)$, $S(t)$ and $P(t)$ are the densities of infected prey population, susceptible prey population and Predator population correspondingly. A is the birth rate of susceptible prey, $\frac{\beta SI}{1 + \alpha_1 I}$ is the saturated incidence rate, β is the rate of infection, α_1 is the saturation constant, d is the death rate of susceptible prey, μ is disease death rate of infective prey, k_1 is the rate of maximum predation, α_2 is the growth rate of predator, k_2 is the reduction rate of predator, C_1 is the environmental protection to prey and C_2 is the environmental protection to predator.

3. EQUILIBRIUM POINTS AND THEIR STABILITY

The proposed system (1) has planar equilibrium point $E_0(S_0, I_0, 0)$ and Interior equilibrium point

$$\begin{aligned}E_1(S^*, I^*, P^*) \text{ where } S_0 &= \frac{\mu}{\beta}(1 + \alpha_1 I_0), I_0 = \frac{A\beta - \mu d}{\mu(\beta + \alpha_1 d)} \\ E_1(S^*, I^*, P^*) \text{ where } S^* &= \frac{A(1 + \alpha_1 I^*)}{d(1 + \alpha_1 I^*) + \beta I^*}, I^* = \frac{m_{22} + \sqrt{m_{22}^2 - 4m_{11}m_{33}}}{m_{22}} \text{ where } m_{22}^2 - 4m_{11}m_{33} > 0\end{aligned}$$

$$m_{11} = (\mu k_2 + k_1 \alpha_2)(d \alpha_1 + \beta), m_{22} = d(\mu k_2 + k_1 \alpha_2) + (d \alpha_1 + \beta)(\mu k_2 c_1 + c_2 k_1 \alpha_2) - \beta A$$

$$m_{33} = d(\mu k_2 c_1 + c_2 k_1 \alpha_2) - \beta A c_1$$

$$P^* = \frac{\alpha_2}{k_2}(c_2 + I^*)$$

Characteristic equation at E_1 is obtained by

$$\begin{vmatrix} -\frac{\beta S^* I^*}{1 + \alpha_1 I^*} - d - \lambda & -\frac{\beta I^*}{(1 + \alpha_1 I^*)^2} & 0 \\ \frac{\beta I^*}{1 + \alpha_1 I^*} & \frac{\beta S^*}{(1 + \alpha_1 I^*)^2} - \mu - \frac{k_1 c_1 P^*}{(c_1 + I^*)^2} - \lambda & \frac{-k_1 I^*}{c_1 + I^*} \\ 0 & \frac{-k_2 P^{*2}}{(c_2 + I^*)^2} & \alpha_2 - \frac{2k_2 P^*}{c_2 + I^*} - \lambda \end{vmatrix} = 0 \quad (2)$$

$$\lambda^3 + a_1 \lambda^2 + a_2 \lambda + a_3 = 0 \quad (3)$$

where

$$a_1 = d + c + \frac{2k_2 P^*}{c_2 + I^*} + \frac{\beta I^*}{1 + \alpha_1 I^*} + \frac{k_1 c_1 P^*}{(c_1 + I^*)^2} - \frac{\beta S^*}{(1 + \alpha_1 I^*)^2} - \alpha_2$$

$$a_2 = \frac{k_1 k_2 P^{*2} I^*}{(c_1 + I^*)(c_2 + I^*)^2} + \frac{2k_2 P^*}{c_2 + I^*} \left(\mu + \frac{k_1 c_1 P^*}{(c_1 + I^*)^2} + d + \frac{\beta I^*}{1 + \alpha_1 I^*} - \frac{\beta S^*}{(1 + \alpha_1 I^*)^2} \right) +$$

$$\alpha_2 \left(d - \mu - \frac{\beta I^*}{1 + \alpha_1 I^*} \right) + \frac{\beta^2 S^* I^*}{(1 + \alpha_1 I^*)^3} + \left(d + \frac{\beta I^*}{1 + \alpha_1 I^*} \right) \left(\mu - \frac{\beta S^*}{(1 + \alpha_1 I^*)^2} + \frac{k_1 c_1 P^*}{(c_1 + I^*)^2} \right)$$

$$a_3 = \left(\frac{\beta I^*}{1 + \alpha_1 I^*} + d \right) \left(\frac{k_1 k_2 P^{*2} I^*}{(c_1 + I^*)(c_2 + I^*)^2} + \frac{\beta S^*}{(1 + \alpha_1 I^*)^2} \alpha_2 - \frac{2k_2 P^*}{c_2 + I^*} \frac{\beta S^*}{(1 + \alpha_1 I^*)^2} + \left(\mu + \frac{k_1 c_1 P^*}{(c_1 + I^*)^2} \right) \frac{2k_2 P^*}{c_2 + I^*} \right)$$

$$+ \frac{\beta^2 S^* I^*}{(1 + \alpha_1 I^*)^3} \frac{2k_2 P^*}{c_2 + I^*} - \frac{\beta^2 S^* I^*}{(1 + \alpha_1 I^*)^3}$$

$a_1 > 0, a_3 > 0$ and $a_1 a_2 > a_3$. Thus, by Routh-Hurwitz criteria equilibrium point E_1 is locally asymptotically stable.

4. GLOBAL STABILITY

To establish the global stability of the equilibrium point $E_1(S^*, I^*, P^*)$, choose a

$$\text{Lyapunov function } v = S - S^* - S^* \ln \frac{S}{S^*} + I - I^* - I^* \ln \frac{I}{I^*} + P - P^* - P^* \ln \frac{P}{P^*} \quad (4)$$

Along the trajectories, the time derivatives of functional $v(t)$

$$\begin{aligned}
 \frac{dv}{dt} &= \left(1 - \frac{S^*}{S}\right) \frac{dS}{dt} + \left(1 - \frac{I^*}{I}\right) \frac{dI}{dt} + \left(1 - \frac{P^*}{P}\right) \frac{dP}{dt} \\
 \frac{dv}{dt} &= \left(\frac{I - I^*}{I}\right) \frac{dI}{dt} + \left(\frac{S - S^*}{S}\right) \frac{dS}{dt} + \left(\frac{P - P^*}{P}\right) \frac{dP}{dt} \\
 \frac{dv}{dt} &= \left(\frac{S - S^*}{S}\right) \left(A - \frac{\beta SI}{1 + \alpha_1 I} - dS\right) + \left(\frac{I - I^*}{I}\right) \left(\frac{\beta SI}{1 + \alpha_1 I} - \mu I - \frac{k_1 IP}{c_1 + I}\right) + \\
 &\quad \left(\frac{P - P^*}{P}\right) \left(\alpha_2 - \frac{k_2 P}{c_2 + I}\right) P \\
 \frac{dv}{dt} &\leq \frac{S - S^*}{S} (dS^* - dS) + \frac{I - I^*}{I} (\mu I^* - \mu I) + \frac{P - P^*}{P} (\alpha_2 P - \alpha_2 P^*) \\
 \frac{dv}{dt} &\leq -\frac{d}{S} (S - S^*)^2 - \frac{\mu}{I} (I - I^*)^2 - \frac{\alpha_2}{P} (P - P^*)^2 < 0
 \end{aligned} \tag{5}$$

Therefore system (1) is globally asymptotically stable.

5. STOCHASTIC STABILITY ANALYSIS

Under stochastic fluctuations system (1) can be characterise as

$$\begin{aligned}
 \frac{dS}{dt} &= A - \frac{\beta SI}{1 + \alpha_1 I} - dS + \sigma_1 (S - S^*) d\xi_1 \\
 \frac{dI}{dt} &= \frac{\beta SI}{1 + \alpha_1 I} - \mu I - \frac{k_1 IP}{c_1 + I} + \sigma_2 (I - I^*) d\xi_2 \\
 \frac{dP}{dt} &= \left(\alpha_2 - \frac{k_2 P}{c_2 + I}\right) P + \sigma_3 (P - P^*) d\xi_3
 \end{aligned} \tag{6}$$

where $\sigma_i, i = 3, 2, 1$ are real constants $\xi_i, i = 3, 2, 1$ are independent of each other is a standard wiener process. The linearized system (6) around E_1 is obtained by Substituting

$w_1 = S - S^*, w_2 = I - I^*, w_3 = P - P^*$, which leads

$$dw(t) = f(w(t))dt + g(w(t))d\xi(t) \tag{7}$$

$$w(t) = \begin{bmatrix} w_1(t) \\ w_2(t) \\ w_3(t) \end{bmatrix} \text{ and}$$

Where

$$f(w(t)) = \begin{bmatrix} -\frac{\beta S^* I^*}{1 + \alpha_1 I^*} - d & -\frac{\beta I^*}{(1 + \alpha_1 I^*)^2} & 0 \\ \frac{\beta I^*}{1 + \alpha_1 I^*} & \frac{\beta S^*}{(1 + \alpha_1 I^*)^2} - \mu - \frac{k_1 c_1 P^*}{(c_1 + I^*)^2} & \frac{-k_1 I^*}{c_1 + I^*} \\ 0 & \frac{-k_2 P^{*2}}{(c_2 + I^*)^2} & \alpha_2 - \frac{2k_2 P^*}{c_2 + I^*} \end{bmatrix} \quad (8)$$

$$g(w(t)) = \begin{bmatrix} \sigma_1 w_1 & 0 & 0 \\ 0 & \sigma_2 w_2 & 0 \\ 0 & 0 & \sigma_3 w_3 \end{bmatrix}$$

Let $C([0, +\infty) \times R^2, R^+)$ be a family of non negative functions $V(t, w)$ defined on $[0, +\infty) \times R^2$ is continuously differentiable function with respect to 'w' twice and with respect to 't'.

With reference to a book by Afanasiev [16] we define a function $V(t, w)$

$$L(V(t, w)) = \frac{\partial V(t, w)}{\partial t} + f_1^T(w) \frac{\partial V}{\partial w}(t, w) + \frac{1}{2} Tr \left[g_1^T(w) \frac{\partial^2 V(t, w)}{\partial^2 w} g_1(w) \right] \quad (9)$$

$$\frac{\partial V}{\partial w} = col \left(\frac{\partial V}{\partial w_1}, \frac{\partial V}{\partial w_2}, \frac{\partial V}{\partial w_3} \right) \quad \text{and} \quad \frac{\partial^2 V(t, w)}{\partial w^2} = \frac{\partial^2 V(t, w)}{\partial w_i \partial w_j}, i, j = 1, 2$$

Theorem1: Suppose $\exists$ a function $V(t, w) \in C^{1,2}([0, +\infty) \times R^2, R^+)$ which satisfy

$$k_1 |w|^p \leq V(t, w) \leq k_2 |w|^p \quad (10)$$

$$LV(t, w) \leq -k_3 |w|^p$$

Where k_3, k_2, k_1 and P are positive constants. Trivial solution of (7) is exponentially p-Stable for $t \geq 0$. The zero solution of (10) is globally asymptotically stable if $p = 2$.

Theorem2: Suppose that

$$\sigma_1^2 \leq 2 \left[d + \frac{\beta I}{1 + \alpha_1 I} \right], \sigma_2^2 \leq 2 \left[\mu - \frac{\beta S}{(1 + \alpha_1 I)^2} + \frac{k_1 c_1 P}{(c_1 + I)^2} \right], \sigma_3^2 \leq 2 \left[\frac{2k_2 P}{1 + c_1 I} - \alpha_2 \right] \quad (11)$$

Then the zero solution of (6) is asymptotically square stable.

Proof: Choose a Lyapunov function $V(w) = \frac{1}{2} [V_1 w_1^2 + V_2 w_2^2 + V_3 w_3^2]$ where V_1, V_2, V_3 are positive constants is to be chosen in the following. It is simple to verify (11)

$$\begin{aligned}
 L(V(w)) = & V_1 \left[- \left(d + \frac{\beta I}{1 + \alpha_1 I} \right) w_1 - \left(\frac{\beta S}{(1 + \alpha_1 I)^2} \right) w_2 \right] w_1 + \\
 & V_2 \left[- \left(\frac{\beta I}{1 + \alpha_1 I} \right) w_1 + \left(\frac{k_1 c_1 P}{(c_1 + I)^2} + \mu - \frac{\beta S}{(1 + \alpha_1 I)^2} \right) w_2 + \frac{k_1 I}{(c_1 + I)} w_3 \right] w_2 + \\
 & \left[- \left(\frac{k_2 P^2}{(1 + c_2 I)^2} w_2 \right) + \left(\frac{2k_2 P}{1 + c_1 I} - \alpha_2 \right) w_3 \right] w_3
 \end{aligned} \quad (12)$$

Choose $V_1 w_2 = V_2 w_1$ and $V_2 w_3 = V_3 w_2$ that is $\left(\frac{\beta S}{(1 + \alpha_1 I)^2} \right) w_2 = \left(\frac{\beta I}{1 + \alpha_1 I} \right) w_1$ and $\frac{k_2 P^2}{(1 + c_2 I)^2} w_2 = \frac{k_1 I}{(c_1 + I)} w_3$

$$\text{We have } \frac{1}{2} \text{Tr} \left[g_1^T(w) \frac{\partial^2 V(t, w)}{\partial^2 w} g_1(w) \right] = \frac{1}{2} [V_1 w_1^2 + V_2 w_2^2 + V_3 w_3^2]$$

$$LV(w) = - \left[d + \frac{\beta I}{1 + \alpha_1 I} \right] V_1 w_1^2 - \left[\mu - \frac{\beta S}{(1 + \alpha_1 I)^2} + \frac{k_1 c_1 P}{(c_1 + I)^2} \right] V_2 w_2^2 - \left[\frac{2k_2 P}{1 + c_1 I} - \alpha_2 \right] V_3 w_3^2 < 0$$

We concluded that by theorem (1) the trivial solution of (7) is globally asymptotically stable

6. NUMERICAL SIMULATIONS

For the set of values

$A=1.2, d=0.2, \beta=1.5, \alpha_1=0.614, \mu=0.9, k_1=1, c_1=0.03, \alpha_2=0.8, k_2=1, c_2=0.04,$

$\sigma_1=0.01, \sigma_2=0.01, \sigma_3=0.01$, Stochastic system corresponding to the deterministic system.

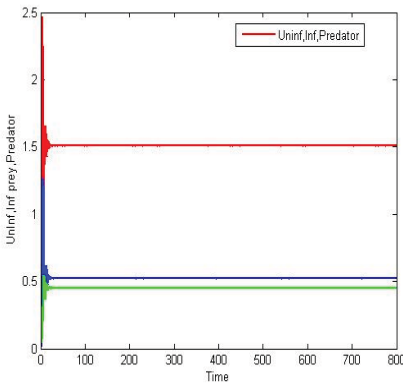


Figure 1a) phase plane of system(1)

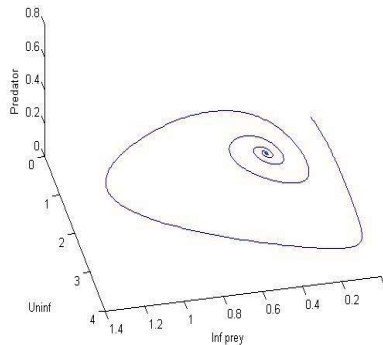
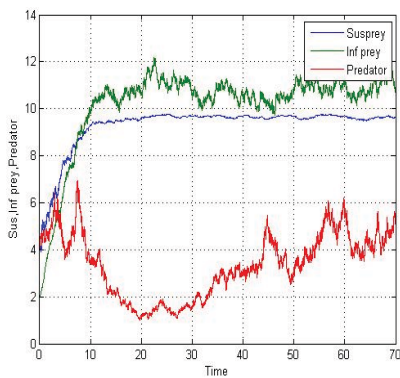
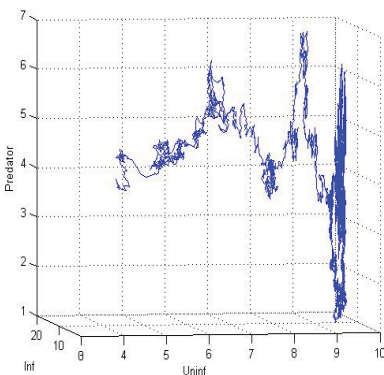


Figure 1b) Phase portrait of system(1)



(a)



(b)

Figure 2. Stochastic variations for the values $\sigma_1 = 0.01, \sigma_2 = 0.01, \sigma_3 = 0.01$

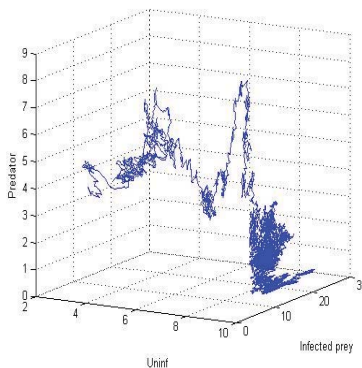
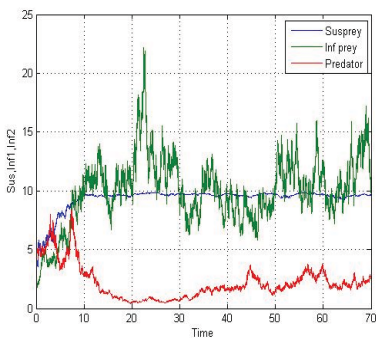


Figure 3. Stochastic variations for the values $\sigma_1 = 0.03, \sigma_2 = 0.01, \sigma_3 = 0.04$

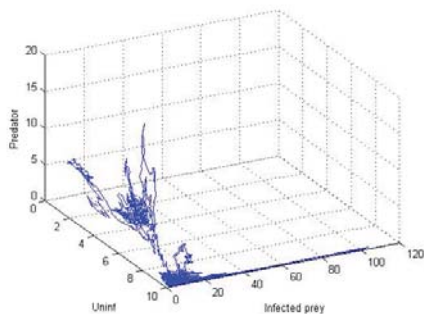
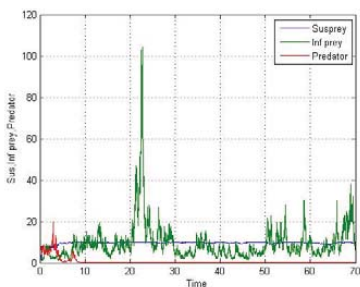


Figure 3. Stochastic variations for the values $\sigma_1 = 0.2, \sigma_2 = 0.2, \sigma_3 = 0.2$

7. CONCLUSIONS

An eco-epidemic model, consists of uninfected prey, infected prey and predator populations is studied. In this model Saturated incidence rate and Holling type-II functional response is considered. The equilibrium points of the system are established and their stability is discussed. The disease modifies the behaviour of the prey population, so infected prey population is more vulnerable to predation. Local stability of interior equilibrium point $E_1(1.5106, 0.5246, 0.4510)$ is analysed. From the numerical simulations, we observe that the co-existence of predator population with infected and susceptible prey population with varying steady state behaviour. Based on the white noise the stochastic system(6) is globally asymptotically stable. In this model we used the main mechanism as predator consumes selectively infected prey. Numerical simulations support the analytical results.

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