

Stability and Hopf Bifurcation Analysis of a Delayed SEIS Model

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ABSTRACT

A delayed SEIS model with saturated incidence rate has been investigated in this paper. The stability of equilibria was analyzed with and without delay. Hopf bifurcation analysis was carried out under certain conditions. Employing Lyapunov functional technique, the global stability pertaining to the endemic equilibrium was made clear contextually. To illustrate the theoretical analysis, some numerical simulations of the model have been incorporated.

Keywords: Hopf bifurcation, time delay, saturated incidence rate, global stability.

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1. INTRODUCTION

Infectious diseases continue to plague humanity and are one of the main causes of death in humans. The preliminary aim of this research in epidemiology is to develop vaccines, treatments and intervention strategies for halting or slowing down the dissemination of infectious diseases and hence reducing the number or frequency of deaths [4, 12]. Mathematical models have an indispensable role to play in the investigation of epidemics for relevant theoretical results. A few diseases, however, live and flourish in the hosts for long periods prior to the hosts becoming contagious [5]. Some researchers considered a class of SIR, SIS models and discussed their stability [9, 10, 13]. For some infectious diseases such as COVID-19, TB, gonorrhea with a latent period, the exposed class must necessarily be reckoned with to facilitate the investigation of SEIR, SEIS compartmental models have been investigated. There has been massive research on models of such types in the literature [7, 8, 15]. Meng et al. [11] discussed the dynamics and therapeutic approach of an SEIS model with latent

patients and active patients. An SEIS model with time delay representing the latent period was investigated by Xu [16].

2. MATHEMATICAL MODEL

This paper addresses an SEIS model in which the total population is classified into three categories: susceptible (S), exposed (E), and infectious (I). The growth of the susceptibles is governed by the logistic equation with carrying capacity κ and intrinsic birth rate r . The incidence rate incorporated in our model is of saturated type, which is introduced by Capasso and Serio [1]. Time delay (τ) is factored in only in exposed class, bearing in mind that a number of contagious diseases display a delayed commencement of transmission, i.e., the moment an individual shows symptoms of infection, it takes quite a time before it begins to infect others. There does exist a period identified as latent period before the diseased condition begins to manifest its impact on others and the model in contemplation room for this occurrence. This model in particular is visualized using the framework composed of equations that figure below:

$$\begin{aligned}\frac{dS}{dt} &= rS\left(1 - \frac{S}{\kappa}\right) - \frac{\beta SI}{1 + \alpha I} - dS + \gamma I \\ \frac{dE}{dt} &= \frac{\beta SI}{1 + \alpha I} - (d + \nu)E(t - \tau) \\ \frac{dI}{dt} &= \nu E(t - \tau) - (d + \gamma)I\end{aligned}\tag{2.1}$$

where β denotes the transmission rate, d is the natural death rate of the population, ν is the rate at which the exposed individuals becomes infectious, γ is the rate at which infected individuals face susceptibility yet again after a tranquil period of temporary immunity and α is a non-negative constant that measures the inhibitory effect.

Theorem 1. *All the solutions of system (2.1) are eventually bounded in the compact subset $\Omega = \{(S, E, I) \in \mathbb{R}_+^3 : N \leq \frac{Mr}{d} + \varepsilon, \varepsilon > 0\}$.*

Proof. Let $(S(t), E(t), I(t))$ be any solution with positive initial conditions (S_0, E_0, I_0) .

Summing up three equations in (2.1) and denoting $N = S(t) + E(t) + I(t)$, we have

$$\frac{dN}{dt} = rS\left(1 - \frac{S}{\kappa}\right) - dN$$

Using standard comparison theorem, we have

$$\limsup_{t \rightarrow \infty} S \leq M, \text{ where } M = \max\{S_0, \kappa\}$$

$$\text{Thus } \frac{dN}{dt} + dN = rS\left(1 - \frac{S}{\kappa}\right) \leq Mr$$

By applying theory of differential inequalities [3], we obtain

$$0 \leq N(S, E, I) \leq \frac{Mr}{d}\left(1 - e^{-dt}\right) + N(S_0, E_0, I_0)e^{-dt},$$

$$\text{and for } t \rightarrow \infty, \text{ we have } 0 \leq N(S, E, I) \leq \frac{Mr}{d}$$

Hence all the solutions of (2.1) are eventually bounded in Ω .

3. EXISTENCE OF EQUILIBRIUM POINTS

The basic reproduction number [2] of the system (2.1) is defined as

$$R_0 = \frac{\kappa\beta v(r-d)}{r(d+v)(d+\gamma)}$$

System (2.1) has two biologically relevant equilibria, namely

- i) Infection-free equilibrium $E_0\left(\frac{\kappa(r-d)}{r}, 0, 0\right)$, which exist when $r > d$
 ii) Endemic equilibrium $E_1(S^*, E^*, I^*)$, which exist if $R_0 > 1$ and is given by

$$S^* = \frac{(d+v)(d+\gamma)(1+\alpha I^*)}{\beta v}, \quad E^* = \frac{(\mu d + \gamma)I^*}{v}, \quad I^* = \frac{-\Lambda_2 + \sqrt{\Lambda_2^2 - 4\Lambda_1\Lambda_3}}{2\Lambda_1}$$

where $\Lambda_1 = r\alpha^2(d+v)^2(d+\gamma)^2$

$$\Lambda_2 = v\kappa\beta^2(d+v+\gamma) - r\alpha(d+v)^2(d+\gamma)^2(R_0 - 2)$$

$$\Lambda_3 = -r(d+v)^2(d+\gamma)^2(R_0 - 1)$$

3.1. Stability of Infection-Free Equilibrium

The Jacobian matrix of system (2.1) at E_0 is given by

$$J_0 = \begin{bmatrix} -(r-d) & 0 & \gamma - \frac{\kappa\beta(r-d)}{r} \\ 0 & -(d+v)e^{-\lambda\tau} & \frac{\kappa\beta(r-d)}{r} \\ 0 & ve^{-\lambda\tau} & -(d+\gamma) \end{bmatrix} \quad (3.1)$$

The eigen values of (3.1) are given by

$$\begin{aligned} & [\lambda + (r-d)] \left\{ \lambda^2 + [(d+v)e^{-\lambda\tau} + (d+\gamma)]\lambda + (d+\gamma)(d+v)(1-R_0)e^{-\lambda\tau} \right\} = 0 \\ \Rightarrow & \lambda = -(r-d) \text{ and } \lambda^2 + [(d+v)e^{-\lambda\tau} + (d+\gamma)]\lambda + (d+\gamma)(d+v)(1-R_0)e^{-\lambda\tau} = 0 \end{aligned} \quad (3.2)$$

For $\tau = 0$, equation (3.2) becomes

$$\lambda^2 + [(d+v) + (d+\gamma)]\lambda + (d+\gamma)(d+v)(1-R_0) = 0 \quad (3.3)$$

It is observed that if $R_0 < 1$, the roots of (3.3) have negative real parts.

Proposition 1. *If $r > d$ and $R_0 < 1$, the infection-free equilibrium E_0 of system (2.1) is locally asymptotically stable if and only if $\tau = 0$.*

3.2. Stability of Endemic Equilibrium

The Jacobian matrix of system (2.1) at E_1 is given by

$$J_1 = \begin{bmatrix} -\frac{rS^*}{\kappa} & 0 & \gamma - \frac{\beta S^*}{(1+\alpha I^*)^2} \\ \frac{\beta I^*}{1+\alpha I^*} & -(d+v)e^{-\lambda\tau} & \frac{\beta S^*}{(1+\alpha I^*)^2} \\ 0 & ve^{-\lambda\tau} & -(d+\gamma) \end{bmatrix} \quad (3.4)$$

The eigen values of (3.4) are given by

$$\lambda^3 + a_1\lambda^2 + a_2\lambda + (b_1\lambda^2 + b_2\lambda + b_3)e^{-\lambda\tau} = 0 \quad (3.5)$$

Where $a_1 = \frac{rS^*}{K} + d + \gamma$

$$a_2 = \frac{r}{K}(d + \gamma)S^*$$

$$b_1 = d + \nu$$

$$b_2 = \frac{r}{K}(d + \nu)S^* + (d + \gamma)(d + \nu) - \frac{\nu\beta S^*}{(1 + \alpha I^*)^2}$$

$$b_3 = \frac{rS^*}{K} \left[(d + \gamma)(d + \nu) - \frac{\nu\beta S^*}{(1 + \alpha I^*)^2} \right] + \frac{\nu\beta^2 S^* I^*}{(1 + \alpha I^*)^3} - \frac{\nu\beta\gamma I^*}{1 + \alpha I^*}$$

For $\tau = 0$, equation (3.5) becomes $\lambda^3 + (a_1 + b_1)\lambda^2 + (a_2 + b_2)\lambda + b_3 = 0$ (3.6)

It can be easily verified that $(a_1 + b_1)(a_2 + b_2) - b_3 > 0$ if $R_0 > 1$.

By Routh-Hurwitz criterion, all the roots of (3.6) have negative real parts.

Proposition 2. If $R_0 > 1$, the endemic equilibrium E_1 of system (2.1) is locally asymptotically stable if and only if $\tau = 0$.

3.3. Hopf Bifurcation Analysis

For $\tau > 0$, assume that equation (3.5) has a purely imaginary root $i\omega$ ($\omega > 0$).

Put $\lambda = i\omega$ in (3.5) and then equating real and imaginary parts, one receives

$$\begin{aligned} a_1\omega^2 &= (b_3 - b_1\omega^2)\cos\omega\tau + b_2\omega\sin\omega\tau \\ \omega^3 - a_2\omega &= b_2\omega\cos\omega\tau - (b_3 - b_1\omega^2)\sin\omega\tau \end{aligned} \quad (3.7)$$

Eliminating τ from (3.7) gives

$$\omega^6 + (a_1^2 - 2a_2 - b_1^2)\omega^4 + (a_2^2 + 2b_1b_3 - b_2^2)\omega^2 - b_3^2 = 0 \quad (3.8)$$

Let $z = \omega^2$, then (3.8) becomes

$$z^3 + s_1z^2 + s_2z + s_3 = 0 \quad (3.9)$$

where $s_1 = a_1^2 - 2a_2 - b_1^2$

$$s_2 = a_2^2 + 2b_1b_3 - b_2^2$$

$$s_3 = -b_3^2$$

Denote $h(z) = z^3 + s_1z^2 + s_2z + s_3$

Since $h(0) = s_3 < 0$ and $\lim_{z \rightarrow \infty} h(z) = +\infty$, equation (3.9) has at least one positive root z_1 . Therefore,

equation (3.8) has at least one positive root, given by $\omega_1 = \sqrt{z_1}$.

Let $\lambda(\tau) = \eta(\tau) + i\omega(\tau)$ be a root of (3.5) near $\tau = \tau_0$ with $\eta(\tau_0) = 0$ and $\omega(\tau_0) = \omega_0$

From (3.7), we get

$$\tau_0 = \frac{1}{\omega_0} \left\{ \arccos \left[\frac{a_1\omega_0^2(b_3 - b_1\omega_0^2) + (\omega_0^3 - a_2\omega_0)b_2\omega_0}{(b_3 - b_1\omega_0^2)^2 + b_2^2\omega_0^2} \right] + 2j\pi \right\}, j = 0, 1, 2, \dots \quad (3.10)$$

Proposition 3. If $R_0 > 1$ and $(2a_2 - a_1^2 + b_1^2) < 3(a_2^2 + 2b_1b_3 - b_2^2)$ then we have the transversality condition $\left. \frac{d\operatorname{Re}(\lambda)}{d\tau} \right|_{\lambda=i\omega_0} > 0$

Substituting $\lambda(\tau)$ into L.H.S of (3.5) and differentiating w.r.t. τ , we get

$$\left[3\lambda^2 + 2a_1\lambda + a_2 + (2b_1\lambda + b_2)e^{-\lambda\tau} - (b_1\lambda^2 + b_2\lambda + b_3)\tau e^{-\lambda\tau} \right] \frac{d\lambda}{d\tau} = (b_1\lambda^2 + b_2\lambda + b_3)\lambda e^{-\lambda\tau}$$

$$\left(\frac{d\lambda}{d\tau} \right)^{-1} = \frac{3\lambda^2 + 2a_1\lambda + a_2}{(b_1\lambda^2 + b_2\lambda + b_3)\lambda e^{-\lambda\tau}} + \frac{2b_1\lambda + b_2}{(b_1\lambda^2 + b_2\lambda + b_3)\lambda} - \frac{\tau}{\lambda}$$

$$\text{Thus } \operatorname{Re} \left(\frac{d\lambda}{d\tau} \right)^{-1} \Big|_{\lambda=i\omega_0} = \frac{(a_2 - 3\omega_0^2)L + 2a_1M - 2b_1^2\omega_0^2 + 2b_1b_3 - b_2^2}{P^2 + Q^2}$$

$$\text{Where } P = (b_3 - b_1\omega_0^2) \sin \omega_0\tau - b_2\omega_0 \cos \omega_0\tau$$

$$Q = (b_3 - b_1\omega_0^2) \cos \omega_0\tau + b_2\omega_0 \sin \omega_0\tau$$

Together with (3.7), it follows that

$$\operatorname{Re} \left(\frac{d\lambda}{d\tau} \right)^{-1} \Big|_{\lambda=i\omega_0} = \frac{3\omega_0^4 - 2(2a_2 - a_1^2 + b_1^2)\omega_0^2 + a_2^2 - b_2^2 + 2b_1b_3}{P^2 + Q^2}$$

Clearly, $\left. \frac{d\operatorname{Re}(\lambda)}{d\tau} \right|_{\lambda=i\omega} > 0$ if $(2a_2 - a_1^2 + b_1^2) < 3(a_2^2 + 2b_1b_3 - b_2^2)$

Based on above analysis and the results of Hale [3], Kuang [6] and Ruan and Wei [14], the following conclusion is reached:

Theorem 2. If $(2a_2 - a_1^2 + b_1^2) < 3(a_2^2 + 2b_1b_3 - b_2^2)$ and Proposition 2 holds true, then the endemic equilibrium E_1 of system (2.1) is unstable for $\tau \geq \tau_0$ and asymptotically stable for $0 \leq \tau < \tau_0$ and system (2.1) go through Hopf bifurcation at E_1 when $\tau = \tau_0$.

3.4. Global Stability of Endemic Equilibrium

Theorem 3. The endemic equilibrium E_1 of system (2.1) is globally asymptotically stable in the interior of Ω if and only if $\tau = 0$.

Proof. Define the Lyapunov functional as

$$V = (S - S^* \ln S) + (E - E^* \ln E) + (I - I^* \ln I)$$

Differentiating with respect to t ,

$$\begin{aligned} \frac{dV}{dt} &= \left(\frac{S-S^*}{S} \right) \frac{dS}{dt} + \left(\frac{E-E^*}{E} \right) \frac{dE}{dt} + \left(\frac{I-I^*}{I} \right) \frac{dI}{dt} \\ &= \left(\frac{S-S^*}{S} \right) \left[rS \left(1 - \frac{S}{K} \right) - \frac{\beta SI}{1+\alpha I} - dS + \gamma I \right] + \left(\frac{E-E^*}{E} \right) \left[\frac{\beta SI}{1+\alpha I} - (d+\nu)E \right] + \left(\frac{I-I^*}{I} \right) \left[\nu E - (d+\gamma)I \right] \\ &= -\frac{r}{K} (S-S^*)^2 + \frac{\gamma}{SS^*} (S-S^*) (S^*I - SI^*) - \frac{(d+\nu)}{E} (E-E^*)^2 - \frac{(d+\gamma)}{I} (I-I^*)^2 \\ &\leq -\frac{r}{K} (S-S^*)^2 + \frac{\gamma}{2SS^*} \left[(S-S^*)^2 + (S^*I - SI^*)^2 \right] - \frac{(d+\nu)}{E} (E-E^*)^2 - \frac{(d+\gamma)}{I} (I-I^*)^2 \\ &\leq -\left[\frac{r}{K} - \frac{\gamma}{2SS^*} \left\{ 1 + \left(\frac{S^*I - SI^*}{S-S^*} \right)^2 \right\} \right] (S-S^*)^2 - \frac{(d+\nu)}{E} (E-E^*)^2 - \frac{(d+\gamma)}{I} (I-I^*)^2 \end{aligned}$$

$$\leq 0 \quad \text{if} \quad \frac{r}{\kappa} > \frac{\gamma}{2SS^*} \left\{ 1 + \left(\frac{S^*I - SI^*}{S - S^*} \right)^2 \right\}$$

$$\text{Clearly, } \frac{dV}{dt} \leq 0 \quad \text{if} \quad \frac{r}{\kappa} > \frac{\gamma}{2SS^*} \left\{ 1 + \left(\frac{S^*I - SI^*}{S - S^*} \right)^2 \right\}$$

Therefore, E_1 is globally asymptotically stable in the interior of Ω if and only if $\tau = 0$.

4. NUMERICAL SIMULATIONS

The section that follows deals with some numerical simulations for supporting the analytical results.

Example 1. When $\tau = 0$, we give the parameter values as follows: $r = 0.9, \kappa = 0.278, \beta = 0.6, d = 0.2, \alpha = 0.9, \nu = 0.3, \gamma = 0.03$. Then system (2.1) has an infection-free equilibrium $E_0(0.2162, 0, 0)$ with $R_0 = 0.3384 < 1$, which is locally asymptotically stable (**Figure 1**).

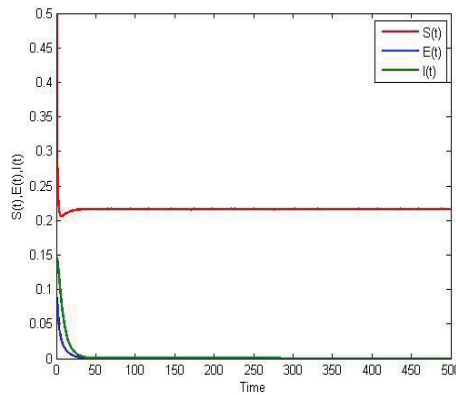


Figure 1. Trajectories of the system (2.1) without delay at E_0

Example 2. When $\tau = 0$, we give the parameter values as follows: $r = 0.7, \kappa = 9.25, \beta = 0.5, d = 0.2, \alpha = 0.06, \nu = 0.09, \gamma = 0.01$. Then system (2.1) has an endemic equilibrium at $E_1(1.3661, 0.3791, 0.1626)$ with $R_0 = 4.8821 > 1$, which is locally asymptotically stable (**Figure 2**).

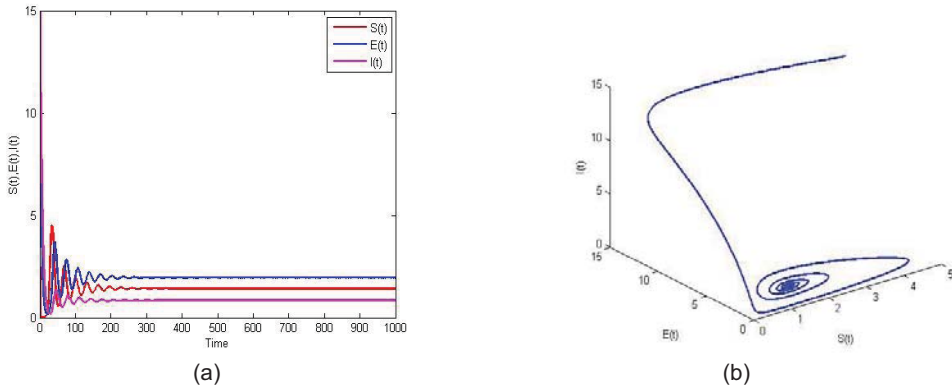


Figure 2. (a) Trajectories and (b) Phase-portraits of the system (2.1) without delay at E_1

Example 3. Imagine the parameter values to be the same as that in Example 2. We can ascertain from (3.10) whenever the time delay (τ) goes past the threshold value $\tau_0 = 1.4956$, E_1 reports loss in stability and a family comprising periodic solutions bifurcate from E_1 (**Figure 3 – Figure 5**). However, E_1 is asymptotically stable when $0 \leq \tau < \tau_0$.

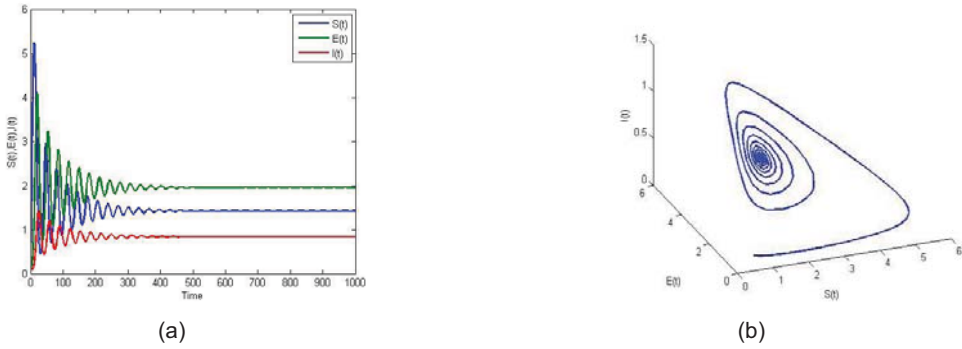


Figure 3. (a) Trajectories and (b) Phase-portraits of the system (2.1) at E_1 when $\tau = 0.8 < \tau_0 = 1.4956$

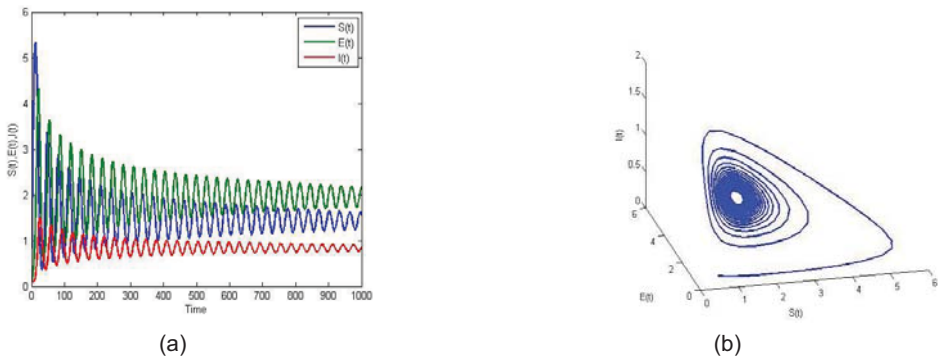


Figure 4. (a) Trajectories and (b) Phase-portraits of the system (2.1) at E_1 when $\tau = \tau_0 = 1.4956$

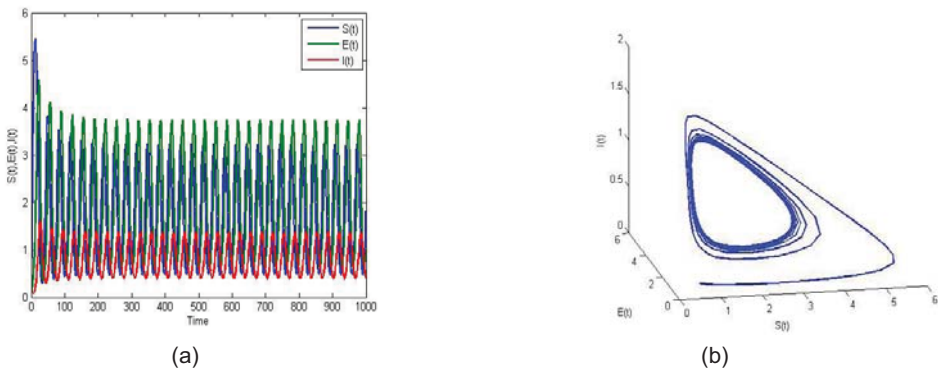


Figure 5. (a) Trajectories and (b) Phase-portraits of the system (2.1) at E_1 when

$$\tau = 2.2 > \tau_0 = 1.4956$$

5. CONCLUSIONS

This paper looked in some detail at effect of delay on an SEIS model with saturated incidence rate, in which the growth of the susceptibles is governed by the logistic equation. The stability of the model was analyzed in terms of basic reproduction number R_0 . If $R_0 < 1$, the infection-free equilibrium E_0 is locally asymptotically stable so the disease dies out. The endemic equilibrium E_1 is locally asymptotically stable if $R_0 > 1$. Employing Lyapunov functional technique, the global stability pertaining to the endemic equilibrium is made clear. Moreover, a behavioral change is observed in the system as it moves from stable to unstable when τ crosses the threshold value τ_0 , through Hopf bifurcation from E_1 . To conclude, numerical simulations were arrived at as a way of validating analytical results.

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