

Dynamic Analysis of Prey-Predator Model with Infection in the Prey Species Under the Effect of Primary and Secondary Toxicants

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ABSTRACT

This paper depicts a predator-prey model to study the synchronous impact of two toxicants with various toxicities, discharged into nature, affecting the biological species. In the model, we consider simultaneous effect of two toxicants, in which primary toxicant is being converted into secondary toxicant and where secondary toxicant is more harmful. Also, transformation rate from primary toxicant to secondary toxicant is directly proportional to concentration of primary toxicant in the environment. Properties of two of these toxicants are different and hence the uptake rate of these toxicants by different species vary. Feasible equilibrium points have been obtained for the model. Using Routh Hurwitz method, conditions for local stability of non-trivial equilibrium points have been established. Also, using a suitable Lyapunov function, global stability of a non linear system at interior equilibrium point has been established. It has been observed in this paper that when conversion rate from primary to secondary toxicant is more, equilibrium is attained at a lower level and survival of population is less. We observe that although there is lesser infection in species in presence of pollution, survival of species becomes difficult with increase in pollution, therefore, either pollution has to be controlled or certain measures should be taken to reduce the conversion rate so that infection decreases and species also survives.

Keywords: prey-predator model, toxicants, Lyapunov Function, local stability, global stability

2020 Mathematics Subject Classification: 92B05, 92D25, 92D40.

1 Introduction

Prey-predator models are basically simulations studied to know the loss-win interactions between species, fighting for their existence. Since years, this interaction has been important in the field of ecology (Berryman, 1992). Various kinds of predator-prey models have been studied by researchers (Niu, Zhang and Teng, 2011) (Shi, Cui and Zhou, 2011) (Aybar, Cui and Zhou, 2018) (Ghosh, Sahoo and Poria, 2017) (Bhatia and Chauhan, 2019) (Bhatia, Chauhan,

Agarwal and Jain, 2020)(Indrajaya and Alghofari, 2016)(Ali, Sarkar and Guin, 2019)(Mondal, Maiti and Samanta, 2019)(Louartassi, Lemnaouar, Benazza and Khalfaoui, 2018)(Paparao and Narayan, 2017). Since species of all kinds are vulnerable to attract the abnormal conditions like disease and infection, due to exposure to external factors, therefore researchers focussed on dynamics of model including infection in the system. Some researchers (Hu and Li, 2012)(Johri, Trivedi, Sisodiya, Sing and Jain, 2012)(Khan, Al-Lawatia and Al-Kharousi, 2016)(Sharma and Samanta, 2015)(Zhou, Cui, Shi and Song, 2010) then focused on pre-predator interaction including two kinds of prey in their model, namely susceptible and infective prey. Later on, many researchers worked on models including disease in predator species as well(Jana, Guria, Das, Kar and Ghorai, 2015)(Xu and Zhang, 2013)(Zhang and Sun, 2005)(Zhou, Shi and Song, 2009).

In the modern era, society is facing problems related to changes in structure and functions of ecosystem occurred by different kind of stresses, one being release of toxicants in environment through different sources which adversely affects the living organisms and biodiversity of the environment(Hallam, Clark and Jordan, 1983a)(Hallam, Clark and Lassiter, 1983b). Many authors focused on studying the effects of pollutant on the ecological system(Kar, and Chaudhuri, 2003)(Sinha, Misra and Dhar, 2008)(Das, Mukherjee and Chaudhuri, 2009)(Naji and Mustafa, 2013)(Huang, Wang and Lewis, 2015)(Dubey and Hussan, 2003)(Chauhan, Misra and Dhar, 2008). Since there are many contaminants present in the environment, which are formed by physical as well as chemical processes, also they are subject to change in nature, at times converting into more toxic contaminants. So (Shukla and Dubey, 1996) (Shukla, Agrawal, Sinha and Dubey, 2003) considered the model with two pollutants.

One of the examples of these kind of toxicants are Nitrogen Oxides (NO_x). At the time of the combustion of fuel(Fig. 1(a)), nitrogen(N) is released which combines with oxygen(O_2) present in the air to form nitric oxide (NO)(Fig. 1(b)). Nitric oxide when present in ambient concentrations are not hazardous, but if inhaled in higher amount, it will rapidly react with oxygen present in lungs, to form NO_2 i.e. Nitrogen Dioxide(Fig. 1(c)), which would act as a pulmonary irritant. Hence, Nitric Oxide acts as a primary toxicant. Longer exposure to Nitrogen Dioxide may cause respiratory infections like flu, tuberculosis, viral hemorrhagic fever etc., which are communicable diseases. Here, Nitrogen Dioxide acts as a secondary toxicant. Both Nitric Oxide and Nitrogen Dioxide are oxides of Nitrogen (NO_x), which reacts to form acid rain and smog.

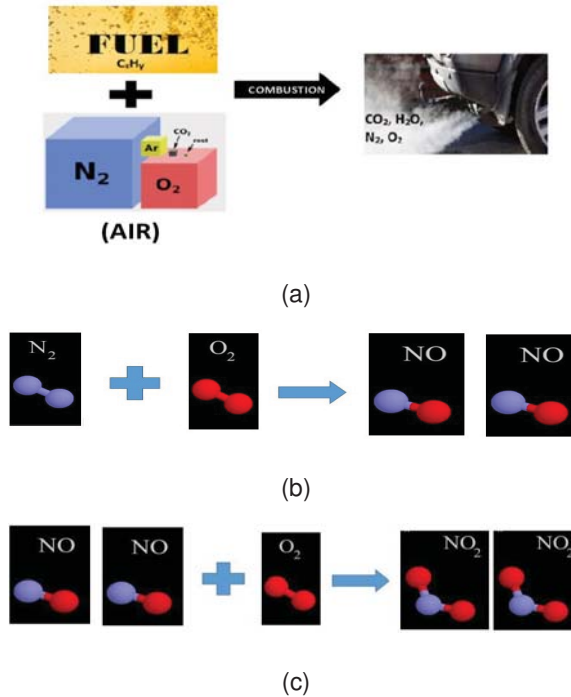


Figure 1: (a) Shows Fuel Combustion, (b) Shows formation of Nitric Oxide, (c) Shows formation of Nitrogen Dioxide

Motivated by the work mentioned above, a predator-prey model has been developed, under the effect of two toxicants, where we have considered the infective prey. We have considered two toxicants as there are toxicants in environment which do not immediately effect the species but when they are converted to other toxicant, they are more hazardous in nature. Each of the toxicant have different toxicity, also, one being dependent on the other, and is more toxic than it. The properties of both of these toxicants are different and hence the uptake rate of these toxicants by different species and concentration of these in the environment also vary. Unlike (Chauhan et al., 2008), we have considered recruitment rate as a function and is dependent on toxicants, which is more practical in approach as they are directly getting affected by the pollutants. To the best of our knowledge, no such work has been done by any other researcher. This paper is sorted out as follows. Section 2 depicts the development of the model with infection in prey species and discusses about the essential properties of the model. Section 3 deals with proof of the existence of equilibrium points. In Section 4, we focus on boundedness and dynamical behaviour of the model. Numerical simulations have been performed in the following section to support our hypothetical examination. An overall conclusion has been given in section 6.

2 Development of the Model

In this section, a non-linear mathematical model has been formulated to inquire simultaneous effect of two contaminants with different toxicities. Consider $X(t) = X_1(t) + X_2(t)$ to be populace density of aggregate prey species, where $X_1(t)$ is susceptible prey and $X_2(t)$ is infective prey. Let populace density of predator species be $Y(t)$. Also, let T_1, T_2 be concentrations of primary and secondary toxicants respectively and U_1, U_2 be uptake concentrations of toxicants by prey populace.

Our eco-epidemiological model has been formulated using the following assumptions:

1. It is assumed that secondary toxicant being transformed from primary toxicant, is more harmful, where primary toxicant is released from an outer source.
2. The uptake rate of both toxicants are different by different organisms.
3. It is presumed that growth rate of prey populace declines as uptake rate of the toxicants inclines.

The following set of differential equations are formulated using the above assumptions :

$$\frac{dX_1}{dT} = r(U_1, U_2)X_1 - \frac{r_0X_1^2}{K(T_1, T_2)} - \beta X_1X_2 - \alpha_1X_1Y \quad (2.1)$$

$$\frac{dX_2}{dT} = \beta X_1X_2 - \alpha_2X_2Y - d_2X_2 \quad (2.2)$$

$$\frac{dY}{dT} = \alpha_1X_1Y + \alpha_2X_2Y - d_3Y \quad (2.3)$$

$$\frac{dT_1}{dT} = Q(t) - \delta_1T_1 - r_1T_1X_1 - r_3T_1X_2 - kT_1 \quad (2.4)$$

$$\frac{dT_2}{dT} = \theta kT_1 - \delta_2T_2 - r_2T_2X_1 - r_4T_2X_2 \quad (2.5)$$

$$\frac{dU_1}{dT} = r_1T_1X_1 + r_3T_1X_2 - m_1U_1 \quad (2.6)$$

$$\frac{dU_2}{dT} = r_2T_2X_1 + r_4T_2X_2 - m_2U_2 \quad (2.7)$$

Then, $X_1(0) \geq 0, X_2(0) \geq 0, Y(0) \geq 0, T_1(0) \geq 0, T_2(0) \geq 0, U_1(0) \geq 0, U_2(0) \geq 0$ are initial conditions of the above ordinary differential equations.

In Table 1, the detailed meaning of the parameters are depicted:

Parameters	Description
$Q(t)$	Discharge rate of primary toxicant in nature
β	Infection rate
α_1	Capture rate of susceptible prey by predator
α_2	Capture rate of infective prey by predator
d_2	The diminishing rate of prey
d_3	The diminishing rate of predator
r_1, r_3	Rate coefficient at which primary toxicant is depleting due to uptake by susceptible prey
r_2, r_4	Rate coefficient at which secondary toxicant is depleting due to uptake by infective prey
δ_1, δ_2	Natural washout out rate of toxicants
m_1, m_2	Rate coefficient at which U_1 and U_2 are depleting naturally
$r(U_1, U_2)$	Function which represents coefficient of growth rate of prey populace density
$K(T_1, T_2)$	Function that depicts carrying capacity
θ	Fractional value of conversion of secondary toxicant from primary toxicant

Table 1: Meaning of the Parameters

If $\theta = 1$, then all primary toxicant will be converted into secondary toxicant. In the model, $r(U_1, U_2)$ decreases with increase of uptake rate of toxicants by prey affected by primary and secondary toxicants i.e. U_1 and U_2 . Then we take:

$$r(0, 0) = r_0 > 0, r(U_1, U_2) > 0 \text{ for } U_1, U_2 > 0$$

$$\frac{\delta r}{\delta U_1} < 0, \frac{\delta r}{\delta U_2} < 0, \text{ for } U_1, U_2 > 0$$

where r_0 is growth rate coefficient which is independent of toxicant. Similarly, $K(T_1, T_2)$ which decreases with increase of concentration of primary and secondary toxicants i.e. T_1 and T_2 . Therefore we take;

$$K(0, 0) = K_0 > 0, K(T_1, T_2) > 0 \text{ for } T_1, T_2 > 0$$

$$\frac{\delta K}{\delta T_1} < 0, \frac{\delta K}{\delta T_2} < 0, \text{ for } T_1, T_2 > 0$$

where, K_0 is carrying capacity which is independent of toxicant. As per assumptions, secondary toxicant being transformed from primary toxicant, is more toxic. In this case, if biomass is exposed one by one amid two toxicants, at same concentration U_t , for same amount of time; then we have,

$$r(0, U_t) < r(U_t, 0) \text{ for } U_t > 0.$$

But, if they are exposed simultaneously at same concentration U_t , amid two toxicants, for same amount of time; then we have,

$$r(U_t, U_t) < r(0, U_t) < r(U_t, 0) < r(0, 0) \text{ for } U_t > 0.$$

Likewise, for carrying capacity we have,

$$K(0, T_t) < K(T_t, 0) \text{ for } T_t > 0.$$

But, if they are exposed simultaneously amid two toxicants at the same concentration U_t , for same amount of time; then we have,

$$K(T_t, T_t) < K(0, T_t) < K(T_t, 0) < K(0, 0) \text{ for } T_t > 0.$$

Hence, from the above conditions it can be said that growth rate and the carrying capacity declines if biomass is exposed all together to both toxicants for same amount of time and same concentration, than when they are exposed individually. Also when uptake rate and concentration in environment of toxicants were nil, growth rate and carrying capacity were higher in comparison to their presence.

3 Existence of Equilibrium Points

Here, we will obtain existence of equilibrium points of the model:

Case I: Let $Q_1(t)$ be constant, i.e., $Q_1(t) = Q_1 > 0$.

In this case, system will have five non negative equilibria, which are:

$$E_0\left(0, 0, 0, \frac{Q_1}{\delta_1+k}, \frac{\theta K Q_1}{\delta_2(\delta_1+k)}, 0, 0\right), E_1(\overline{X}_1, 0, 0, \overline{T}_1, \overline{T}_2, \overline{U}_1, \overline{U}_2), E_2(\check{X}_1, \check{X}_2, 0, \check{T}_1, \check{T}_2, \check{U}_1, \check{U}_2), \\ E_3(\acute{X}_1, 0, \acute{Y}, \acute{T}_1, \acute{T}_2, \acute{U}_1, \acute{U}_2) \text{ and } E^*(X_1^*, X_2^*, Y^*, T_1^*, T_2^*, U_1^*, U_2^*).$$

The existence of $E_0\left(0, 0, 0, \frac{Q_1}{\delta_1+k}, \frac{\theta K Q_1}{\delta_2(\delta_1+k)}, 0, 0\right)$ is obvious. We shall prove existence of E_1 , E_2 , E_3 and E^* .

Existence of E_1 : Here, $\overline{X}_1, \overline{T}_1, \overline{T}_2, \overline{U}_1, \overline{U}_2$ are as follows:

$$\overline{X}_1 = \frac{r(U_1, U_2)K(T_1, T_2)}{r_0} \\ \overline{T}_1 = \frac{Q_1}{\delta_1 + r_1 X_1 + k} = h_1(X_1)(say) \\ \overline{T}_2 = \frac{\theta k Q_1}{(\delta_1 + r_1 X_1 + k)(\delta_2 + r_2)} = h_2(X_1)(say) \\ \overline{U}_1 = \frac{r_1 X_1 Q_1}{(\delta_1 + r_1 X_1 + k)m_1} = h_3(X_1)(say) \\ \overline{U}_2 = \frac{r_2 X_1 Q_1 \theta k}{(\delta_1 + r_1 X_1 + k)(\delta_2 + r_2)m_1} = h_4(X_1)(say)$$

Let,

$$H(X_1) = r(h_3(X_1), h_4(X_1))K(h_1(X_1), h_2(X_1)) - r_0 X_1$$

Then, we note that

$$H(0) = r(h_3(0), h_4(0))K(h_1(0), h_2(0)) > 0$$

and

$$H(K_0) = r(h_3(K_0), h_4(K_0))K(h_1(K_0), h_2(K_0)) - r_0 K_0 < 0$$

Therefore, this shows existence of the root, say, $\overline{X_1}$, i.e., $H(\overline{X_1}) = 0$ where $0 < \overline{X_1} < K_0$.

Now, for uniqueness of $\overline{X_1}$, we must have $\frac{dH}{dX_1} > 0$. As,

$$\frac{dH}{dX_1} = K(h_1(X_1), h_2(X_2)) \left(\frac{dr}{dU_1} \frac{dh_3}{dX_1} + \frac{dr}{dU_2} \frac{dh_4}{dX_1} \right) + r(h_3(X_1), h_4(X_2)) \left(\frac{dK}{dT_1} \frac{dh_1}{dX_1} + \frac{dK}{dT_2} \frac{dh_2}{dX_1} \right) - r_0$$

Then, $\overline{X_1}$ exists uniquely if

$$K(h_1(X_1), h_2(X_2)) \left(\frac{dr}{dU_1} \frac{dh_3}{dX_1} + \frac{dr}{dU_2} \frac{dh_4}{dX_1} \right) + r(h_3(X_1), h_4(X_2)) \left(\frac{dK}{dT_1} \frac{dh_1}{dX_1} + \frac{dK}{dT_2} \frac{dh_2}{dX_1} \right) > r_0$$

Existence of E_2 :

Here, the values of $\check{X}_1, \check{X}_2, \check{T}_1, \check{T}_2, \check{U}_1, \check{U}_2$ are as follows:

$$\begin{aligned} \check{X}_1 &= \frac{d_2}{\beta} \\ \check{X}_2 &= \frac{d_3\beta - \alpha_1 d_2}{\beta \alpha_2} \\ \check{T}_1 &= \frac{Q_1 \beta \alpha_2}{\delta_1 \beta \alpha_2 + r_1 d_2 \alpha_2 + r_3 (d_3 \beta - \alpha_1 d_2) + k \beta \alpha_2} \\ \check{T}_2 &= \frac{Q_1 \theta k \beta^2 \alpha_2^2}{(\delta_1 \beta \alpha_2 + r_1 d_2 \alpha_2 + r_3 (d_3 \beta - \alpha_1 d_2) + k \beta \alpha_2)(\delta_2 \beta \alpha_2 + r_2 d_2 \alpha_2 + r_4 (d_3 \beta - \alpha_1 d_2))} \\ \check{U}_1 &= \frac{Q_1 (r_1 d_2 \alpha_2 + r_3 (d_3 \beta - \alpha_1 d_2))}{m_1 (\delta_1 \beta \alpha_2 + r_1 d_2 \alpha_2 + r_3 (d_3 \beta - \alpha_1 d_2) + k \beta \alpha_2)} \\ \check{U}_2 &= \frac{Q_1 \theta k \alpha_2 \beta (r_2 d_2 \alpha_2 + r_4 (d_3 \beta - \alpha_1 d_2))}{m_2 (\delta_1 \beta \alpha_2 + r_1 d_2 \alpha_2 + r_3 (d_3 \beta - \alpha_1 d_2) + k \beta \alpha_2)(\delta_2 \beta \alpha_2 + r_2 d_2 \alpha_2 + r_4 (d_3 \beta - \alpha_1 d_2))} \end{aligned}$$

Hence, E_2 exists uniquely if $d_3 \beta > \alpha_1 d_2$.

Existence of E_3

Now, for E_3 , values of $(\acute{X}_1, 0, \acute{Y}, \acute{T}_1, \acute{T}_2, \acute{U}_1, \acute{U}_2)$ are as follows:

$$\begin{aligned} \acute{X}_1 &= \frac{d_3}{\alpha_1} \\ \acute{Y} &= \frac{\beta d_3 - d_2 \alpha_1}{\alpha_1 \alpha_2} \\ \acute{T}_1 &= \frac{Q_1 \alpha_1}{\delta_1 \alpha_1 + r_1 d_3 + k \alpha_1} \\ \acute{T}_2 &= \frac{\theta k Q_1 \alpha_1^3}{(\delta_1 \alpha_1 + r_1 d_3 + k \alpha_1)(\delta_2 \alpha_1^2 + r_2 d_3)} \\ \acute{U}_1 &= \frac{r_1 Q_1 d_3}{(\alpha_1 \delta_1 + r_1 d_3 + k \alpha_1) m_1} \\ \acute{U}_2 &= \frac{r_1 Q_1 d_3 \alpha_1^2 \theta k}{(\alpha_1 \delta_1 + r_1 d_3 + k \alpha_1)(\delta_2 \alpha_1^2 + r_2 d_3) m_2} \end{aligned}$$

Hence, E_3 exists uniquely if $d_3 \beta > \alpha_1 d_2$.

Existence of $E^*(X_1^*, X_2^*, Y^*, T_1^*, T_2^*, U_1^*, U_2^*)$ is obtained as follows: From equation (2.1), we get,

$$r(U_1, U_2)X_1 - \frac{r_0 X_1^2}{K(T_1, T_2)} - \beta X_1 X_2 - \alpha_1 X_1 Y = 0,$$

which implies,

$$r_0 X_1 = (r(U_1, U_2) - \beta X_2 - \alpha_1 Y)K(T_1, T_2)$$

From equation (2.2), we get,

$$Y^* = \frac{\beta X_1 - d_2}{\alpha_2}$$

Similarly from equation (2.3), we get,

$$X_2^* = \frac{d_3 - \alpha_1 X_1}{\alpha_2}.$$

Substituting values of X_2^* and Y^* in $r_0 X_1^* = (r(U_1, U_2) - \beta X_2 - \alpha_1 Y)K(T_1, T_2)$, we get the function as follows:

$$F(X_1) = r_0 X_1 - \left(r(U_1, U_2) - \frac{\beta(d_3 - \alpha_1 X_1)}{\alpha_2} - \frac{\alpha_1(\beta X_1 - d_2)}{\alpha_2} \right) K(T_1, T_2) > 0 \quad (3.1)$$

Now solving equations (2.4) and (2.6), we get,

$$T_1^* = \frac{Q_1 \alpha_2}{(\delta_1 \alpha_2 + r_1 X_1 \alpha_2 + r_3(d_3 - \alpha_1 X_1) + k \alpha_2)} = f_1(X_1) \text{ (say)}$$

and

$$U_1^* = \frac{Q_1 \alpha_2 (r_1 X_1 \alpha_2 + r_3(d_3 - \alpha_1 X_1))}{m_1 (\delta_1 \alpha_2 + r_1 X_1 \alpha_2 + r_3(d_3 - \alpha_1 X_1) + k \alpha_2)} = f_3(X_1) \text{ (say)}$$

Similarly solving equations (2.5) and (2.7), we get,

$$T_2^* = \frac{\theta k Q_1 \alpha_2^2}{(\delta_1 \alpha_2 + r_1 X_1 \alpha_2 + r_3(d_3 - \alpha_1 X_1) + k \alpha_2)(\delta_2 \alpha_2 + r_2 X_1 \alpha_2 + r_4(d_3 - \alpha_1 X_1))} = f_2(X_1) \text{ (say)}$$

and

$$U_2^* = \frac{\theta k Q_1 (r_2 X_1 + r_4(d_3 - \alpha_1 X_1))}{(\delta_1 + r_1 X_1 + r_3(d_3 - \alpha_1 X_1) + k)(\delta_2 + r_2 X_1 + r_4(d_3 - \alpha_1 X_1))m_2} = f_4(X_1) \text{ (say)}$$

Then, $F(X_1) = r_0 X_1 - \left(r(f_3(X_1), f_4(X_1)) - \frac{\beta(d_3 - \alpha_1 X_1)}{\alpha_2} - \frac{\alpha_1(\beta X_1 - d_2)}{\alpha_2} \right) K(f_1(X_1), f_2(X_1))$,
and, we note that

$$F(0) = - \left(r(f_3(X_1), f_4(X_1)) - \left(\frac{\beta d_3 - \alpha_1 d_2}{\alpha_2} \right) \right) K(f_1(X_1), f_2(X_1)) < 0 \quad (3.2)$$

if, $r > \frac{\beta d_3 - \alpha_1 d_2}{\alpha_2}$

$$F(K_0) = r_0 K_0 - \left(r(f_3(K_0), f_4(K_0)) - \frac{\beta(d_3 - \alpha_1 K_0)}{\alpha_2} - \frac{\alpha_1(\beta K_0 - d_2)}{\alpha_2} \right) K(f_1(K_0), f_2(K_0)) > 0 \quad (3.3)$$

Thus, there exists X_1^* such that $0 < X_1^* < K_0$, and $F(X_1^*) = 0$. Hence, this shows existence of $E^*(X_1^*, X_2^*, Y^*, T_1^*, T_2^*, U_1^*, U_2^*)$, when $r > \frac{\beta d_3 - \alpha_1 d_2}{\alpha_2}$.

Now, for uniqueness of X_1^* , we must have,

$$\frac{dF}{dX_1} = r_0 - K(f_1(X_1), f_2(X_2)) \left(\frac{dr}{dU_1} \frac{df_3}{dX_1} + \frac{dr}{dU_2} \frac{df_4}{dX_1} - \frac{\beta d_3 \alpha_1}{\alpha} - \frac{\alpha_1 \beta}{\alpha_2} \right) - \left(r(f_3(X_1), f_4(X_1)) - \frac{\beta(d_3 - \alpha_1 X_1)}{\alpha_2} - \frac{\alpha_1(\beta X_1 - d_2)}{\alpha_2} \right) \left(\frac{dK}{dT_1} \frac{df_1}{dX_1} + \frac{dK}{dT_2} \frac{df_2}{dX_1} \right) > 0$$

We note that $f_1(X_1)$, $f_2(X_1)$, $f_3(X_1)$ and $f_4(X_1)$ increases as value of Q_1 increases for given X_1 . Thus, with increase in value of Q_1 , $r(f_3(X_1), f_4(X_1))$ and $k(f_1(X_1), f_2(X_1))$ decreases. Since we know that $X_1^* = \frac{r(U_1, U_2)k(T_1, T_2) - (\beta X_2 + \alpha_1 Y)k(T_1, T_2)}{r_0}$, i.e., population density is dependent upon the product of carrying capacity and growth rate, then, it can be said that X_1^* decreases as Q_1 increases. Also, with increase in transformation rate, concentration and toxicity of secondary toxicant increases, which leads to decrease in biomass density(i.e., X_1^*). Hence, it can be said that the secondary toxicant with higher toxicity is more hazardous in comparison to two toxicants with same toxicities.

4 Boundedness and Dynamical Behaviour

This segment deals with study of boundedness and dynamical behaviour of the model for which we have the theorem as follows:

Theorem 4.1. *The set*

$$\Omega = \left\{ (X_1, X_2, Y, T_1, T_2, U_1, U_2) : 0 \leq X_1 + X_2 + Y \leq \frac{(r_0 + \delta)k_0}{\delta}, T_1 + U_1 \leq \frac{Q}{\mu_1}, T_2 + U_2 \leq \frac{\theta k Q}{\mu_1 \mu_2} \right\} \quad (4.1)$$

where, $\delta = \min(d_2, d_3)$, $\mu_1 = \min(\delta_1 + k, m_1)$ and $\mu_2 = \min(\delta_2, m_2)$ is the region of attraction for all solutions.

Proof. Consider,

$$\frac{dX_1}{dt} + \frac{dX_2}{dt} + \frac{dY}{dt} = rX_1 - \frac{r_0(X_1^2)}{k} - d_2X_2 - d_3Y \leq r_0X_1 - \delta(X_2 + Y) - \delta X_1 + \delta X_1 \quad (4.2)$$

which implies

$$X_1 + X_2 + Y \leq \frac{(r_0 + \delta)k_0}{\delta}$$

where, $\delta = \min(d_2, d_3)$.

Now consider,

$$\frac{dT_1}{dt} + \frac{dU_1}{dt} = Q - \delta_1 T_1 - k T_1 - m_1 U_1 \leq Q - \mu_1 (T_1 + U_1) \quad (4.3)$$

where, $\mu_1 = \min(\delta_1 + k, m_1)$.

This implies, $\lim_{t \rightarrow \infty} \text{Sup}(T_1 + U_1) \leq \frac{Q}{\mu_1}$. Hence, $0 \leq T_1 + U_1 \leq \frac{Q}{\mu_1}$.

Also,

$$\frac{dT_2}{dt} + \frac{dU_2}{dt} = \theta k T_1 - \delta_1 T_2 - m_2 U_2 \quad (4.4)$$

$$\leq \theta k \frac{Q}{\mu_1} - \mu_2 (T_2 + U_2) \quad (4.5)$$

where, $\mu_2 = \min(\delta_2, m_2)$,

This implies,

$$\lim_{t \rightarrow \infty} \text{Sup}[T_2 + U_2] \leq \frac{\theta k Q}{\mu_1 \mu_2} \quad (4.6)$$

Hence, $0 \leq T_2 + U_2 \leq \frac{\theta k Q}{\mu_1 \mu_2}$. This completes the proof. $\square$

4.1 Stability Analysis of the Equilibrium Points

This segment depicts local stability of equilibrium points by Routh-Hurtwitz criteria and global stability of E^* by Lyapunov's Direct method.

4.1.1 Local Stability of Equilibrium Points

We can easily calculate that the eigenvalues of characteristic equations corresponding to E_0 are $r_0, -d_2, -d_3, -\delta_2, -m_1, -m_2, -\delta_1 - k$. Therefore, it is noted that E_0 is a saddle point. It has unstable manifold locally in X_1 space and stable manifold locally in $X_2 - Y - T_1 - T_2 - U_1 - U_2$ space.

Now, we will study the local stability of E^* .

Theorem 4.2. E^* is locally stable, if the below mentioned conditions hold true:

1. $a_i > 0$, for all $i = 1, 2, 3, 4, 5, 6, 7$
2. $a_1 a_2 > a_3$
3. $a_5 > a_1 a_4$
4. $-a_2(a_1 a_2 a_5 - a_1^2 a_6) + a_2 a_3 a_5 - a_1 a_2 a_5 > 0$
5. $a_1(a_4 a_5 - a_3 a_6) - a_5^2 + a_3 a_7 > 0$
6. $a_1 a_5 a_4 > a_1 a_3 a_6 + a_5$
7. $a_3(a_1 a_2 a_3 a_6 - a_3^2 a_6 - a_1^2 a_4 a_6 + a_1 a_5 a_6 - a_1 a_2^2 a_7 + a_2 a_3 a_7 + a_1 a_4 a_7 - a_5 a_7) < 0$
8. $a_1 a_2 a_5 a_6 - a_3 a_5 a_6 - a_1^2 a_6^2 - a_1 a_2 a_4 a_7 + a_3 a_4 a_7 + 2 a_1 a_6 a_7 - a_7^2 > 0$
9. $a_1 a_2 a_3 a_4 a_7 - a_3^2 a_4 a_7 - a_1^2 a_4^2 a_7 - a_1 a_2^2 a_5 a_7 + a_2 a_3 a_5 a_7 + 2 a_1 a_4 a_5 a_7 - a_5^2 a_7 + a_1^2 a_2 a_6 a_7 - a_1 a_3 a_6 a_7 - a_1 a_2 a_7^2 a_3 a_7^2 < 0$
10. $a_1 a_2 a_3 a_6 a_7 - a_3^2 a_6 a_7 - a_1^2 a_4 a_6 a_7 + a_3 a_5 a_6 a_7 - a_1 a_6^2 a_7^2 + a_2 a_3 a_7^2 + a_1 a_4 a_7^2 - a_5 a_7^2 > 0$

$$11. a_1a_2a_5a_6a_7 - a_3a_5a_6a_7 - a_1a_6^2a_7^2 - a_1a_2a_4a_7^2 + a_3a_4a_7^2 + 2a_1a_6a_7^2 - a_7^3 < 0$$

where a'_i s are the coefficients of characteristic equation corresponding to E^* .

Proof. Characteristic equation corresponding to E^* is as follows:

$$a_0\lambda^7 + a_1\lambda^6 + a_2\lambda^5 + a_3\lambda^4 + a_4\lambda^3 + a_5\lambda^2 + a_6\lambda + a_7 = 0 \quad (4.7)$$

where,

$$a_0 = 1$$

$$a_1 = A + B + C + D + G$$

$$a_2 = \xi^* \alpha_2 X_2 r_2 T_2 \eta^* r_1 T_1$$

$$a_3 = \xi^* [r_2 X_1 + r_4 X_2 + 2(A + C) + B + Dr_2 T_2 + \alpha_2 X_2 + 1\beta \alpha_2 r_4 T_2 + r_1 X_1 + r_3 X_2 r_1 T_1 + 2\beta \alpha_1 r_3 t_1] + \eta(\theta k + \beta X_2 r_3 T_1) + (A + B + C + 1)r_1 T_1 + YX[\alpha_1 \alpha_2 \beta(1 + Y^*) + F\alpha_1^2 + X_2 + E\beta^2 X_2] + G\alpha_2^2 X_2 Y + (A + B + C + D)[\alpha_1^2 Y X_1 + \beta X_2^* X_1^* + \alpha_2^2 X_2 Y]$$

$$a_4 = \xi^* ([\alpha_2 X_2 + (CD + 2AC + AD + AB + BC) B + \theta k r_2 X_1 r_2 T_2 + (2A + B + 2C + D + r_2 X_1 + r_4 X_2) \beta X_2 r_4 T_2 + (CD + AC)r_1 T_1 + r_2 X_1 r_4 X_2 + \beta X_1 + B + 2(r_1 X_1 + r_3 X_2) + (A + 1)\beta X_1 r_3 T_1] + \eta^* ([\theta k + (1 + A + B + C)\beta X_2 r_3 T_1 + (AB + AC + BC)r_1 T_1 + (A + B + C + D)(X_1 \alpha_1 (\beta X_2 \alpha_2 Y) + G\alpha_2 X - 2Y + AB + AC + AD + BC + BD + CD)(X_1(\alpha_1 + \beta X_2 + \beta \alpha_2 X - 2\alpha_1 Y) + \alpha_2^2 X_2 Y) + ABCD + G])$$

$$a_5 = \xi^* [(DB + B + ACD)r_2 T_2 + (r_2 X_1 + r_4 X_2 + A + C + D)r_1 T_1] + \eta^* [\theta k \alpha_2 X_2 + (A + C + AB + AC + BC)\beta X_2 r_3 T_1 + (AB + AC + AD + BC + BD + CD)(\alpha_1 \alpha_2 \beta X_2 Y) + G\alpha_2^2 X_2 Y]$$

$$a_6 = \xi^* [\alpha_2 X_2 r_2 X_1 + r_4 X_2 + CD + AC + AD + A + B\theta k BD(\beta X_2 r_4 T_2) + ACD(\beta X_2 r_4 T_2) + (AB)\beta X_2 r_4 T_2 + B\beta X_2(r_2 X_1 r_2 T_2 + r_4 T_2 r_2 X_2) + \eta^* [\theta k + AB + AC + BC + A + B\alpha_2 X_2 + (ACD + BCD + ABC + ABD + ABCD)(X_1(\alpha_1 X_2 \beta \alpha_2 Y + \beta X_2 \alpha_1 Y + \alpha_1^2 Y + \beta X_2) + ABCDAG)]$$

$$a_7 = \xi^* [\beta \alpha_2 r_4 t_2 (Dr_2 X_1 + Dr_4 X_2) + (Dr_2 X_1 + Dr_4 X_2 + r_2 X_1 + r_4 X_2)r_2 T_2 + ACD(\alpha_1 X_2) + (AC(r_1 X_1 + r_3 X_2 + D)r_1 T_1 + \theta k + (Br_2 X_1 + Br_4 X_2)\alpha_2 X_2)] + \eta^* (\theta k(AB + ABC(\alpha_2 X_2)) + ABCD(X_1(\alpha_1 \beta X_2 \alpha_2 Y + E\beta^2 X_2 + \beta \alpha_2 X_2 \alpha_1 Y)))$$

and,

$$A = m_2, B = m_1, C = \delta_2 + r_2 X_1 + r_4 X_2, D = \delta_1 + r_1 X_1 + r_3 X_2 + k, G = -r(U_1, U_2) + \frac{2r_0 X_1}{K(T_1, T_2)} + \beta X_2 + a_1 Y, \xi^* = r'(U_1, U_2), \eta^* = K'(T_1, T_2).$$

If all conditions stated in theorem hold, then by Routh Hurwitz criteria, E^* is locally stable. $\square$

Remark: Local stability analysis of other equilibrium points E_1, E_2 and E_3 can be done on similar lines as in the previous theorem.

4.1.2 Global stability of E^*

Now, by using Lyapunov's direct method we will show global stability of interior equilibrium point E^* , for which we have the theorem as follows:

Theorem 4.3. (Shukla et al., 2003) Let $r(U_1, U_2)$ and $k(T_1, T_2)$ satisfy the following conditions in Ω ,

$$K_m \leq K(T_1, T_2) \leq K_0 \quad (4.8)$$

$$0 \leq \frac{-\delta r}{\delta U_1}(U_1, U_2) \leq \rho_1 \quad (4.9)$$

$$0 \leq \frac{-\delta r}{\delta U_2}(U_1, U_2) \leq \rho_2 \quad (4.10)$$

$$0 \leq \frac{-\delta k}{\delta T_1}(T_1, T_2) \leq k_1 \quad (4.11)$$

$$0 \leq \frac{-\delta k}{\delta T_2}(T_1, T_2) \leq k_2 \quad (4.12)$$

for some constants $K_m, \rho_1, \rho_2, k_1, k_2$ which are positive. Then, if the following inequalities holds in Ω ,

$$6(\beta + \beta k_0)^2 \leq \frac{r_0}{k(T_1^*, T_2^*)}(\alpha_2 Y^* + d_2) \quad (4.13)$$

$$3\left(\alpha_1 + \alpha_1 \frac{(r_0 + \delta)k_0}{\delta}\right)^2 \leq \frac{a_1 r_0}{k(T_1^*, T_2^*)}(d_3) \quad (4.14)$$

$$6\left(\frac{(r_0 + \delta)k_0}{\delta}\rho_1 + r_1 \frac{Q}{\mu_1}\right)^2 \leq \frac{r_0}{k(T_1^*, T_2^*)}(\delta_1 + k + r_1 X_1^* + r_4 X_2^*) \quad (4.15)$$

$$\frac{9}{2}(r_0 k_0 \rho_1)^2 \leq \frac{r_0}{k(T_1^*, T_2^*)}(\delta_2 + r_1 X_1^* + r_4 X_2^*) \quad (4.16)$$

$$\frac{9}{2}\left(\frac{k_1}{k_m^2} + \frac{r_0 \theta k Q}{\mu_1, \mu_2} + \frac{r_1 Q}{\mu_1}\right)^2 \leq \frac{r_0 m_1}{k(T_1^*, T_2^*)} \quad (4.17)$$

$$\frac{9}{2}\left(\frac{k_1}{k_m^2} + \frac{r_0 \theta k Q}{\mu_1, \mu_2}\right)^2 \leq \frac{r_0 m_2}{k(T_1^*, T_2^*)} \quad (4.18)$$

$$2\left(\alpha_2 k_0 + \alpha_2 \frac{(r_0 + \delta)k_0}{\delta}\right)^2 \leq \beta \alpha_1 X_1^{*2} + \beta \alpha_2 X_1^* X_2^* + \alpha_2 d_3 Y^* + d_2 d_3 \quad (4.19)$$

$$4\left(r_2 \frac{Q}{\mu_1}\right)^2 \leq (d_2 + \alpha_2 Y^*)(\delta_2 + k + r_1 X_1^* + r_4 X_2^*) \quad (4.20)$$

$$3\left(r_4 \frac{\theta k Q}{\mu_1, \mu_2} + r_3 \frac{Q}{\mu_1}\right)^2 \leq (d_2 + \alpha_2 Y^*)(m_1) \quad (4.21)$$

$$3\left(r_2 \frac{\theta k Q}{\mu_1, \mu_2}\right)^2 \leq (d_2 + \alpha_2 Y^*)(m_2) \quad (4.22)$$

$$3(\theta k)^2 \leq (\delta_1 + r_1 X_1^* + r_3 X_2^* + k)(\delta_2 + r_1 X_1^* + r_4 X_2^*) \quad (4.23)$$

$$3(r_4 X_1^* + r_3 X_2^*)^2 \leq (\delta_1 + r_1 X_1^* + r_3 X_2^* + k)(m_1) \quad (4.24)$$

$$\frac{9}{4}\left(r_2 X_1^* + r_4 X_2^*\right)^2 \leq (\delta_2 + r_1 X_1^* + r_4 X_2^*)(m_2) \quad (4.25)$$

then E^* is globally stable.

Proof. A positive definite function W around E^* is considered as :

$$W(X_1, X_2, Y, T_1, T_2, U_1, U_2) = \left(X_1 - X_1^* - X_1^* \ln \frac{X_1}{X_1^*} \right) + \frac{1}{2}(X_2 - X_2^*)^2 + \frac{1}{2}(Y - Y^*)^2 + \frac{1}{2}(T_1 - T_1^*)^2 + \frac{1}{2}(T_2 - T_2^*)^2 + \frac{1}{2}(U_1 - U_1^*)^2 + \frac{1}{2}(U_2 - U_2^*)^2 \quad (4.26)$$

then,

$$\begin{aligned} \frac{dW}{dt} = & \left(\frac{X_1 - X_1^*}{X_1} \right) \frac{dX_1}{dt} + (X_2 - X_2^*) \frac{dX_2}{dt} + (Y - Y^*) \frac{dY}{dt} + (T_1 - T_1^*) \frac{dT_1}{dt} \\ & + (T_2 - T_2^*) \frac{dT_2}{dt} + (U_1 - U_1^*) \frac{dU_1}{dt} + (U_2 - U_2^*) \frac{dU_2}{dt} \end{aligned} \quad (4.27)$$

substituting values of $\frac{dX_1}{dt}, \frac{dX_2}{dt}, \frac{dY}{dt}, \frac{dT_1}{dt}, \frac{dT_2}{dt}, \frac{dU_1}{dt}, \frac{dU_2}{dt}$ in (4.27), we get;

$$\begin{aligned} \frac{dW}{dt} = & (X_1 - X_1^*) \left(r(U_1, U_2) - \frac{r_0 X_1}{k(T_1, T_2)} - \beta X_2 - \alpha_1 Y \right) + (X_2 - X_2^*) (\beta X_1 X_2 - \alpha_2 X_2 Y - d_2 X_2) \\ & + (Y - Y^*) (\alpha_1 X_1 Y - \alpha_2 X_2 Y - d_3 Y) + (T_1 - T_1^*) (Q_1 - \delta_1 T_1 - r_1 T_1 X_1 - r_3 T_1 X_2 - k T_1) \\ & + (T_2 - T_2^*) (\theta k T_1 - \delta_2 T_2 - r_2 T_2 X_1 - r_4 T_2 X_2) + (U_1 - U_1^*) (r_1 X_1 T_1 - r_3 X_2 T_1 - m_1 U_1) \\ & + (U_2 - U_2^*) (r_2 X_1 T_2 + r_4 X_2 T_2 - m_2 U_2) \end{aligned} \quad (4.28)$$

Now, let $(X_1 - X_1^*) = z_1, (X_2 - X_2^*) = z_2, (Y - Y^*) = z_3, (T_1 - T_1^*) = z_4, (T_2 - T_2^*) = z_5, (U_1 - U_1^*) = z_6, (U_2 - U_2^*) = z_7$

which implies,

$$\begin{aligned} \frac{dW}{dt} = & z_1 \left(r - \frac{r_0 X_1}{k} - \beta X_2 - \alpha_1 Y \right) + z_2 (\beta X_1 X_2 - \alpha_2 X_2 Y - d_2 X_2) + z_3 (\alpha_1 X_1 Y - \alpha_2 X_2 Y - d_3 Y) \\ & + z_4 (Q_1 - \delta_1 T_1 - r_1 T_1 X_1 - r_3 T_1 X_2 - k T_1) + z_5 (\theta k T_1 - \delta_2 T_2 - r_2 T_2 X_1 - r_4 T_2 X_2) \\ & + z_6 (r_1 X_1 T_1 - r_3 X_2 T_1 - m_1 U_1) + z_7 (r_2 X_1 T_2 + r_4 X_2 T_2 - m_2 U_2) \end{aligned} \quad (4.29)$$

hence, from above equation we get,

$$\begin{aligned} \frac{dW}{dt} = & (b_{11} z_1^2 + b_{12} z_1 z_2 + b_{13} z_1 z_3 + b_{14} z_1 z_4 + b_{15} z_1 z_5 + b_{16} z_1 z_6 + b_{17} z_1 z_7 + b_{22} z_2^2 \\ & + b_{23} z_2 z_3 + b_{24} z_2 z_4 + b_{26} z_2 z_6 + b_{27} z_2 z_7 + b_{33} z_3^2 + b_{44} z_4^2 + b_{45} z_4 z_5 + \\ & b_{46} z_4 z_6 + b_{55} z_5^2 + b_{56} z_5 z_7 + b_{66} z_6^2 + b_{77} z_7^2) \end{aligned} \quad (4.30)$$

where,

$$\begin{aligned} b_{11} = & \frac{-r_0}{3k(T_1^*, T_2^*)}, b_{22} = \frac{1}{2}(\beta X_1^* - \alpha_2 Y_2^* - d_2), b_{33} = \alpha_1 X_1^* + \alpha_2 X_2^* - d_3, b_{44} = -\frac{1}{2}(\delta_1 - r_1 X_1^* - k - \\ & r_3 X_2^*), b_{55} = -\frac{3}{2}(\delta_2 - r_2 X_1^* - r_4 X_2^*), b_{66} = -\frac{3}{2}m_1, b_{77} = -\frac{3}{2}m_2, b_{12} = -\beta + \beta X_2, b_{13} = -\alpha_1 + \\ & \alpha_1 Y, b_{14} = -r_1 T_1 - r_0 X_1 \eta_1(T_1, T_2), b_{15} = -r_0 X_1 \eta_2(T_1^*, T_2), b_{16} = r_1 T_1 + \xi_1(U_1, U_2) - r_2 T_2, b_{17} = \\ & r_2 T_2 + \xi_2(U_1^*, U_2), b_{23} = -a_2 \alpha_2 X_2 - \alpha_2 Y, b_{24} = r_2 T_1, b_{26} = -r_4 T_2 + r_3 T_1, b_{27} = r_4 T_2, b_{45} = \\ & \theta k, b_{46} = r_4 X_1^* + r_3 X_2^*, b_{57} = r_2 X_1^* + r_4 X_2^* \end{aligned}$$

and,

$$\begin{aligned}\eta_1(T_1, T_2) &= \begin{cases} \frac{1}{k(T_1, T_2)} - \frac{1}{k(T_1^*, T_2)}, \text{ for } T_1 \neq T_1^* \\ -\frac{1}{k^2(T_1^*, T_2)} \frac{\delta k}{\delta T_1}(T_1^*, T_2), \text{ for } T_1 = T_1^* \end{cases} \\ \eta_2(T_1^*, T_2) &= \begin{cases} \frac{1}{k(T_1^*, T_2)} - \frac{1}{k(T_1^*, T_2^*)}, \text{ for } T_2 \neq T_2^* \\ -\frac{1}{k^2(T_1^*, T_2^*)} \frac{\delta k}{\delta T_2}(T_1^*, T_2^*), \text{ for } T_2 = T_2^* \end{cases} \\ \xi_1(U_1, U_2) &= \begin{cases} \frac{r(U_1, U_2) - r(U_1^*, U_2)}{U_1, U_1^*}, \text{ for } U_1 \neq U_1^* \\ \frac{\delta r}{\delta U_1}(U_1, U_2), \text{ for } U_1 = U_1^* \end{cases} \\ \xi_1(U_1^*, U_2) &= \begin{cases} \frac{r(U_1^*, U_2) - r(U_1^*, U_2^*)}{U_2, U_2^*}, \text{ for } U_2 \neq U_2^* \\ \frac{\delta r}{\delta U_2}(U_1^*, U_2), \text{ for } U_2 = U_2^* \end{cases}\end{aligned}$$

and by equations (4.8-4.12), above conditions and mean value theorem, we note that

$$\begin{aligned}|\xi_1(U_1, U_2)| &\leq \rho_1, \quad |\xi_1(U_1^*, U_2)| \leq \rho_2 \\ |\eta_1(U_1, U_2)| &\leq \frac{k_1}{K_m^2}, \quad |\eta_2(U_1^*, U_2)| \leq \frac{k_2}{K_m^2}\end{aligned}$$

Now, we see that by Sylvester's Criteria E^* is globally stable, if the below mentioned conditions hold:-

$$b_{12}^2 < b_{11}b_{22}, b_{13}^2 < b_{11}b_{33}, b_{14}^2 < b_{11}b_{44}, b_{15}^2 < b_{11}b_{55}, b_{16}^2 < b_{11}b_{66}, b_{23}^2 < b_{22}b_{33}, b_{24}^2 < b_{22}b_{44}, b_{26}^2 < b_{22}b_{66}, b_{27}^2 < b_{22}b_{77}, b_{45}^2 < b_{44}b_{55}, b_{46}^2 < b_{44}b_{66} \text{ and } b_{57}^2 < b_{55}b_{77}$$

By substituting values of $b'_{i,j}$ in above inequalities, we can see that $\frac{dW}{dt}$ is a negative definite function, if all conditions stated in theorem are true. Then by Lyapunov's direct method, E^* is globally asymptotically stable.

It can be seen from conditions given in theorem that if transformation rate coefficient (k), magnitude of transformation from primary to secondary toxicant (θ) and emission rate of primary toxicant in environment (Q_0) are kept minimal, then conditions (4.13-4.15) can be easily satisfied.

Also, considering results of section 3, it is notable here that equilibrium of biomass density is attained at lower level and survival of population is less when secondary toxicant is more toxic, than in presence of two toxicants with same toxicity. Thus, secondary toxicant can act pernicious, in case of its high presence of toxicity, resulting in extinction of biomass density. $\square$

Case II: Let $Q_1 = Q_1(t) = Q_{01}(t) + \varsigma \vartheta(t)$ be the periodic function with $\vartheta(t + \varpi) = \vartheta(t)$.

Then, we can write the given system as:

$$\frac{dn}{dt} = H(n) + \varsigma L(t), \quad n(0) = n_0$$

where,

$$n = (n_1, n_2, n_3, n_4, n_5, n_6, n_7)^T = (X_1, X_2, Y, T_1, T_2, U_1, U_2)^T$$

$$H(n) = \begin{pmatrix} r(n_6, n_7)n_1 - \frac{r_0 n_1^2}{k(n_4, n_5)} - \beta n_1 n_2 - \alpha_1 n_1 n_3 \\ \beta n_1 n_2 - \alpha_2 n_2 n_3 - d_2 n_2 \\ \alpha_1 n_1 n_3 + \alpha_2 n_2 n_3 - d_3 n_3 \\ Q_1 - \delta_1 n_4 - r_1 n_4 n_1 - r_3 n_4 n_2 - k n_4 \\ \theta k n_4 - \delta_2 n_5 - r_2 n_5 n_1 - r_4 n_5 n_2 \\ r_1 n_4 n_1 + r_3 n_1 n_2 - m_1 n_1 \\ r_2 n_4 n_1 + r_4 n_5 n_2 - m_2 n_6 \end{pmatrix}$$

$$L(t) = (0, 0, 0, \varsigma \vartheta(t), 0, 0, 0)^T, n_0(X_{01}, X_{02}, Y_{01}, T_{01}, T_{02}, U_{01}, U_{02})^T$$

Considering $n(t, \varrho, \varsigma)$ to be the solution of $H(n) + \varsigma L(t)$, so that $n(0, \varrho, \varsigma) = n^* + \varrho$, where

$$n = (X_1^*, X_2^*, Y^*, T_1^*, T_2^*, U_1^*, U_2^*)^T, \varrho = (\varrho_1, \varrho_2, \varrho_3, \varrho_4, \varrho_5, \varrho_6, \varrho_7)^T$$

Corresponding to E^* , let H^* be the variational matrix. Then, considering the analysis done by (Freedman and Shukla, 1991), following theorem can be stated as:

Theorem 4.4. *If H^* has no characteristic values with zero real parts, then system (2.1-2.7) with*

$Q_1 = Q_1(t) = Q_{01}(t) + \varsigma \vartheta(t)$ will have periodic function $(X_1(t, \varsigma), X_2(t, \varsigma), Y(t, \varsigma), T_1(t, \varsigma), T_2(t, \varsigma), U_1(t, \varsigma), U_2(t, \varsigma))$ of period ϖ such that

$$(X_1(t, 0), X_2(t, 0), Y(t, 0), T_1(t, 0), T_2(t, 0), U_1(t, 0), U_2(t, 0)) = (X_1^*, X_2^*, Y^*, T_1^*, T_2^*, U_1^*, U_2^*)$$

Theorem 4.5. *If H^* has no characteristic values with zero real parts, then stability behaviour of equilibrium point E^* and of periodic solution of system (2.1-2.7) is same, for sufficiently small ς .*

Also,

$$n(t, \varrho, \varsigma) = n^* + e^{H^* t} \left(\int_0^t e^{-H^* p} L(p) dp - (e^{-H^* \vartheta} - I)^{-1} e^{H^* \vartheta} \int_0^{\vartheta} e^{-H^* p} C(p) dp \right) \varsigma + o(\varsigma)$$

This analysis implies that minute periodic release of pollutant in the environment leads to periodic behaviour in the system.

5 Numerical Simulation

Numerical simulations have been performed in this section, using MATLAB. Using the hypothetical and biologically feasible parametric values, i.e., $r = 0.7, \beta = 0.379, \alpha_1 = 0.002, \alpha_2 = 0.005, d_2 = 0.1, d_3 = 0.11, Q = 5, \delta_1 = 0.14, \delta_2 = 0.17, r_1 = 0.2, r_2 = 0.8, r_3 = 0.03, r_4 = 0.02, k = 5, m_1 = 0.2, m_2 = 0.3$ and function $r(U_1, U_2) = l_0 - \frac{l_1 U_1}{1 + l_2 U_1} - \frac{l_3 U_2}{1 + l_4 U_2}$ and $k(T_1, T_2) = n_0 - \frac{n_1 T_1}{1 + n_2 T_1} - \frac{n_3 T_2}{1 + n_4 T_2}$ (Shukla et al., 2003), where $l_0 = 0.7, l_1 = 0.01, l_2 = 2.2, n_0 = 4, n_1 = 1.1$ and $n_2 = 1.02$, following observations have been made:

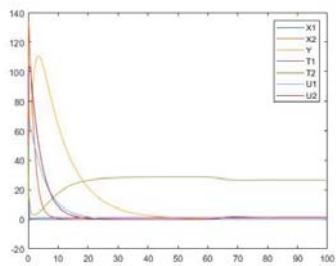
Parameters	$\theta = 1$	$\theta = 0.5$	$\theta = 0$
X_1	0.0158	0.0364	0.7573
X_2	0.000	0.0000	0.8198
Y	0.0032	0.0033	0.0047
T_1	0.9728	0.9727	0.9656
T_2	26.6324	12.211	0.0000
U_1	0.0015	0.0035	0.1490
U_2	1.1195	1.1852	0.0000

Table 2: Value of Equilibrium Points

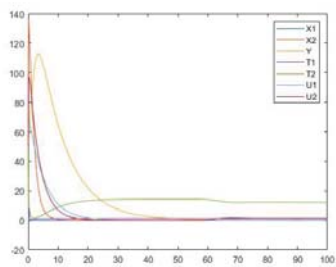
1. It has been observed that when $\theta = 0$, i.e., when no primary toxicant was transformed into secondary toxicant, biomass density of susceptible prey, infective prey and predator was $X_1 = 0.7573$, $X_2 = 0.8198$ and $Y = 0.0047$, respectively.
2. When $\theta = 0.5$, i.e., when some proportion of primary toxicant was transformed into secondary toxicant, biomass density of prey and predator reduced to $X_1 = 0.0364$, $X_2 = 0.000$ and $Y = 0.0033$.
3. And when $\theta = 1$, i.e., when whole proportion of primary toxicant was transformed into secondary toxicant, biomass density of prey and predator reduced further to $X_1 = 0.0158$, $X_2 = 0.0000$ and $Y = 0.0032$.

Therefore, it can be concluded that although there is a sharp decrease in infectives but overall population density also declines sharply in presence of secondary toxicant, hence secondary toxicant is more harmful and toxic than primary toxicant. So in order to maintain the population size, and to reduce infection, pollution has to be controlled and measures should be taken so that primary toxicant does not get converted into secondary toxicant.

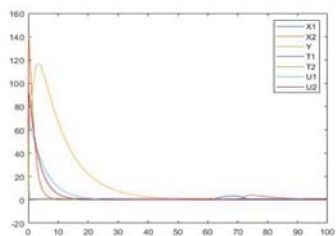
4. From Fig. 3 and 4 we can see that system approaches interior equilibrium point ($= 0.2612, 0.0001, 0.0034, 0.9718, 1.1551, 0.0253, 0.8043$). This implies that for the above mentioned parameters, E^* is globally stable. The mentioned parameters also satisfy conditions of theorem 4.3.



(a)



(b)



(c)

Figure 2: (a),(b),(c) Trajectories when $\theta = 1$, $\theta = 0.5$ and $\theta = 0$

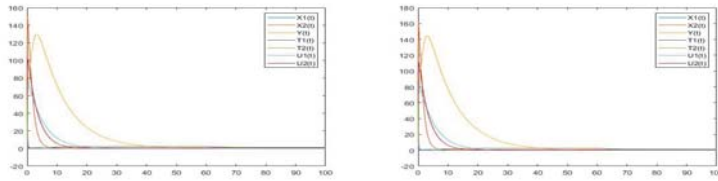


Figure 3: Trajectories with initial condition $[55, 55, 55, 55, 55, 55]$ and $[50, 50, 50, 50, 50, 50]$ respectively.

6 Conclusion

In this paper, we developed a model in which we considered infection in prey species. In the model, we considered simultaneous effect of two toxicants, each having different toxicity, also one being dependent on the other and is more toxic than it. Authors like (Chauhan et al., 2008), who worked on similar lines, considered only one type of toxicant and where recruitment was constant and independent of pollutants, depicted that equilibrium points were attained at a lower level than in a system with no pollution. Whereas in our model, we noticed that there was even more decline in the biomass density, as recruitment rate which we considered was dependent on the pollution and there were two toxicants in the system.

We also noticed that biomass density declines more with increase in the toxicity of the secondary toxicant, than in the case where two toxicants have same toxicities. Therefore, care must be taken for incorporating proper recruitment rate in the system and measures should be taken so that primary toxicant does not get converted into secondary toxicant.

Using the analysis done by (Freedman and Shukla, 1991), we observed that periodic release of very small amount of pollutant in the environment, results in periodic behaviour in the system.

We discussed about essential properties of the model. Existence of equilibrium points was proved. Boundedness and dynamical behaviour of the model was also shown, which includes local stability and global stability of non-negative equilibria of the model. Through numerical simulation, we showed that when no primary toxicant was transformed into secondary toxicant, biomass density of prey and predator was more; when some proportion of primary toxicant was transformed into secondary toxicant, biomass density of prey and predator reduced and when most or almost whole proportion of primary toxicant was transformed into secondary toxicant, biomass density of predator and prey reduced further.

Thus, it can be concluded that, presence of toxicants affects the dynamics of the predator-prey model and it leads to severe depletion of species, if conversion rate of toxicant from primary to secondary is high. It can be said that when the transformation rate was high, secondary toxicant acts deleterious and shows detrimental effects on the populace. Therefore measures should be taken for controlling pollution and for preserving species.

Acknowledgment

No funding received.

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