

Dynamics of an Exploited Prey Predator Model Induced by Sigmoidal Functional Response in Strong Allee Effect

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ABSTRACT

This work deals with a prey-predator system with an Allee effect induced by two advanced functional response viz. sigmoidal and modified Leslie-Gower for predation and outlay respectively. Here both the communities are harvested by the combined harvesting effort E . Initially, we examined some basic properties like the positivity, boundedness, persistence and permanence of the system to validate the system in order to ensconce the long term survival of the species. It is observed that the system is asymptotically stable around some steady states under certain conditions and globally stable. Also the growth rate of prey should be restricted otherwise the system will be unstable through bifurcation. Taking E (harvesting effort) as control parameter, we have determine optimal harvesting policy. We have examine the Allee effects (L) leading to investigate about the sustainability of the system. Investigation reveals that the gradual increment of Allee lead the system unstable from a stable state and after a certain threshold level of L , the system may be ruined. Finally, the MSY analysis explores some new findings leading to obtain sustainable maximum yield.

Keywords: Prey-predator model, Sigmoidal functional response, Allee effect, Persistence, Bifurcation, Harvesting, MSY.

Mathematics Subject Classification: 92D40, 37G15, 91B76.

Journal of Economic Literature(JEL) Classification : C02, Q01, Q22, Q27, Q57.

1. INTRODUCTION AND MODEL FORMULATION

Biomathematics and Mathematical ecology are the important interdisciplinary research subjects for study among many young investigators [1-3, 7, 11]. These fields are actually the combination of two core science subjects Biology and Mathematics. A large number of researchers doing there work in the field of eco-epidemiological ratio dependent model or in the endemic susceptible infected model system. Some are doing also on different kinds of time delay model [7, 13, 19-25] where harvesting plays an important role [3, 4, 7, 13, 22, 25].

One of the significant biological phenomena is Allee effect. It is correlated between the size of population and the mean isolated fitness. The term was specially introduced after the work of Allee [5]. It is the relation between the density of population and per capita growth rate. Allee effect appears when size of populations is relatively small. Then the individual species suffers from mitigation in some countenance of its fitness. Allee effects are of two types in the nature: strong and weak. It is said to be strong when the per capita growth rate is negative. It is occurred when the densities of the population fall down a critical limit. In case of weak Allee effect, per capita growth rates are small and have no critical threshold to exceed. Therefore, the weak Allee effect is also termed as non-critical Allee effect. In a natural environment, Allee effect arises from the cooperation or facilitation among individuals in the species. This effect usually caused in the species by some mechanisms like the mate limitation, cooperative defence, habitat alteration, cooperative feeding, dispersal or an unsuitable environmental condition. Thus Allee effect is one of the important and interesting topics to study for the scientist which is yet required to investigate in different directions. Systems with such types of Allee effect are also investigated by several researchers like McCarthy [9], Scheuring [12], Rocha *et al.*, [15, 17, 18]. Fowler *et al.* [6] discussed a well known dynamic and isolated model with the presence of Allee effect, where stabilizing effect of the population dynamics with the effect of Allee had been rigorously studied. Jang [8] analyzed an Allee effected discrete time Beverton Holt stock model occurring with in the host population and investigated the asymptotic dynamics of the system. Zhou *et al.* [10] describes a model of prey-predator community interaction with Allee effects, and they experiences a destabilizing force on the system.

Das *et al.* [13] investigated a system of modified Leslie Gower type under combined harvesting effort. But they completely ignored the dynamics of non-delay system. We have considered the similar model excluding the time delay phenomenon. The ecological set up of our proposed model are almost same as proposed in Das *et al.* [13]. Assuming $x(t)$ and $y(t)$ are the prey and predator population any instant t , we compose our harvesting prey dynamics without the time delay (same as proposed model of Das *et al.* [13] in the following way:

$$\frac{dx}{dt} = rx \left(1 - \frac{x}{K} \right) \left(\frac{x}{L} - 1 \right) - \frac{\alpha x^2 y}{a^2 + x^2} - q_1 E x$$

Here $r (> 0)$ denotes prey's intrinsic growth rate and K represents the corresponding carrying capacity. Due to critical depensation [22, 23], the term $L (< K)$ is the population level below which the population goes to the extinction. This term arises due to the strong Allee effect and it restricts the natural growth of prey population. The sigmoidal functional response term $f(x, y) = \alpha x^2 y / (a^2 + x^2)$ have used in the prey dynamics because to consume the prey population by predator population. In this functional response term α and a are respectively denoted as the rate of predation and the half saturation type constant respectively. The dynamics of predator is described in following way:

$$\frac{dy}{dt} = sy \left(1 - \frac{\beta y}{x + M} \right) - q_2 E y$$

Here the predator population's growth is taken as modified Leslie Gower type and s prevails the intrinsic growth of predator. β represents the amount of prey required to support one predator at the equilibrium state. M denotes the alternative food supplement in absence of prey population or the residual loss of the predator species due to scarcity of prey species x . It is assumed that both the species are harvested with catchability coefficient q_1 and q_2 respectively, E denote the combined fishing effort.

Therefore, the two dimensional dynamical system becomes:

$$\left. \begin{aligned} \frac{dx}{dt} &= rx \left(1 - \frac{x}{K} \right) \left(\frac{x}{L} - 1 \right) - \frac{\alpha x^2 y}{a^2 + x^2} - q_1 E x \\ \frac{dy}{dt} &= sy \left(1 - \frac{\beta y}{x + M} \right) - q_2 E y \end{aligned} \right\} \quad (2.1)$$

It is also assume that for the feasibility of the system (2.1), it must be within the first quadrant and defined by the Cartesian product $\Phi = \{(x(t), y(t)) \in R^2 / x(t) \geq 0, y(t) \geq 0\} = R_0^+ \times R_0^+$

Das *et. al* [13] mainly concentrated on the simultaneous effect of multiple delays. They did not investigate the effect of Allee in details. There the functional responses were also different and they investigated optimal policies mainly, But we are interested to investigate the sensitivity of different parameters, especially parameter due to Allee effect, consumption rate, Effort etc. The strong Allee effect, weak Allee effect and the dynamics related to these effects in the system are also investigated by few researchers [14, 16, 17]. Pal *et. al* [14] discussed the qualitative analysis of double delay in prey.

Biological resources like Fish, Hen, Deer, Tortoise etc. are useful to the human beings for their economic and cultural importance. Fish is not used only for food, it has some great other ecological importance too. Due to exploitation through the unplanned way of harvesting, species are going to extinct day by day. Exhaustion of the resources can bring up an ecological imbalance too. So, we have to harvest the species maintaining a certain level of stock. To maintain or sustain the species, harvesting should be permissible up to a certain maximum level. Maximum sustainable yield (MSY) policy is introduced to manage such type of unplanned harvesting or over exploitation. For a single species fishery model, Schaefer [26] introduced MSY policy with logistic growth and proportional harvesting. We can find the classical theory of MSY or MSTY for a single species model or the interacting model in some work of Legovic *et. al.* [27]. For single species fishery, the MSY policy with strong Allee effect has investigated recently by Kar and Matsuda [23]. Ghosh *et al.* [28] established that the total maximum yield may be sustainable under equal harvesting effort for a multi-trophic harvested system. But is it possible to apply MSY policy every where? Can both the prey predator subsist in the maximum sustainable yield? Here we have tried to answer some of these questions in our proposed model both theoretically and numerically. Thus, in our paper we have discussed an exploited prey predator system under a combine harvesting effort where the prey development is affected by the presence of Allee.

2. SOME PRELIMINARY RESULTS

2.1 Positivity of the system

Here our intention is to examine the system (2.1), the solution trajectories of which are confined in a positive region. By the succession of two theorems we will establish this.

Theorem 2.1 Every trajectories of the entire system (2.1) are positive and belongs to the positive quadrant ($\text{int } R_+^2$).

(Proof is given in the Appendix – A.)

2.2 Persistence and extinction of the system

In biological systems, persistence apprehends whether a population will continue to subsist or will be eradicated. The boundedness of any system may be fulfilled by this property. It exhibit the stability of the dynamical system. Thus, the formulation of model becomes stronger through persistence. Following result gives the condition for the uniform strong persistence of the system.

Theorem 2.2 All the biomass of the system (2.1) has its limit supremum as well as limit infimum as $t \rightarrow +\infty$, i.e. $\limsup_{t \rightarrow +\infty} x(t) \leq K$, $\liminf_{t \rightarrow +\infty} x(t) \geq L$, $\limsup_{t \rightarrow +\infty} y(t) \leq \Omega(K + M)$ and $\liminf_{t \rightarrow +\infty} y(t) \geq \Omega(L + M)$

where $\Omega = (s - q_2 E) / (s\beta)$.

(Proof is given in the Appendix – B.)

Thus we see that the system is uniformly persistent as the populations are bounded with in some constant lower and upper bound.

3. EQUILIBRIA OF THE DYNAMICAL SYSTEM: ENTITY AND STABILITY

Here, we become curly to envisage all the feasible non negative equilibria of the two dimensional prey-predator system (2.1). In our proposed system (2.1) has the following three feasible non negative equilibria:

1) The trivial equilibrium at $E_0(0,0)$.

2) The predator free axial equilibrium points $E_1(x_1, 0)$ and $E_2(x_2, 0)$ where $x_1 = \frac{1}{2}(K + L + D_0)$,

$$x_2 = \frac{1}{2}(K + L - D_0) \quad \text{and} \quad D_0 = \left((K + L)^2 - 4 \left(KL + \frac{qEKL}{r} \right) \right)^{1/2} \quad \text{exists if}$$

$$E < \{r(K - L)^2\} / 4q_1KL.$$

3) The interior point of equilibrium $E_*(x^*, y^*)$ exists if the harvesting effort doesn't exceed the

$$\text{biotechnical productivity of predator (i.e.; } E < \frac{s}{q_2} \text{) where } y^* = \frac{(s - q_2 E)(x^* + M)}{s\beta}$$

(3.1)

and x^* satisfy the following equation

$$A_4 x^4 + A_3 x^3 + A_2 x^2 + A_1 x + A_0 = 0. \quad (3.2)$$

$$\text{with } A_4 = rs\beta, \quad A_3 = -rs\beta(K+L), \quad A_2 = rs\beta a^2 + rs\beta KL + \alpha KL(s - q_2 E) + q_1 EKLs\beta \\ A_1 = \alpha MKL(s - q_2 E) - rs\beta a^2(K+L), \quad A_0 = (rs\beta + q_1 E)a^2 K L s \beta.$$

Stability analysis around equilibria:

We verify stability of our system (2.1) around existing equilibria. The variational matrix at (x, y) is

$$J(x, y) = \begin{bmatrix} r\left(\frac{2x}{L} + \frac{2x}{K} - \frac{3x^2}{KL} - 1\right) - \frac{2\alpha a^2 xy}{(a^2 + x^2)^2} - q_1 E & -\frac{\alpha x^2}{a^2 + x^2} \\ \frac{s\beta y}{(x+M)^2} & s - \frac{2s\beta y}{x+M} - q_2 E \end{bmatrix} \quad (3.3)$$

The investigation for stability at the points specified above are summarised below.

Theorem 3.1. *The system is always unstable around the trivial equilibrium poin..*

(Proof is given in the appendix – C.)

Theorem 3.2. *A necessary condition that the predator free equilibrium point $E_1(x_1, 0)$ or $E_2(x_2, 0)$ is*

locally asymptotically stable if $\frac{1}{2}(K+L) < x_1$ or x_2 accordingly.

(Proof is given in the Appendix – D.)

Theorem 3.3. *The nontrivial steady state $E_*(x^*, y^*)$ is a stable node if*

$$r\left(\frac{2x}{L} + \frac{2x}{K} - \frac{3x^2}{KL} - 1\right) + \left(s - \frac{2s\beta y}{x+M} - q_2 E\right) < \frac{2\alpha a^2 xy}{(a^2 + x^2)^2} + q_1 E \quad \text{and} \\ r\left(\frac{2x}{L} + \frac{2x}{K} - \frac{3x^2}{KL} - 1\right) \left(s - \frac{2s\beta y}{x+M} - q_2 E\right) > \frac{\alpha \beta x^2 y^2}{(a^2 + x^2)(x+M)^2} + \left(s - \frac{2s\beta y}{x+M} - q_2 E\right) \left(\frac{2\alpha a^2 xy}{(a^2 + x^2)^2} + q_1 E\right)$$

(Proof is given in the Appendix – E.)

4. GLOBAL STABILITY ANALYSIS

Our interest in this section is to find out the sufficient condition of global asymptotic stability. Here we

check the global stability around the coexisting equilibrium point $E_*(x^*, y^*)$ of the system

(2.1) by generating a congruous Lyapunov function defined by:

$$V(x, y) = \frac{1}{\alpha} [(x - x^*) - x^* \log(x/x^*)] + \frac{1}{s\beta} [(y - y^*) - y^* \log(y/y^*)] \quad (4.1)$$

Differentiating with respect to time along the trajectories we get

$$\frac{dV}{dt} = \frac{1}{\alpha} (x - x^*) \left[r\left(1 - \frac{x}{K}\right) \left(\frac{x}{L} - 1\right) - \frac{\alpha xy}{a^2 + x^2} - q_1 E \right] + \frac{1}{s\beta} (y - y^*) \left[s\left(1 - \frac{\beta y}{x+M}\right) - q_2 E \right]$$

$$\begin{aligned} \frac{dV}{dt} &= - \left[\frac{a^2 y - x x^* y}{(a^2 + x^2)(a^2 + x^{*2})} + \frac{x + x^* - L - K}{KL} \right] (x - x^*)^2 \\ &\quad - \left[\frac{a^2 x^* + x x^{*2}}{(a^2 + x^2)(a^2 + x^{*2})} - \frac{y^*}{(x + M)(x^* + M)} \right] (x - x^*)(y - y^*) - \left[\frac{M}{(x + M)(x^* + M)} \right] (y - y^*)^2 \\ &= -X^T A X \end{aligned}$$

where $X^T = [(x - x^*) \ (y - y^*)]$ and

$$A = \begin{bmatrix} \frac{a^2 y - x x^* y^*}{R} + \left(\frac{x + x^* - L - K}{KL} \right) \frac{r}{\alpha} & \frac{1}{2} \left(\frac{a^2 x^* + x x^{*2}}{R} - \frac{y^*}{T} \right) \\ \frac{1}{2} \left(\frac{a^2 x^* + x x^{*2}}{R} - \frac{y^*}{T} \right) & \frac{1}{x + M} \end{bmatrix}$$

with $R = (a^2 + x^2)(a^2 + x^{*2})$ and $T = (x + M)(x^* + M)$.

The system is asymptotically stable in global sense in the vicinity of $E_*(x^*, y^*)$ if $dV/dt < 0$ i.e., if

A is positive definite. Thus the condition of global stability is

$$\frac{1}{(x + M)} \left[\frac{a^2 y - x x^* y^*}{R} + \frac{r}{\alpha} \left(\frac{x + x^* - L - K}{KL} \right) \right] > \frac{1}{4} \left(\frac{a^2 x^* + x x^{*2}}{R} - \frac{y^*}{T} \right)^2.$$

So we summarise the above analysis as below:

Theorem 4.1: The proposed system is asymptotically stable in global sense if the condition

$$\frac{1}{(x + M)} \left[\frac{a^2 y - x x^* y^*}{R} + \frac{r}{\alpha} \left(\frac{x + x^* - L - K}{KL} \right) \right] > \frac{1}{4} \left(\frac{a^2 x^* + x x^{*2}}{R} - \frac{y^*}{T} \right)^2 \text{ holds.}$$

5. OPTIMAL HARVESTING POLICY

In the modern era, one of the common issues is the exploitation of natural resource. In ecological view point, the human beings are using this in an unscientific way. Necessity of optimal harvesting policy is to determine the control strategies leading to manage the unscientific exploitation of natural resources. In this context, our main objective is to determine optimal harvesting policy for our proposed model system such that it maximizes the profit. Our emphasis is over the profit making healthy system. The details study is based on the revenue acquired by harvesting and a suitable optimal harvesting policy. Here the cost is taken as quadratic function. The main causes are it gives us to execute analytical disclosure for the optimal harvest.

Maximize the entire exemption seasoned share of the fishery is our principle goal. Here the present value symbolically denoted as J is given by

$$J(E) = \int_0^{\infty} \left((p_1 - v_1 q_1 E x) q_1 E x + (p_2 - v_2 q_2 E y) q_2 E y - cE \right) e^{-\delta t} dt \quad (5.1)$$

where δ denotes the fugacious perennial recess rate. $p_i, i = 1, 2, 3$ are represented the price per unit biomass of the corresponding species and $-v_i, i = 1, 2, 3$ denotes the economic constraints and c denoted the constant fishing cost per unit effort.

We use Pontryagin's Maximum Principle to solve the problem subjected to the system equation (2.1).

The control variable E is restricted in the region $0 \leq E \leq E_{\max}$. We consider E_{δ} as the optimal control where the state variables are x_{δ} and y_{δ} . Our goal is to derive E_{δ} for which

$J(E_{\delta}) = \max\{J(E) : E \in U\}$, where U is the control set defined by

$$U = \{E : [t_0, t_f] \rightarrow [0, 1] : mE \text{ is Lebesgue measurable}\}.$$

The Hamiltonian for the system is

$$H = (p_1 - v_1 q_1 E x) q_1 E x + (p_2 - v_2 q_2 E y) q_2 E y - cE + \lambda_1 \left[rx \left(1 - \frac{x}{K} \right) \left(\frac{x}{L} - 1 \right) - \frac{\alpha x^2 y}{a^2 + x^2} - q_1 E x \right] \\ + \lambda_2 \left[sy \left(1 - \frac{\beta y}{x + M} \right) - q_2 E y \right]$$

where the adjoint variables are denoted by $\lambda_1(t)$ and $\lambda_2(t)$.

Also the conditions of transversality are $\lambda_i(t_f) = 0, i = 1, 2$.

Now $\frac{\partial H}{\partial E} = p_1 q_1 x - 2v_1 q_1^2 E x^2 + p_2 q_2 y - 2v_2 q_2^2 E y^2 - c - \lambda_1 q_1 x - \lambda_2 q_2 y = 0$ at $E^*(t)$ gives

$$E^* = \frac{p_1 q_1 x^* + p_2 q_2 y^* - c - \lambda_1 q_1 x^* - \lambda_2 q_2 y^*}{2(v_1 q_1^2 x^{*2} + v_2 q_2^2 y^{*2})} \text{ and the corresponding adjoint equations are}$$

$$\frac{d\lambda_1}{dt} = \delta\lambda_1 - \frac{\partial H}{\partial x} = \delta\lambda_1 - \left[p_1 q_1 E - 2v_1 q_1^2 E^2 x + \lambda_1 \left(\frac{2rx}{L} + \frac{2rx}{K} - \frac{3rx^2}{KL} - r - \frac{2\alpha a^2 xy}{(a^2 + x^2)^2} - q_1 E \right) \right. \\ \left. + \lambda_2 \left(\frac{s\beta y^2}{(x + M)^2} \right) \right] \quad (5.2)$$

$$\frac{d\lambda_2}{dt} = \delta\lambda_2 - \frac{\partial H}{\partial y} = \delta\lambda_2 - \left[p_2 q_2 E - 2v_2 q_2^2 E^2 y + \lambda_1 \left(-\frac{\alpha x^2}{(a^2 + x^2)} \right) + \lambda_2 \left(s - \frac{2s\beta y}{x + M} - q_2 E \right) \right] \quad (5.3)$$

So the above analysis may be stated as follows:

Theorem 5.1 *There exist an optimal control E_{δ} with the associated solutions x_{δ} and y_{δ} which maximizes the present value function $J(E)$ over U . Even more, there exists the adjoint functions λ_1 and λ_2 are satisfying the upward system of differential equations (5.2)-(5.3) with transversality*

conditions $\lambda_i(t_f) = 0, i = 1, 2, 3$. Also the optimal control is given by:

$$E^* = \frac{p_1 q_1 x^* + p_2 q_2 y^* - c - \lambda_1 q_1 x^* - \lambda_2 q_2 y^*}{2(v_1 q_1^2 x^{*2} + v_2 q_2^2 y^{*2})}.$$

6. ANALYSIS OF MSY AND ITS IMPACT

In ecological modelling, the term “Maximum sustainable Yield (MSY)” plays an important role in the dynamics of the system. At first we have considered the system without harvesting of prey (i.e. only predator harvested) and analyzed leading to obtain MSY. Then we have considered the system (2.1) where both the species are harvested. Finally, we have analyzed the system in absence of Allee effect.

The only predator harvesting model of our the system will be

$$\left. \begin{aligned} \frac{dx}{dt} &= rx \left(1 - \frac{x}{K} \right) \left(\frac{x}{L} - 1 \right) - \frac{\alpha x^2 y}{a^2 + x^2} \\ \frac{dy}{dt} &= sy \left(1 - \frac{\beta y}{x + M} \right) - q_2 E y \end{aligned} \right\} \quad (6.1)$$

If possible, let the equilibrium point of system (6.1) be $P_1^*(x_1^*, y_1^*)$.

$$\text{Here } y_1^* = \frac{(s - q_2 E)(x_1^* + M)}{s\beta} \quad (6.2)$$

which exist only if the effort level of predator population is less than the bio-technical productivity of predator population i.e. $E < \frac{s}{q_2}$ and x_1^* satisfy the following equation

$$A'_4 x^4 + A'_3 x^3 + A'_2 x^2 + A'_1 x + A'_0 = 0 \quad (6.3)$$

where $A'_4 = rs\beta$, $A'_3 = -rs\beta(K + L)$, $A'_2 = rs\beta a^2 + rs\beta KL + \alpha KL(s - q_2 E)$,

$$A'_1 = \alpha MKL(s - q_2 E) - rs\beta a^2(K + L), \quad A'_0 = a^2 s^2 \beta^2 KL.$$

If we differentiate the y_1^* with respect to E , then we get

$$\frac{dy_1^*}{dE} = -\frac{q_2(x_1^* + M)}{s\beta} + \frac{(s - q_2 E)}{s\beta} \frac{dx_1^*}{dE}. \quad (6.4)$$

According to Ghosh *et. al.* [28], in an exploited system prey biomass decreases in presence of Allee

effect (i.e. $\frac{dx^*}{dE} < 0$). Thus, it is clear from (6.4) that if the effort increases then predator population

decreases. Here the yield is $Y_y(E) = \frac{q_1 E(s - q_2 E)(x_1^* + M)}{s\beta}$. As the expression $q_1 E(s - q_2 E)$ is a

quadratic function of E , so $Y_y(E)$ must have at least one maximum value in the effort interval

$\left(0, \frac{s}{q_2}\right)$. The remaining part of the expression of $Y_y(E)$ will enhance the yield as effort increases.

Therefore, MSY exist only when predator species is harvested. But when we consider the case for combined harvesting for both the species, the result indicates a different reflection. The coordinate for the co-existence equilibria $P^*(x^*, y^*)$ is already given by equations (3.1) and (3.2) in the section 3. The analytical findings is same like the previous one. We analyze the model under the above situations.

7. NUMERICAL SIMULATION AND RESULT DISCUSSION

For the purpose of numerical simulations to validate the theoretical results, we need to have some real word observations. But it is very difficult to obtain such numerical values of different parameters of the system on real-world observations. The qualitative simulations can be described the major results. In our paper we have considered the values of a number of parameters leading to analyze our Allee model and we observe that some parameters are really sensitive based on the hypothetical data set. Thus, only to solve the system numerically, we have analyzed the system with the parameters set $r = 1.5, \alpha = 0.6, K = 100, a = 5, L = 2.3, s = 0.9, \beta = 0.4, M = 0.05, q_1 = 0.05, q_2 = 0.03, E = 0.2$.

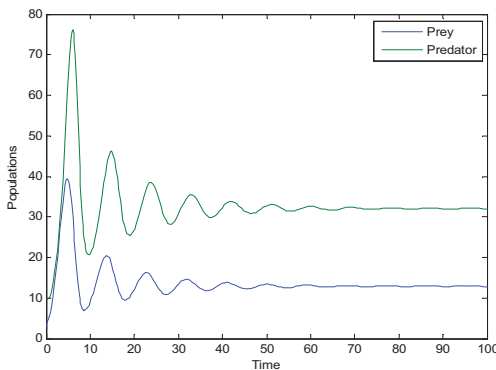


Figure 1: Solution trajectories of the system (2.1) without Allee effect.

In the Figure 1 we have depicted the time series plot without the Allee effect. It is clearly observed that when there is no effect of Allee, the species initially fluctuates and then becomes stable and sustain safely. On the other hand, when Allee is employed, it is noticed that the prey population increases and predator population decrease with time as shown in Figure 2. It is also very clear to us from the model, the parameter L and r imposes opposite effects on the dynamics of the system. Figure 2, shows that prey and predator decreases with the increase of L which is quiet natural. In strong Allee effect, there will be a threshold population level and if the population size is less than that threshold level (i.e. $0 < x < L$), the system will become ruin. Here we observed that, the species coexists and remains stable up to a certain limit of L as shown in Figure 3(a) for $L = 0.1$ and in Figure

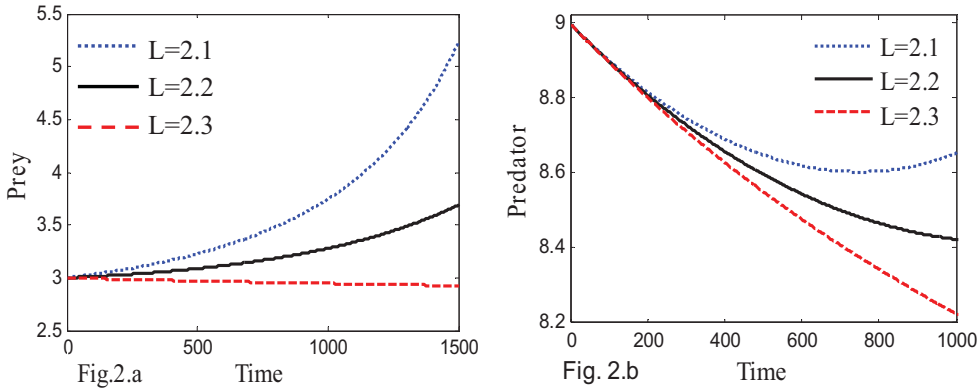
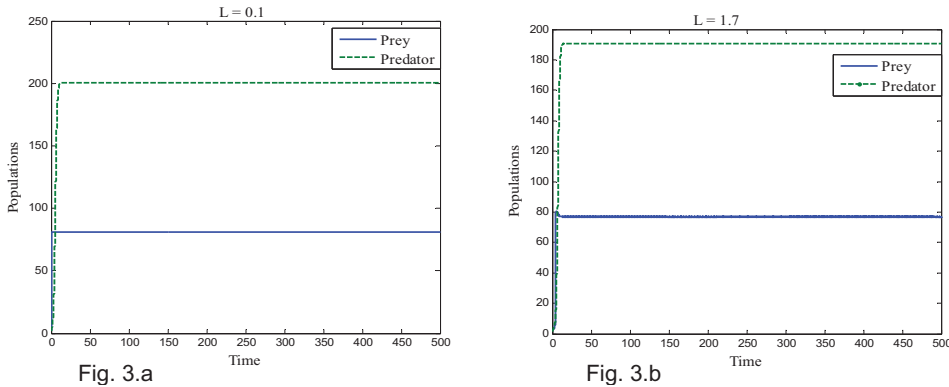


Figure 2: Change of the prey and predator population with time and Allee.


 Figure 3: Curve of prey and predator population with respect to time with Allee effect $L=0.1$ and $L=1.7$.

3(b) for $L=1.7$ respectively. This is the case of weak Allee effect (i.e. $x > L$) when population size is above the threshold level of L . But we further noticed that if the Allee effect increases, threshold level becomes higher than the population size and species are going to zero level as revealed in the Figure 4(a) and 4(b), i.e., as the threshold level (L) increases, species are going to extinct. Thus, there will be a critical or threshold value of L ($L^*=1.80550$ (approx)) with respect to the considered set of parameters, below which the system sustain and stable, and above which the system do not run after certain time. Thus, we should consider the Allee effect in the system up to a certain limit (L^*) for sustainability of the system. This does not mean that, Allee effect free system is ideal. We theoretically proved that the Allee effect in our system is upper bounded by the existing species, i.e., $L < x, \forall t$.

In Figure 5, we see that the prey population are increasing but predator populations are decreasing with time. It is also observed that both the species are increasing with respect to intrinsic growth rate r . Intrinsic growth rate of prey plays a vital role on the dynamical interaction between the species. From our theoretical observations we can say that prey population is directly proportional to r and hence it is obvious that the prey population will increase with the increase of r . In abundance of prey,

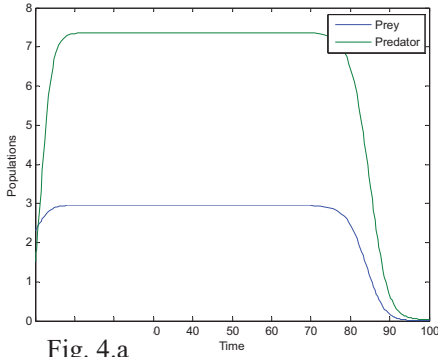


Fig. 4.a

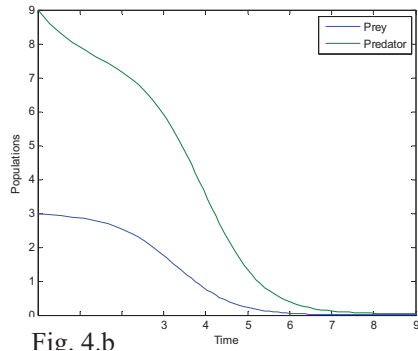


Fig. 4.b

Figure 4: Curve of different populations with respect to time when Allee effect $L = 1.80550$ and $L=2.3$ respectively.

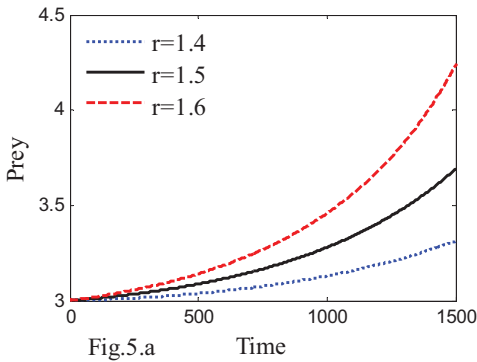


Fig.5.a

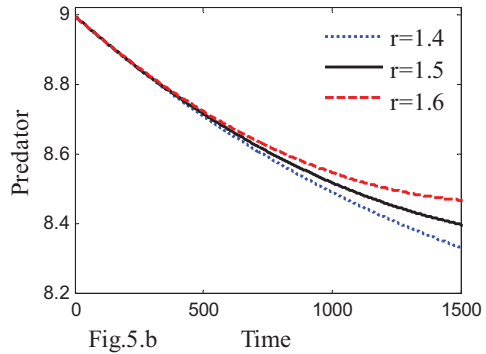


Fig.5.b

Figure 5: Change of the prey (5.a) and predator (5.b) populations with time for different values of r .

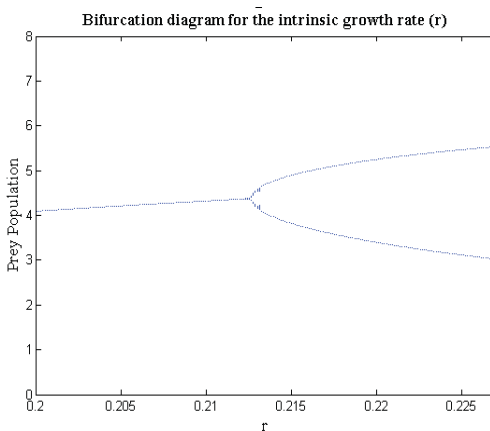


Figure 6: Bifurcation diagram of prey for the intrinsic growth rate r .

predator consumes more prey and increases them also with the increase of r . We have also examined numerically the bifurcation phenomenon for r . In Figure 6, we see that as r increases, the prey population becomes stable up to a certain value of r and then becomes split in two differ-

nt directions simultaneously, i.e. bifurcated at $r^* = 2.14$ (approx). So the intrinsic growth rate should be kept under a critical value (r^*), called bifurcation value of r^* .

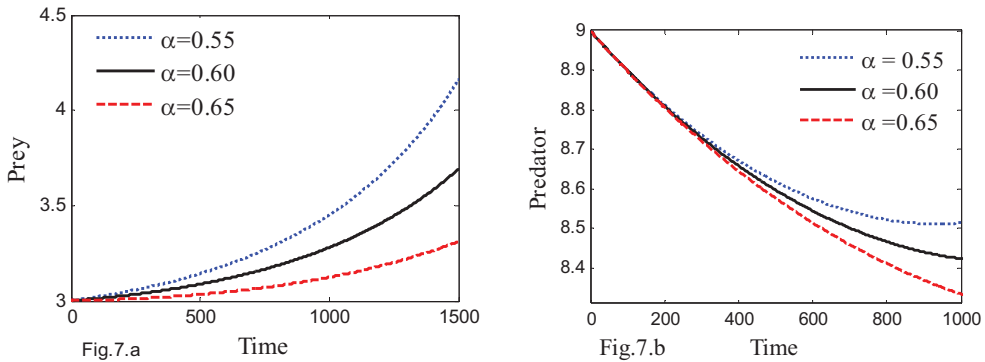


Fig.7. Change of prey and predator populations with time for $\alpha = 0.55$ (Blue), $\alpha = 0.60$ (Black), and $\alpha = 0.65$ (Red).

Figure 7 depicts the variation of prey and predator populations with respect time and the maximal per capita consumption rate α . It is shown in the Figures 7.a and 7.b that the prey and predator are decreasing with the increase of α . Also the rate of increase of prey with time decreases with the increase of α and that increases in case of predator as shown in the Figures 7.a and 7.b respectively.

We also observe that effect of α on predator is not significant initially, but as time elapsed this effect becomes significant gradually. It is very simple that predation by the predator reduce prey and increase the predator. Naturally, increasing rate of prey or decreasing rate of predator with respect to time will reduce. But the increase of predator biomass for the consumption of prey requires some time which is called consumption delay, and for that delay, predator's noticeably

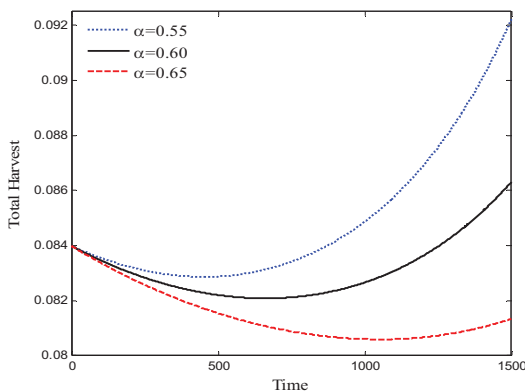


Figure 8: Variation of the Total Harvest with time for $\alpha = 0.55$ (Blue), $\alpha = 0.60$ (Black), and $\alpha = 0.65$ (Red).

increases with time. Though, this delay has not incorporated in our model, but our observations reveal the existence of such delay and that part retains for further researchers. When the populations are decreasing, the Total Harvest ($TH = q_1Ex + q_2Ey$) should be decreasing and that is depicted in

Figure 8. This indicates that the net revenues from the fishery will be downwards and for the sake of monetary social benefit, we try to arrange the system with minimum predation rate.

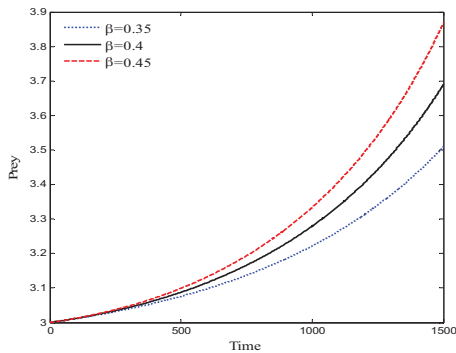


Fig 9.a

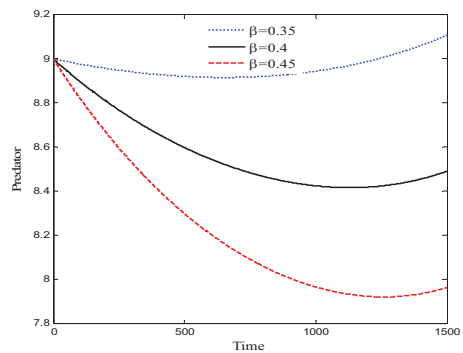


Fig. 9.b

Figure 9: Change of the prey and predator population with time for $\beta = 0.4$ (Black solid line), $\beta = 0.35$ (Blue dotted line) and $\beta = 0.45$ (Red dashed line).

Figure 9 reveals the change of prey and predator populations with time and the predators' intra specific competition rate (β). It is shown in the Figures 9.a that the prey is increasing with β and time, and in Figure 9.b the predator is decreasing with time and β . It is observed that the rate of increase of prey with time increases with the increase of β (Fig 9.a) and the rate of decrease of predator with time decreases with the increase of β significantly (Figure.9.b). High competition among predators for food naturally reduces predator directly (Figure 9.b). Reduction of predator reduces the consumption of prey which in turn increase prey population. It is interesting to observe

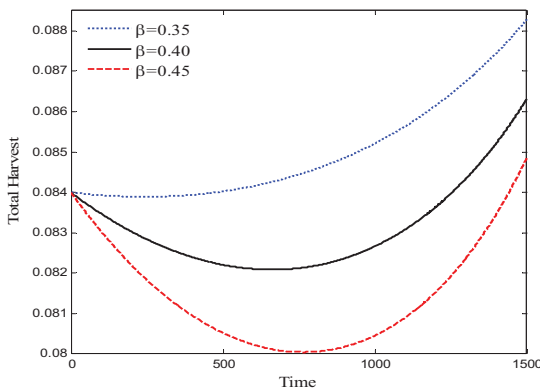


Figure 10: Variation of the Total harvest with time. The solid line corresponds to $\beta = 0.4$ and the dotted line and the dashed line corresponds to $\beta = 0.35$ and $\beta = 0.45$ respectively.

in Figure 10 that the Total Harvest (TH) decreases with the increase of β . This should reflect on the net revenues from the fishery. Thus to get monetary social benefit in parallel to the sustainability of the system, competition rate among predations should be minimum.

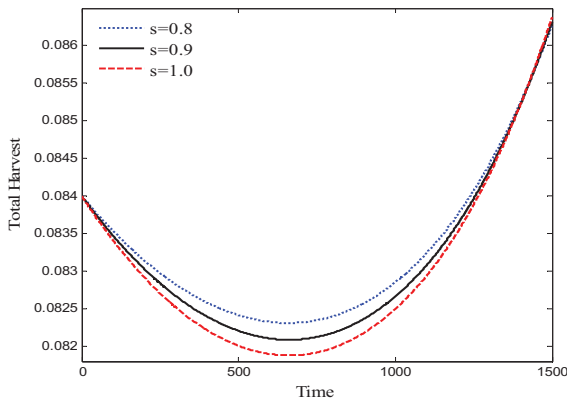


Figure 11: Variation of the Total harvest with Time for $s = 0.8$ (Blue dotted line), $s = 0.9$ (Black solid line) and $s = 1.0$ (Red dashed line).

In Figure 11, we see that TH (i.e., Total Harvest) decreases initially with time but after a certain time period it increases gradually. Another phenomenon observed that the increase of the growth rate (s) of predator reduce TH. This is interesting because s increases predator but TH is not increased. All the previous figures display that the prey populations are increasing and predator populations are decreasing normally with time. But the increasing rate of prey is more than the decreasing rate of predator. So the TH finally increases with time. But with the increase of s predator increases. Naturally, competition among predator become high. Not only that, consumption of prey increases. As a result the Total Harvest decreases.

Now to analyze MSY, let us take the following set of numerical data as: $r = 2.5, K = 80, \alpha = 0.5, a = 5, L = 2.3, s = 0.9, \beta = 0.4, M = 0.5, q_1 = 0.795, q_2 = 0.595$, in appropriate units. When only predator is harvested as formulated in system 6.1, MSY occurs at $E_{MSY} = 0.76$ as shown in Figure 12 with equilibrium stock $(80.0, 97.9)$ and $MSY = 46.25$.

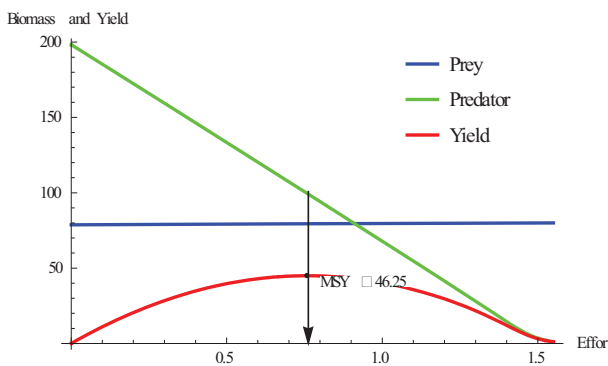


Figure 12: MSY exist in case of only predator harvesting system and the corresponding stock is $(80.0, 97.9)$.

Further it is noticed that when effort crosses 1.512, the predator exhausted. The result matched with our theoretical findings because for these values of s, q_2 and E , $s - q_2 E = 0$ that implies $y_1^* = 0$. There also prey increases with very small rate.

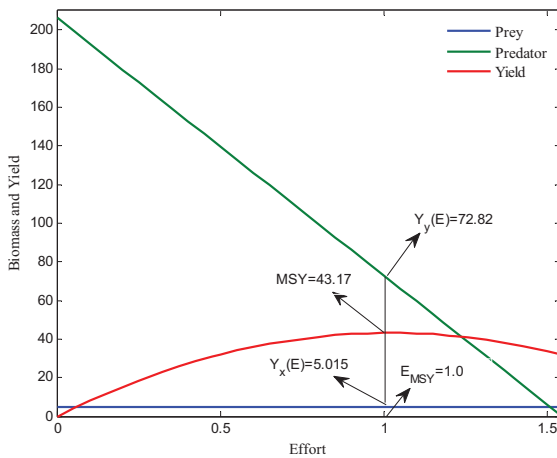


Figure13: The graphical presentation of biomass and Yield with respect to the effort when Allee is employed.

In our present model system (2.1) where Allee is employed, we see from the Figure 13 that the predator biomass exists at MSY level but not prey. Therefore combined harvesting effect for both the species is not possible for sustain the stock.

When we simulate our model without the effect of Allee, both the prey and predator biomass exists at MSY level. From the Figure 14 it is very clear that MSY occurs at $E_{MSY} = 1.18$ with

$x_{MSY} = 40.77$, $y_{MSY} = 21.95$. So, observations explores that the combined harvesting effort for both the species can be employed to sustain the stock level when the Allee effect is absent in the system, otherwise only predator have to harvest.

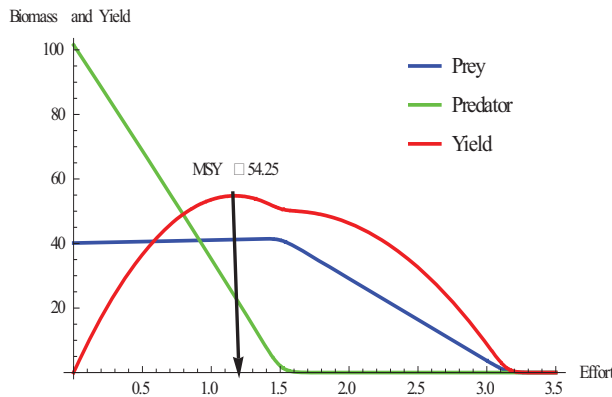


Figure 14: In absence of Allee effect the exploited system MSY exist and the corresponding stock is $(40.77, 21.95)$.

8. CONCLUDING REMARKS

The Allee effect is an exciting contemporary research experience and exposes a better understanding in real world ecological phenomenon [29, 30]. In our research work, a prey predator exploited model under a combine harvesting effort is investigated where the development of prey is influenced by the Allee effect. It is observed that the size of the population is bounded by the initial data independent

bounds after sufficiently long time. So, the structured population is uniformly (strong) persistent. Stability of different equilibrium points are discussed through the analysis of the nature of the roots of the corresponding characteristic equation. It is established that the trivial equilibrium is always unstable. The predator free equilibrium and interior equilibrium are locally asymptotically stable under certain conditions. Moreover, the long run sustainability of the system is ensured by the global stability of system, the conditions of which is derived at coexistence equilibria. All these investigations are general analysis and the results are almost similar with the results obtained by other works [3,12]. Our intention is to investigate the Allee effect (L) on this ecological system. It is well accepted that there is no real ecological system which is free from the effect of Allee. It is observed that the system is stable without Allee effect and when this effect is small, the system will be stable under certain parametric condition. Further increment of Allee decreases the densities of both the species and the system deviates from stability with the gradual increment of the Allee causing extinction of the system. Hence there is an admissible or critical level of Allee (L^*) in the system to sustain both species with stability for the considered set of parameters which is called the threshold level and we have obtained this threshold value $L^*=1.8055$ for the above set of parameters. That threshold level is required to determine the stability of the system. For long term sustainability of any system like ours, Allee effect should be less than L^* but population size should be always greater than that threshold value of L i.e. L^* , which is the case of weak Allee effect, otherwise any system may loose its stability and sustainability.

We have also examined the effect of some parameters like r, α, β, E etc. on the dynamical behaviour of the system. The increasing trend of the growth rate (r) of prey increase the prey and decrease the predator population up to a certain critical limit ($r^* = 2.14$), below which the system is stable. But when r becomes higher than r^* the prey population bifurcates from stable solution and the system become unstable. So the intrinsic growth rate should be kept under a critical bifurcation value of r . The effect of consumption parameter (α) and predation parameter (β) on the dynamics of the system are almost similar. For the increment α , both the species decreases and when β increases, predator decreases but prey increases. In both the cases total harvest (TH) decreases. So the alternative food may be employed in such type of ecological system to control predation leading to obtain constant rate of harvesting. Another interesting phenomenon observed that the intrinsic growth rate of predator though increase themselves but the total harvest decreases. So the predator's growth should be restricted or under control to get maximum sustainable yield. All these results are useful and required to maintain for the greater beneficiary of different ecological system like fishery management in Marine reserve, Wildlife reservation management etc. The MSY analysis explores that the combined harvesting effort for both the species can be employed to sustain the stock level when the Allee effect is absent in the system. But the exploited system in presence of the Allee effect where both the species are harvested can not reaches MSY level with sustainable stock of both species. Further, to attain the MSY level in such system, only predator species have to harvest.

However, the whole analysis discussed above are based on the deterministic framework. But, under frequent ecological fluctuations, if the model be considered in a stochastic environment, it will be more realistic. The development of further research related to this work may be in this direction .

9. Acknowledgement

Mr. Haldar gratefully acknowledges Dr. Sima Banerjee, Principal, Hooghly Women's College for her unconditional help. This research work is not funded by any funding agency.

10. APPENDIX

10.1. Appendix – A (Proof of Theorem 2.1)

From the first set of equation (2.1) we get,

$$\begin{aligned}\frac{dx}{dt} &= x(t) \left[r \left(1 - \frac{x}{K} \right) \left(\frac{x}{L} - 1 \right) - \frac{\alpha x y}{a^2 + x^2} - q_1 E \right] \\ \text{or, } \frac{dx}{x} &= \left[r \left(1 - \frac{x}{K} \right) \left(\frac{x}{L} - 1 \right) - \frac{\alpha x y}{a^2 + x^2} - q_1 E \right] dt \\ &= \phi(x, y) dt, \text{ where } \phi(x, y) = \left[r \left(1 - \frac{x}{K} \right) \left(\frac{x}{L} - 1 \right) - \frac{\alpha x y}{a^2 + x^2} - q_1 E \right].\end{aligned}$$

Integrating over the region $[0, t]$, we can write $x(t) = x(0)e^{\int_0^t \phi(x, y) dt} > 0, \forall t$ where $x(0) > 0, y(0) > 0$.

Again, from the second equation of the system we can write $y(t) = y(0)e^{\int_0^t \xi(x, y) dt} > 0, \forall t$

where $\xi(x, y) = \left(s \left(1 - \frac{\beta y}{x + M} \right) - q_2 E \right)$ and $x(0) > 0, y(0) > 0$.

Therefore all the solutions of system (2.1) are positive.

10.2. Appendix – B (Proof of Theorem 2.2)

From the first equation of the system (2.1), we have

$$rx \left(1 - \frac{x}{K} \right) \left(\frac{x}{L} - 1 \right) > 0 \Rightarrow \left(1 - \frac{x}{K} \right) > 0 \text{ and } \left(\frac{x}{L} - 1 \right) > 0.$$

Therefore, $L < x < K$. Hence, $\limsup x = K$ and $\liminf x = L$. (3.1)

Again, from the second equation of the system (2.1), we can write

$$\dot{y}(t) \leq y(t) \left(s - q_2 E - \frac{s\beta}{(K + M)} y(t) \right) \text{ and } \dot{y}(t) \geq y(t) \left(s - q_2 E - \frac{s\beta}{(L + M)} y(t) \right).$$

From the lemma (4.1), We have,

$$\limsup_{t \rightarrow +\infty} y(t) = \frac{(s - q_2 E)(K + M)}{s\beta} \text{ and } \liminf_{t \rightarrow +\infty} y(t) = \frac{(s - q_2 E)(L + M)}{s\beta}.$$

Therefore, we can write $\limsup_{t \rightarrow +\infty} x(t) \leq K$, $\liminf_{t \rightarrow +\infty} x(t) \geq L$, $\limsup_{t \rightarrow +\infty} y(t) \leq M_1$ and $\liminf_{t \rightarrow +\infty} y(t) \geq N_1$.

Hence the require result is done.

10.3 Appendix – C (Proof of Theorem 3.1)

At the point E_0 , the Jacobian matrix of the system (2.1) is, $J|_{E_0} = \begin{pmatrix} -r - q_1 E & 0 \\ 0 & s - q_2 E \end{pmatrix}$.

Therefore, the eigenvalues of $J|_{E_0}$ are $\lambda_1 = -(r + q_1 E) < 0$ and $\lambda_2 = s - q_2 E$. We now that the system will be stable if both the eigen values are negative. Here $r + q_1 E > 0$, so λ_1 is negative. We know that BTP of any population should be always greater than the corresponding harvesting effort, i.e. $s/q_1 > E$. Therefore, $s - q_2 E > 0$, i.e. $\lambda_2 > 0$. Thus the system is not stable at E_0 . So we conclude that E_0 is always unstable. Hence the result.

10.4 Appendix – D (Proof of Theorem 3.2)

The Jacobian of the system (2.1) evaluated at $E_1(x_1, 0)$ or $E_2(x_2, 0)$ is

$$J|_{E_1 \text{ or } E_2} = \begin{pmatrix} -\frac{2rx'^2}{KL} + rx' \left(\frac{1}{K} + \frac{1}{L} \right) & -\frac{\alpha x'^2}{a^2 + x'^2} \\ 0 & 0 \end{pmatrix} \text{ where } x' \text{ is equal to } x_1 \text{ or } x_2$$

corresponding to $E_1(x_1, 0)$ or $E_2(x_2, 0)$.

Clearly one eigen value of the system is zero and the other eigen value is

$\lambda_2 = -\frac{2rx'^2}{KL} + rx' \left(\frac{1}{K} + \frac{1}{L} \right)$. So the equilibrium points $E_1(x_1, 0)$ or $E_2(x_2, 0)$ is locally

asymptotically stable if the second eigen value is negative. i.e.; $-\frac{2rx'^2}{KL} + rx' \left(\frac{1}{K} + \frac{1}{L} \right) < 0$. By a

simple algebraic manipulation, we can write $\frac{1}{2}(K + L) < x'$. Hence the desire result is done.

10.5 Appendix – E (Proof of Theorem 3.3)

The eigen values of the matrix $J|_{E_*}$ are the roots $\lambda_i (i = 1, 2)$ of the equation

$$\lambda^2 - \lambda \left[r \left(\frac{2x}{L} + \frac{2x}{K} - \frac{3x^2}{KL} - 1 \right) - \frac{\alpha x^2 y}{(a^2 + x^2)^2} - q_1 E + \left(s - \frac{2s\beta y}{x + M} - q_2 E \right) \right] + \left[\left\{ r \left(\frac{2x}{L} + \frac{2x}{K} - \frac{3x^2}{KL} - 1 \right) - \frac{\alpha x^2 y}{(a^2 + x^2)^2} - q_1 E \right\} \left(s - \frac{2s\beta y}{x + M} - q_2 E \right) - \frac{\alpha \beta s x^2 y^2}{(a^2 + x^2)(x + M)^2} \right] = 0 \quad (4.2)$$

In the above equation,

$$\lambda_1 + \lambda_2 = r \left(\frac{2x}{L} + \frac{2x}{K} - \frac{3x^2}{KL} - 1 \right) - \frac{\alpha x^2 y}{(a^2 + x^2)^2} - q_1 E + \left(s - \frac{2s\beta y}{x + M} - q_2 E \right),$$

$$\lambda_1 \lambda_2 = \left\{ r \left(\frac{2x}{L} + \frac{2x}{K} - \frac{3x^2}{KL} - 1 \right) - \frac{\alpha x^2 y}{(a^2 + x^2)^2} - q_1 E \right\} \left(s - \frac{2s\beta y}{x + M} - q_2 E \right) - \frac{\alpha \beta s x^2 y^2}{(a^2 + x^2)(x + M)^2}.$$

Assuming that (x^*, y^*) is a stable node, we must have $\lambda_1 + \lambda_2 < 0$ and $\lambda_1 \lambda_2 > 0$.

This requires that the conditions

$$r \left(\frac{2x}{L} + \frac{2x}{K} - \frac{3x^2}{KL} - 1 \right) + \left(s - \frac{2s\beta y}{x + M} - q_2 E \right) < \frac{\alpha x^2 y}{(a^2 + x^2)^2} + q_1 E \quad \text{and}$$

$$r \left(\frac{2x}{L} + \frac{2x}{K} - \frac{3x^2}{KL} - 1 \right) \left(s - \frac{2s\beta y}{x + M} - q_2 E \right) > \frac{\alpha \beta s x^2 y^2}{(a^2 + x^2)(x + M)^2} + \left(s - \frac{2s\beta y}{x + M} - q_2 E \right) \left(\frac{\alpha x^2 y}{(a^2 + x^2)^2} + q_1 E \right)$$

hold simultaneously. Hence the theorem is proved.

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