

On Integer Solutions to Exponential Diophantine Equations Arising from Centered Tetradeccagonal and Pronic Bases

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ABSTRACT

This study investigates two forms of exponential Diophantine equations that are built upon special numerical structures: centered tetradeccagonal numbers and pronic numbers. The primary focus lies in determining the existence of integer solutions for each equation. It is shown that for one of the equations, there exists a unique solution in the set of whole numbers for all values of the parameter greater than one. In contrast, the second equation exhibits a consistent pattern of solutions across all positive integers. Additionally, several particular instances are examined to further support and illustrate the general results.

Keywords: Mihăilescu Theorem, Exponential Diophantine equation, Integer solutions, Centered tetradeccagonal number, Pronic number.

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1 Introduction

Diophantine equations are integral to advancing cryptography and ensuring network security. An exponential Diophantine equations contains the unknown exponents with the known or unknown bases. Jesmanowicz and Terai determined a conjecture which is connected with the uniqueness of the solution of

$$k^e + m^f = l^z \quad (1.1)$$

where $\gcd(k, m, l) = 1$ and $k, m, l \in \mathbb{Z}_+$ under some constraints. Jesmanowicz (Jesmanowicz, 2013) also discovered that if k, m, l are Pythagoren triplets, then $(e, f, z) = (2, 2, 2)$ is the only solution of (1.1). Further, Terai demonstrated that if $p, q, r, l, m, n \in \mathbb{Z}_+$ are fixed then the equation $p^l + q^m = r^n$ with $p, q, r, l, m, n \geq 2$, then (e, f, z) of (1.1) takes the unique values

(l, m, n) . In (Acu, 2007), D.Acu established the solution $(x, y, z) \in \{(3, 0, 3), (2, 1, 3)\}$, for the Diophantine equation $2^x + 5^y = z^2$. The existence of solution of the equation $2^x + p^y = z^2$ is established for prime p in (Suvarnamani, 2011) and (Sroysang, 2013) is a special case of (Sroysang, 2013). Tangjai and Chubthaisong (W. Tangjai, 2021) proposed the solution for $3^x + p^y = z^2$ where $p \equiv 2 \pmod{3}$. In addition, the explicit solution of the Diophantine equation $(4p^2 + 1)^x + (5p^2 - 1)^y = (3p)^z$ is established by Bertok in (C, 2016) where as earlier this equation was analysed for some values of p by Terai.

W. Orosram and C. Comemuang of (Orosram and Comemuang, 2020) dealt with the equation $8^x + n^y = z^2$. Two different classes of exponential Diophantine equations are considered in (Borah and Dutta, 2022). In (N. Viriyapong, 2022), it is found that there is a unique solution for $s^x + 19^y = z^2$ where $s \equiv 2 \pmod{57}$ and $s + 1 \neq z^2$ for any $z \in \mathbb{Z}$ whereas in (N. Viriyapong, 2021), the authors found the solution of $s^x + 13^y = z^2$ with respect to the condition $s \equiv 2 \pmod{57}$ and the same authors discussed the non-existence of solution in $\mathbb{W}$ of the equation $147^x + 741^y = z^2$ (N. Viriyapong, 2024). Murat Alan (Alan, 2020) studied that the Diophantine equation $(n^2 + n + 1)^x + n^y = (n + 1)^z$ has no solution in $\mathbb{W}$ except $(1, 1, 2)$ for $n \notin \{1, 2, 3\}$. The integer solutions, restricted to positive values, of the Diophantine equation

$$(a(a - l)m^2 + 1)^x + (alm^2 - 1)^y = (am)^z$$

is analysed in (Jinyan He, 2022), in which $a, l, m \in \mathbb{Z}_+$ with $a \equiv_2 1$ and $am^2 \equiv -2 \pmod{p}$, where p is the prime factor of l . It is also observed that this equation has only $(1, 1, 2)$ as the positive integer solution. In order to prove this, the authors used elementary methods, Bilu-Hanrot-Voutier Theorem on primitive divisors of Lehmer numbers and few results based on generalized Ramanujan-Nagell equations.

Various research works have been carried out in detecting the solution of many exponential Diophantine equations. Still, there are many unsolved problems. In this paper, we solved an exponential equation with bases as centered tetradecagonal number $(7q^2 + 7q + 1)$ and four times of Pronic number $(q^2 + q)$ of rank q . We have analysed the equation for different values of q and also established some special cases where there are two solutions exists for the equation $(C_{14,q})^r + (4Pr_q)^m = n^2$ in case of $q = 1$.

2 Preliminaries

Notations:

- $C_{14,q} = 7q^2 + 7q + 1$ – Centered tetradecagonal number of rank q with 14 sides
- $Pr_q = q(q + 1)$ – Pronic number of rank q

In this paper, $c \equiv_n d$ implies that $c \equiv d \pmod{n}$ which means c is congruent to d modulo n where $c, d, n \in \mathbb{Z}$ such that $n \geq 1$.

Now, Consider the following conjecture proposed by E.Catalan in 1844. This can be found in (Catalan, 1844) and later the proof of this conjecture is found by Mihailescu (Mihailescu, 2004).

Proposition 2.1. (Mihăilescu Theorem) *The Diophantine equation $b^a - c^d = 1$ has the unique solution $(b, c, a, d) = (3, 2, 2, 3)$ where $b, c, a, d \in \mathbb{Z}$ with $\min\{b, c, a, d\} > 1$.*

The following propositions are found from (Ivan Niven, n.d.)

Proposition 2.2. *n is odd if and only if $n^2 \equiv 1 \pmod{4}$*

Proposition 2.3. *n is even if and only if $n^2 \equiv 0 \pmod{4}$*

3 Main Results

Theorem 3.1. *The Diophantine equation $(C_{14,q})^r + (4Pr_q)^m = n^2$ has a unique non-negative integer solution $(r, m, n) = (0, 1, 2q + 1)$ for each positive integer $q > 1$.*

Proof. For $q \in \mathbb{Z}_+$ and $q > 1$, Re-frame the given equation as

$$(7q^2 + 7q + 1)^r + (4(q^2 + q))^m = n^2 \quad (3.1)$$

According to the proposition 1, (3.1) has no solution if $m = 0$. Hence $m \geq 1$. Since $(7q^2 + 7q + 1)^r$ is odd and $(4(q^2 + q))^m$ is even, it is obvious that n is odd. Let $n = 2s + 1$. Then (3.1) becomes

$$(7q^2 + 7q + 1)^r + (4(q^2 + q))^m = 4(s^2 + s) + 1 \quad (3.2)$$

Also, since n is odd, $n^2 \equiv_4 1$ by proposition 2.

Hence from (3.2), there may arise two cases:

Case A: For Some $q \in \mathbb{Z}_+$, $(7q^2 + 7q + 1)^r \equiv 1 \pmod{4}$ for all even and odd values of r .

Case B: For Some $q \in \mathbb{Z}_+$,

$$(7q^2 + 7q + 1)^r \equiv \begin{cases} 1 \pmod{4}, & \text{if } r \in \mathbb{Z}_+ \text{ is even number} \\ 3 \pmod{4}, & \text{if } r \in \mathbb{Z}_+ \text{ is an odd number} \end{cases}$$

Claim : $r = 2\mu$ for all $\mu \in \mathbb{Z}_+$.

Suppose that $r \in \mathbb{W}$ is odd. Then $r = 2\mu + 1$ for any $\mu \in \mathbb{Z}_+$. Thus, (3.1) becomes,

$$\begin{aligned} (7q^2 + 7q + 1)^{2\mu+1} + (4(q^2 + q))^m &= n^2 \\ \Rightarrow (4(q^2 + q))^m &= n^2 - [(7q^2 + 7q + 1)^{\frac{\mu}{2}+1}]^2 \\ (2q)^m(2q + 2)^m &= (n - (7q^2 + 7q + 1)^{\frac{\mu}{2}+1})(n + (7q^2 + 7q + 1)^{\frac{\mu}{2}+1}) \end{aligned} \quad (3.3)$$

Consider $n - (7q^2 + 7q + 1)^{\frac{\mu}{2}+1} = (2q)^m$ and $n + (7q^2 + 7q + 1)^{\frac{\mu}{2}+1} = (2q + 2)^m$. This gives

$$2(7q^2 + 7q + 1)^{\frac{\mu}{2}+1} = (2q + 2)^m - (2q)^m \quad (3.4)$$

(3.4) results in contradiction to the value of μ . Hence, r is even.

Let $r = 2\mu$, for any $\mu \in \mathbb{Z}_+$. Then (3.1) yields

$$\begin{aligned} (7q^2 + 7q + 1)^{2\mu} + (4(q^2 + q))^m &= n^2 \\ \Rightarrow (4(q^2 + q))^m &= n^2 - [(7q^2 + 7q + 1)^\mu]^2 \\ (2q)^m(2q + 2)^m &= (n - (7q^2 + 7q + 1)^\mu)(n + (7q^2 + 7q + 1)^\mu) \end{aligned} \quad (3.5)$$

Without loss of generality, Take $n - (7q^2 + 7q + 1)^\mu = (2q)^m$ and $n + (7q^2 + 7q + 1)^\mu = (2q + 2)^m$. For otherwise, we arrive at a contradiction to the value of m, μ . The above equation together gives

$$\begin{aligned} 2(7q^2 + 7q + 1)^\mu &= (2q + 2)^m - (2q)^m \\ 2(7q^2 + 7q + 1)^\mu &= 2^m[(q + 1)^m - (q)^m] \end{aligned} \quad (3.6)$$

From (3.6), it is found that $m = 1$ for all values of $\mu \in \mathbb{Z}_+$ as both $(7q^2 + 7q + 1)^\mu$ and $((q + 1)^m - q^m)$ are odd for all m, μ . On substituting $m = 1$ in $(7q^2 + 7q + 1)^\mu = [(q + 1)^m - q^m]$, one may have $\mu = 0$ which implies that $r = 0$. Utilizing the values $m = 1$ and $r = 0$ in $n - (7q^2 + 7q + 1)^\mu = (2q)^m$, it is observed that $n = 2q + 1$.

This completes the proof. $\square$

Corollary 3.2. Let $q \in \mathbb{N}$ with $q > 1$ be defined by $q = 2\eta(\eta + 1)$, where $\eta \in \mathbb{Z}_+$ and $\eta \geq 1$. Then (r, m, ω) admits a unique solution $(0, 1, \sqrt{2q + 1})$ in $\mathbb{Z}_+$ to the equation $(C_{14,q})^r + (4Pr_q)^m = \omega^4$.

Proof. The given equation can be written as $(C_{14,q})^r + (4Pr_q)^m = (\omega^2)^2$. Let $n = \omega^2$. By Theorem 4, it is found that $n = 2q + 1$. Hence $\omega = \sqrt{2q + 1}$. It is observed that $2q + 1$ is a perfect square for all $q \in \mathbb{N}$ such that $q = 2\eta(\eta + 1)$, where $\eta \in \mathbb{Z}_+$ with $\eta \geq 1$. $\square$

Corollary 3.3. For $q = 0$, the Diophantine equation $(C_{14,q})^r + (4Pr_q)^m = n^2$ has an infinite number of integer solutions, which are non-negative and of the form $(r, m, n) = (k_1, k_2, 1)$, where $k_1, k_2 \in \mathbb{Z}_+$.

Theorem 3.4. For each $q \in \mathbb{N}$ and $q > 1$, the equation $4(C_{14,q})^r + (16Pr_q)^m = n^2$ contains $(r, m, n) = (0, 1, 4q + 2)$ as the unique solution in $\mathbb{W}$

Proof. Rewriting the given equation,

$$4(7q^2 + 7q + 1)^r + (16(q^2 + q))^m = n^2 \quad (3.7)$$

From (3.7), it is clear that n is even. Hence $n^2 \equiv_4 0$. This implies that $4(7q^2 + 7q + 1)^r \equiv_4 0 \pmod{4}$ for all $q, r \in \mathbb{Z}_+$ with $q > 1$. Thus, r can be even or odd.

Suppose that r take odd values, then $r = 2\mu' + 1$ for $\mu' \in \mathbb{Z}$. Hence (3.7) becomes,

$$(16(q^2 + q))^m = n^2 - 4(7q^2 + 7q + 1)^{2\mu'+1} \quad (3.8)$$

$$\begin{aligned} &= n^2 - (2(7q^2 + 7q + 1))^{\frac{\mu'}{2}+1} \\ \Rightarrow (4q)^m(4q + 4)^m &= (n - 2(7q^2 + 7q + 1)^{\frac{\mu'}{2}+1})(n + 2(7q^2 + 7q + 1)^{\frac{\mu'}{2}+1}) \end{aligned} \quad (3.9)$$

Consider, $4q = (n - 2(7q^2 + 7q + 1)^{\frac{\mu'}{2}+1})$ and $4q + 4 = (n + 2(7q^2 + 7q + 1)^{\frac{\mu'}{2}+1})$

$$\begin{aligned} \Rightarrow 4(7q^2 + 7q + 1)^{\frac{\mu'}{2}+1} &= (4q + 4)^m - (4q)^m \\ 4(7q^2 + 7q + 1)^{\frac{\mu'}{2}+1} &= 4^m[(q + 1)^m - q^m] \end{aligned}$$

The above equation leads to contradiction to the value of μ' . Hence, r must be even for all $\mu' \in \mathbb{Z}_+$.

Let $r = 2\mu'$, where $\mu' \in \mathbb{Z}_+$. On substituting $r = 2\mu'$ in (3.7)

$$\begin{aligned} (16(q^2 + q))^m &= n^2 - 4(7q^2 + 7q + 1)^{2\mu'} \\ &= n^2 - (2(7q^2 + 7q + 1))^{\mu'} \\ \Rightarrow (4q)^m (4q + 4)^m &= (n - 2(7q^2 + 7q + 1))^{\mu'} (n + 2(7q^2 + 7q + 1))^{\mu'} \end{aligned}$$

Without loss of generality, assume that $4q = (n - 2(7q^2 + 7q + 1))^{\mu'}$ and $4q + 4 = (n + 2(7q^2 + 7q + 1))^{\mu'}$.

The above equations are combined together to form the following equation

$$\begin{aligned} 4(7q^2 + 7q + 1)^{\mu'} &= (4q + 4)^m - (4q)^m \\ 4(7q^2 + 7q + 1)^{\mu'} &= 4^m [(q + 1)^m - (q)^m] \end{aligned} \quad (3.10)$$

From (3.10), it is obtained that $m = 1$ as both $(7q^2 + 7q + 1)^{\mu'}$ and $(q + 1)^m - (q)^m$ are odd for all μ', m and as a result $r = 0$. Using these two values in the equation $4q = (n - 2(7q^2 + 7q + 1))^{\mu'}$, one may have $n = 4q + 2$. Thus, $(r, m, n) = (0, 1, 4q + 2)$ in $\mathbb{W}$ is the only solution to the given equation (3.7). $\square$

Corollary 3.5. *There exists no solution (r, m, ω) for the Diophantine equation $4(C_{14,q})^r + (16Pr_q)^m = \gamma^4$ in $\mathbb{W}$*

Proof. Rewrite the given equation in the form $4(C_{14,q})^r + (16Pr_q)^m = (\gamma^2)^2$. Let $n = \gamma^2$. By Theorem 5, it is clear that $n = 4q + 2$. Thus, $\gamma = \sqrt{4q + 2}$. Since $4q + 2 = 2(2q + 1)$, 2 is a factor of $4q + 2$ but 2^2 is not a factor of the same. Hence, it is found that $4q + 2$ is not a perfect square for any positive integer q , which implies that the equation $4(C_{14,q})^r + (16Pr_q)^m = \gamma^4$ does not have an integer solution, which are non-negative. $\square$

Corollary 3.6. *When $q = 0$, the Diophantine equation $4(C_{14,q})^r + (16Pr_q)^m = n^2$ has an infinite number of solutions of the form $(r, m, n) = (k_1, k_2, 2)$, where $k_1, k_2, r, m, n \in \mathbb{Z}_+$.*

3.1 Special Case

The following theorems 3.7 and 3.10 give the solution to equations (3.1) and (3.7) when $q = 1$.

Theorem 3.7. *The Exponential Diophantine equation $15^r + 8^m = n^2$ has exactly three non-negative integer solutions $(r, m, n) = (0, 1, 3), (1, 0, 4), (2, 2, 17)$.*

Proof. In order to prove Theorem 6, we need to prove two lemmas.

Lemma 3.8. *For all $r, n \in \mathbb{W}$, only $(r, n) = (1, 4)$ satisfies the equation $15^r + 1 = n^2$*

Proof. Suppose that $r = 0$, then $n^2 = 2$ which is impossible. Hence, $r \geq 1$. By preposition 1, it is observed that r must be equal to 1. Thus, $n^2 = 16$, which implies $n = 4$. Hence $(r, n) = (1, 4)$ is the unique non-negative integer solution of the given equation. $\square$

Lemma 3.9. *For all $m, n \in \mathbb{W}$, $(m, n) = (1, 3)$ is the one and only solution of the Diophantine equation $1 + 8^m = n^2$*

Proof. If $m = 0$, then $n^2 = 2$ which is impossible. Hence, $m \geq 1$. According to Proposition 1, it is clear that $m = 1$. Thus, $n^2 = 9$, which implies $n = 3$. Hence $(m, n) = (1, 3)$ satisfies the given equation. $\square$

Combining the preceding lemmas yields the conclusion that equation $15^r + 8^m = n^2$ has the solutions $(0, 1, 3)$ and $(1, 0, 4)$.

Also, since $(15, 8)$ forms a Pythagorean pair, it is observed that $(8, 15, 17)$ forms a Pythagorean triple. Hence, $(2, 2, 17)$ is also a solution due to the fact that $15^2 + 8^2 = 17^2$. $\square$

Theorem 3.10. *The Exponential Diophantine equation $4(15^r) + 32^m = n^2$ has a unique solutions $(r, m, n) = (0, 1, 6)$, where $r, m, n \in \mathbb{W}$.*

Proof. Consider the exponential equation of the form $4 + 32^m = n^2$. When $m = 0$, the equation yields $n^2 = 5$, which is not a perfect square. Therefore, it must be that $m \geq 1$. According to Proposition 1, the only viable value is $m = 1$, leading to the solution $(m, n) = (1, 6)$. Similarly, the equation $4 \cdot 15^r + 1 = n^2$ admits no solution for any pair $(r, n) \in \mathbb{W}$. Hence, it follows that the combined equation $4(15^r) + 32^m = n^2$ possesses a unique solution given by $(r, m, n) = (0, 1, 6)$ in $\mathbb{W}$. $\square$

3.2 Numerical illustrations

The following table presents some interesting illustrations.

q	$(C_{14,q})^r + (4Pr_q)^m = n^2$	$(C_{14,q})^r + (4Pr_q)^m = \omega^4$	$4(C_{14,q})^r + (16Pr_q)^m = n^2$
1	$(0,1,3), (1,0,4), (2,2,17)$	$(1,0,2)$	$(0,1,6)$
2	$(0,1,5)$	No Solution	$(0,1,10)$
3	$(0,1,7)$	No Solution	$(0,1,14)$
4	$(0,1,9)$	$(0,1,3)$	$(0,1,18)$
5	$(0,1,11)$	No Solution	$(0,1,22)$
6	$(0,1,13)$	No Solution	$(0,1,26)$
7	$(0,1,15)$	No Solution	$(0,1,30)$
8	$(0,1,17)$	No Solution	$(0,1,34)$
9	$(0,1,19)$	No Solution	$(0,1,38)$
10	$(0,1,21)$	No Solution	$(0,1,42)$

Table 1: Some illustrations of given equations along with their solutions

4 Conclusion

We have analysed the nature of integer solution of two classes of Exponential Diophantine Equations $(C_{14,q})^r + (4Pr_q)^m = n^2$ and $4(C_{14,q})^r + (16Pr_q)^m = n^2$ in which a family of centered tetradecagonal numbers and pronic numbers are involved as bases and the analysis confirms

that solutions exist for these equations in $\mathbb{W}$ and of the form $(0, 1, 2q + 1)$ and $(0, 1, 4q + 2)$ respectively. Further, some special cases of the above equations are also analysed and numerical illustrations are presented for some values of $q \geq 1$.

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