

APPROXIMATION OF BEST PROXIMITY PAIR FOR NONCYCLIC RELATIVELY NONEXPANSIVE MAPPINGS

Dipti Thakur¹ and Rashmi Bhagat²

¹School of studies in Mathematics
 Pt. Ravishankar Shukla University, Raipur (C.G.)
 dipti.thakur15@gmail.com

²School of studies in Mathematics
 Pt. Ravishankar Shukla University, Raipur (C.G.)
 rashmibhagat32@gmail.com

ABSTRACT

In this article, we study the Thakur et al. (Thakur, Thakur and Postolache, 2014) iterative process for finding the best proximity pair for noncyclic relatively nonexpansive mappings. Using this iterative process, we establish the convergence results in uniformly convex Banach spaces. We support our main result by a numerical example in which, we approximate the best proximity pair using MATLAB program.

Keywords: Noncyclic relatively nonexpansive mappings, uniformly convex Banach space, best proximity pair, iteration.

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1 Introduction

Let $\mathcal{A}$ and $\mathcal{B}$ be two nonempty and disjoint subsets of a normed linear space $(\mathcal{X}; \|\cdot\|)$. A mapping $\mathcal{T} : \mathcal{A} \cup \mathcal{B} \rightarrow \mathcal{A} \cup \mathcal{B}$ is called cyclic provided that $\mathcal{T}(\mathcal{A}) \subseteq \mathcal{B}$ and $\mathcal{T}(\mathcal{B}) \subseteq \mathcal{A}$. Then $\|x - \mathcal{T}x\| \geq \text{dist}(\mathcal{A}, \mathcal{B}) = \inf\{d(x, y) : (x, y) \in \mathcal{A} \times \mathcal{B}\}$ for any $x \in \mathcal{A} \cup \mathcal{B}$. In this case, it is natural to ask how can we find a solution for a minimization problem

$$\min_{x \in \mathcal{A} \cup \mathcal{B}} d(x, \mathcal{T}x). \quad (1.1)$$

A point $p \in \mathcal{A} \cup \mathcal{B}$ is said to be a best proximity point for the cyclic mapping $\mathcal{T} : \mathcal{A} \cup \mathcal{B} \rightarrow \mathcal{A} \cup \mathcal{B}$ if p is a solution for the minimization problem (1.1). Equivalently, $p \in \mathcal{A} \cup \mathcal{B}$ is called a best proximity point for $\mathcal{T}$, whenever

$$d(p, \mathcal{T}p) = \text{dist}(\mathcal{A}, \mathcal{B}) = \inf\{d(x, y) : (x, y) \in \mathcal{A} \times \mathcal{B}\}.$$

Inspired of cyclic mappings, we say that the mapping $\mathcal{T} : \mathcal{A} \cup \mathcal{B} \rightarrow \mathcal{A} \cup \mathcal{B}$ is noncyclic whenever $\mathcal{T}(\mathcal{A}) \subseteq \mathcal{A}$, $\mathcal{T}(\mathcal{B}) \subseteq \mathcal{B}$. For noncyclic case, we can consider the following minimization problem :

$$\min_{x \in \mathcal{A}} d(x, \mathcal{T}x), \quad \min_{y \in \mathcal{B}} d(y, \mathcal{T}y), \quad \min_{(x, y) \in \mathcal{A} \cup \mathcal{B}} d(x, y). \quad (1.2)$$

We say that a pair $(x^*, y^*) \in \mathcal{A} \times \mathcal{B}$ is a best proximity pair for the noncyclic mapping $\mathcal{T}$ if it is a solution of (1.2), or equivalently, $x^* = \mathcal{T}x^*$, $y^* = \mathcal{T}y^*$ and $d(x^*, y^*) = \text{dist}(\mathcal{A}, \mathcal{B})$.

Now, we recall that the mapping $\mathcal{T} : \mathcal{A} \cup \mathcal{B} \rightarrow \mathcal{A} \cup \mathcal{B}$ is relatively nonexpansive if $d(\mathcal{T}x, \mathcal{T}y) \leq d(x, y)$ for any $(x, y) \in \mathcal{A} \times \mathcal{B}$. These mappings are interesting because they do not need to be nonexpansive or even continuous.

On the other hand, various iterative methods have been introduced and studied for approximating fixed point for nonexpansive mappings. In 2014, Thakur et al. (Thakur et al., 2014) introduced a following three step iteration scheme for approximation of fixed points for the nonexpansive mappings. Let $\mathcal{C}$ be a nonempty closed convex subset of a norm space $\mathcal{X}$. Let $\mathcal{T}$ be a nonexpansive self mapping on $\mathcal{C}$, where the sequence $\{x_n\}$ is generated from arbitrary $x_0 \in \mathcal{C}$ by

$$\begin{cases} x_{n+1} &= (1 - \alpha_n)\mathcal{T}x_n + \alpha_n\mathcal{T}y_n, \\ y_n &= (1 - \beta_n)z_n + \beta_n\mathcal{T}z_n, \\ z_n &= (1 - \gamma_n)x_n + \gamma_n\mathcal{T}x_n, \end{cases} \quad (1.3)$$

where $\{\alpha_n\}$, $\{\beta_n\}$, and $\{\gamma_n\}$ are real sequence in $(0, 1)$. They proved convergence results for nonexpansive mappings using the iteration (1.3). They also proved that the iteration (1.3) converges faster than Picard, Mann (Mann, 1953), Ishikawa (Ishikawa, 1974), Noor (Noor, 2000), S-iteration (Agarwal, O Regan and Sahu, 2007), Abbas et al. (Abbas and Nazir, 2014) iteration process.

Afterwards, In 2005, Eldred et al. (Eldred, Kirk and Veeramani, 2005) introduced the notion of proximal normal structure and used to study mappings that are relatively nonexpansive mapping. They also approximate the fixed point using Krasnoselskii's type iteration method. In 2017, Eldred and Praveen (Eldred and Praveen, 2017) generalized the results of Eldred et al. (Eldred et al., 2005) by using the Mann iteration method for noncyclic relatively nonexpansive mappings to find the best proximity point. Let $\mathcal{X}$ be a Banach space then the sequence $\{x_n\}$ is generated from arbitrary $x_0 \in \mathcal{X}$ by

$$\begin{cases} x_{n+1} &= (1 - \alpha_n)x_n + \alpha_n\mathcal{T}y_n, \forall n \in \mathbb{N}_0, \end{cases} \quad (1.4)$$

where $\mathbb{N}_0 = \{0, 1, 2, \dots\}$ and $\alpha_n \in [0, 1]$.

In 2020, Abdeljawad et al. (Abdeljawad, Ullah, Ahmad, De La Sen and Ulhaq, 2020) used the S-iteration process to approximate the best proximity point of noncyclic relatively nonexpansive mappings. Let X be a Banach space then the sequence $\{x_n\}$ is generated from arbitrary $x_0 \in \mathcal{X}$ by

$$\begin{cases} y_n &= (1 - \beta_n)x_n + \beta_n\mathcal{T}x_n, \\ x_{n+1} &= (1 - \alpha_n)\mathcal{T}x_n + \alpha_n\mathcal{T}y_n, \forall n \in \mathbb{N}_0, \end{cases} \quad (1.5)$$

where $\alpha_n, \beta_n \in [0, 1]$.

In same year, Gabeleh et al. (Gabeleh, Ezhil Manna, Eldred and Olela Otafudu, 2020) studied the strong and weak convergence of best proximity points using the Ishikawa iteration scheme for noncyclic relatively nonexpansive mappings in uniformly convex Banach spaces. Let X be a Banach space then the sequence $\{x_n\}$ is generated from arbitrary $x_0 \in \mathcal{X}$ by

$$\begin{cases} y_n &= (1 - \beta_n)x_n + \beta_n\mathcal{T}x_n, \\ x_{n+1} &= (1 - \alpha_n)x_n + \alpha_n\mathcal{T}y_n, \forall n \in \mathbb{N}_0, \end{cases} \quad (1.6)$$

where $\alpha_n, \beta_n \in [0, 1]$.

In 2021, Shanjit et al. (Shanjit, Rohen, Chandok and Bina Devi, 2021) proved a sufficient condition that every nonempty closed convex bounded pair $(\mathcal{C}, \mathcal{D})$ in a reflexive Banach space

satisfying Opial's condition has proximal normal structure. They also proved some convergence results of the best proximity point for noncyclic relatively nonexpansive mappings using Ishikawa's and Halpern's iteration processes. Let $\mathcal{C}$ be a nonempty closed convex subset of a Banach space $\mathcal{X}$ for any fixed $x_0 \in \mathcal{C}$, the sequence $\{x_n\}$ in $\mathcal{C}$ then Halpern's iteration defined by

$$\begin{cases} x_{n+1} &= (1 - \alpha_n)u + \alpha_n \mathcal{T}y_n, \forall n \in \mathbb{N}_0, \end{cases} \quad (1.7)$$

where $\alpha_n \in [0, 1]$, $u \in \mathcal{C}$ referred to as anchor, and $x_0 \in \mathcal{C}$ an initial guess taken arbitrarily.

Motivated by the aforementioned developments, In this paper, we use the Thakur et al. (Thakur et al., 2014) iteration process to approximate best proximity pair for noncyclic relatively non-expansive mappings. We establish strong convergence result in the setting of Banach space. We support our proof by a numerical example in which we approximate the best proximity pair. Also we use MATLAB coding to show that the convergence behaviour of best proximity pair.

2 Preliminaries

In this section, we recall some basic definitions and concepts which will be used in our conclusions. In the sequel, we shall say that a pair $(\mathcal{C}, \mathcal{D})$ of subsets of a Banach space $\mathcal{X}$ satisfies a property if both $\mathcal{C}$ and $\mathcal{D}$ satisfy that property. For example, $(\mathcal{C}, \mathcal{D})$ is convex if and only if both $\mathcal{C}$ and $\mathcal{D}$ are convex. $(\mathcal{C}, \mathcal{D}) \subseteq (\mathcal{A}, \mathcal{B}) \Leftrightarrow \mathcal{C} \subseteq \mathcal{A}$ and $\mathcal{D} \subseteq \mathcal{B}$. We shall also adopt the notation

$$\begin{aligned} \text{dist}(\mathcal{C}, \mathcal{D}) &= \inf\{\|x - y\| : x \in \mathcal{C}, y \in \mathcal{D}\}; \\ d(x, \mathcal{D}) &= \inf\{\|x - y\| : y \in \mathcal{D}\}; \\ \mathcal{C}_0 &= \{x \in \mathcal{C} : \|x - y'\| = d(\mathcal{C}, \mathcal{D}) \text{ for some } y' \in \mathcal{D}\}; \\ \mathcal{D}_0 &= \{y \in \mathcal{D} : \|x' - y\| = d(\mathcal{C}, \mathcal{D}) \text{ for some } x' \in \mathcal{C}\}. \end{aligned}$$

For a nonempty subset $\mathcal{C}$ of $\mathcal{X}$ a metric projection operator $\mathcal{P}_{\mathcal{C}} : \mathcal{X} \rightarrow 2^{\mathcal{C}}$ is defined as

$$\mathcal{P}_{\mathcal{C}}(x) = \{y \in \mathcal{C} : \|x - y\| = \text{dist}(\{x\}, \mathcal{C})\}, \forall x \in \mathcal{X}.$$

Where $2^{\mathcal{C}}$ denote the set of all nonempty subsets of $\mathcal{C}$. It is well known that if $\mathcal{C}$ is a nonempty, closed and convex subset of a reflexive and strictly convex Banach space $\mathcal{X}$. Then the metric projection $\mathcal{P}_{\mathcal{C}}$ is single valued from $\mathcal{X}$ to $\mathcal{C}$. Moreover, if Proximal pair may be empty but, in particular if $(\mathcal{C}, \mathcal{D})$ is nonempty, bounded, closed and convex pair in a reflexive Banach space $\mathcal{X}$ then $(\mathcal{C}_0, \mathcal{D}_0)$ is also nonempty, closed and convex.

Definition 2.1. (Eldred et al., 2005) A pair $(\mathcal{C}, \mathcal{D})$ of subsets of a normed linear space is said to be a proximal pair if for every $(x, y) \in \mathcal{C} \times \mathcal{D}$, there exists $(x', y') \in \mathcal{C} \times \mathcal{D}$ such that

$$\|x - y'\| = \|x' - y\| = \text{dist}(\mathcal{C}, \mathcal{D}).$$

Definition 2.2. (Suzuki, Kikkawa and Vetro, 2009) Let $\mathcal{C}$ and $\mathcal{D}$ be a nonempty pair of subset of metric space $(\mathcal{X}, d)$. The pair $(\mathcal{C}, \mathcal{D})$ is said to satisfy the property UC if the following holds:

- If $\{x_n\}$ and $\{x'_n\}$ are sequence in $\mathcal{C}$ and $\{y_n\}$ is sequence in $\mathcal{D}$ such that $\lim_{n \rightarrow \infty} d(x_n, y_n) = \text{dist}(\mathcal{C}, \mathcal{D})$ and $\lim_{n \rightarrow \infty} d(x'_n, y_n) = \text{dist}(\mathcal{C}, \mathcal{D})$, then $\lim_{n \rightarrow \infty} d(x_n, x'_n) = 0$ holds.

Remark 1. (Di Bari, Suzuki and Vetro, 2008) Let $(\mathcal{C}, \mathcal{D})$ is nonempty, closed pair of subset of uniformly convex Banach space such that $\mathcal{C}$ is convex. Then $(\mathcal{C}, \mathcal{D})$ has the property UC .

Definition 2.3. (Raj, 2011) A pair $(\mathcal{C}, \mathcal{D})$ be a pair of nonempty subsets in a Banach space $(\mathcal{X}, d)$ with $\mathcal{C}_0 \neq \emptyset$. The pair $(\mathcal{C}, \mathcal{D})$ is said to have the P-property if and only if

$$\left. \begin{aligned} d(x_1, y_1) &= \text{dist}(\mathcal{C}, \mathcal{D}) \\ d(x_2, y_2) &= \text{dist}(\mathcal{C}, \mathcal{D}) \end{aligned} \right\} \implies d(x_1, x_2) = d(y_1, y_2).$$

Remark 2. (Abkar and Gabeleh, 2012) Let $\mathcal{C}$ and $\mathcal{D}$ be two nonempty, bounded, closed and convex subsets of a uniformly convex Banach space $\mathcal{X}$. Then $(\mathcal{C}, \mathcal{D})$ has the P-property.

Proposition 2.1. (Chidume, 2009) Let $\mathcal{X}$ be a uniformly convex Banach space $\alpha \in (0, 1)$ and $\epsilon > 0$. Then for each $d > 0$ and $x, y \in \mathcal{X}$ be such that $\|x\| \leq d$, $\|y\| \leq d$, $\|x - y\| \geq \epsilon$, then there exists some $\delta = \delta(\frac{\epsilon}{d}) > 0$ such that $\|\alpha x + (1 - \alpha)y\| \leq [1 - 2\delta(\frac{\epsilon}{d}) \min\{\alpha, 1 - \alpha\}]d$.

Suppose $(\mathcal{C}, \mathcal{D})$ be a nonempty, weakly compact convex pair of subsets in a Banach space $\mathcal{X}$. Consider the map $\mathcal{P} : \mathcal{C} \cup \mathcal{D} \rightarrow \mathcal{C} \cup \mathcal{D}$ defined as

$$\mathcal{P}(x) = \begin{cases} \mathcal{P}_{\mathcal{C}}(x) & \text{if } x \in \mathcal{D}, \\ \mathcal{P}_{\mathcal{D}}(x) & \text{if } x \in \mathcal{C}. \end{cases} \quad (2.1)$$

If $\mathcal{X}$ is strictly convex Banach space, then the map $\mathcal{P}$ is a single valued map and satisfies $\mathcal{P}(\mathcal{C}) \subseteq \mathcal{D}$, $\mathcal{P}(\mathcal{D}) \subseteq \mathcal{C}$.

Proposition 2.2. (Raj and Veeramani, 2009) Let $\mathcal{C}, \mathcal{D}$ be a nonempty weakly compact convex subset of a strictly convex Banach space $\mathcal{X}$. Let $\mathcal{T} : \mathcal{C} \cup \mathcal{D} \rightarrow \mathcal{C} \cup \mathcal{D}$ be a relatively nonexpansive map and $\mathcal{P} : \mathcal{C} \cup \mathcal{D} \rightarrow \mathcal{C} \cup \mathcal{D}$ be a map defined as (2.1). Then $\mathcal{T}(\mathcal{P}(x)) = \mathcal{P}(\mathcal{T}x)$, for all $x \in \mathcal{C}_0 \cup \mathcal{D}_0$.

Lemma 2.3. (Schu, 1991) Let $0 < a < t_n < b < 1$ for every $n \geq 1$ assume that $\{u_n\}$ and $\{v_n\}$ are sequence in uniformly convex Banach space $\mathcal{X}$ such that $\|u_n\| \leq 1$, $\|v_n\| \leq 1$. Define $\{b_n\}$ in $\mathcal{X}$ by $b_n = (1 - t_n)u_n + t_nv_n$ if $\lim_{n \rightarrow \infty} \|b_n\| = 1$, then $\lim_{n \rightarrow \infty} \|u_n - v_n\| = 0$.

Lemma 2.4. (Thakur et al., 2014) Let $\mathcal{C}$ be a closed convex subset of a uniformly convex Banach space $\mathcal{X}$. Let $\mathcal{T}$ be a nonexpansive self mapping on $\mathcal{C}$, $\{x_n\}$ defined by equation (3.1), where $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\gamma_n\}$ are in $[\epsilon, 1 - \epsilon]$ for all $n \in \mathbb{N}$ and for some $\epsilon \in (0, 1)$ and $F(\mathcal{T}) \neq \emptyset$. Then $\lim_{n \rightarrow \infty} \|x_n - \mathcal{T}x_n\| = 0$.

The following lemma gives guarantees of the nonemptiness of $\mathcal{C}_0$ and $\mathcal{D}_0$.

Lemma 2.5. (Kirk, Reich and Veeramani, 2003) Let $\mathcal{X}$ be a reflexive Banach space, let $\mathcal{C}$ be a nonempty bounded closed convex subset of $\mathcal{X}$, and let $\mathcal{D}$ be a nonempty closed convex subset of $\mathcal{X}$. Then $\mathcal{C}_0$ and $\mathcal{D}_0$ are nonempty and satisfy

$$P_{\mathcal{D}}(\mathcal{C}_0) \subseteq \mathcal{D}_0 \text{ and } P_{\mathcal{C}}(\mathcal{D}_0) \subseteq \mathcal{C}_0.$$

Theorem 2.6. (Eldred et al., 2005) Let $(\mathcal{C}, \mathcal{D})$ be a nonempty, weakly compact convex pair in strictly convex Banach space, and suppose $(\mathcal{C}, \mathcal{D})$ has proximal normal structure. Suppose $\mathcal{T} : \mathcal{C} \cup \mathcal{D} \rightarrow \mathcal{C} \cup \mathcal{D}$ satisfies,

- $\mathcal{T}(\mathcal{C}) \subseteq \mathcal{C}$ and $\mathcal{T}(\mathcal{D}) \subseteq \mathcal{D}$,
- $\|\mathcal{T}u - \mathcal{T}v\| \leq \|u - v\|$ for all $u \in \mathcal{C}, v \in \mathcal{D}$.

Then, there exists $u_0 \in \mathcal{C}$ and $v_0 \in \mathcal{D}$ such that $\mathcal{T}u_0 = u_0$, $\mathcal{T}v_0 = v_0$ and $\|u_0 - v_0\| = \text{dist}(\mathcal{C}, \mathcal{D})$.

3 Convergence results

In this section we prove some convergence results for noncyclic relatively nonexpansive mappings using Thakur et al. (Thakur et al., 2014) iteration.

Let $(\mathcal{C}, \mathcal{D})$ be a nonempty closed and convex pair in strictly convex Banach space $\mathcal{X}$. For $x_1 \in \mathcal{C}_0$, put $x'_1 := \mathcal{P}(x_1) \in \mathcal{D}_0$. Define the sequence pair $\{(x_n, x'_n)\}$ as follows:

$$\begin{cases} x_{n+1} &= (1 - \alpha_n)\mathcal{T}x_n + \alpha_n\mathcal{T}y_n, \\ y_n &= (1 - \beta_n)z_n + \beta_n\mathcal{T}z_n, \\ z_n &= (1 - \gamma_n)x_n + \gamma_n\mathcal{T}x_n, \end{cases} \quad \begin{cases} x'_{n+1} &= (1 - \alpha_n)\mathcal{T}x'_n + \alpha_n\mathcal{T}y'_n, \\ y'_n &= (1 - \beta_n)z'_n + \beta_n\mathcal{T}z'_n, \\ z'_n &= (1 - \gamma_n)x'_n + \gamma_n\mathcal{T}x'_n, \end{cases} \quad (3.1)$$

where α_n, β_n and $\gamma_n \in (0, 1)$.

Lemma 3.1. *Let $(\mathcal{C}, \mathcal{D})$ be a nonempty, closed, bounded and convex pair in a uniformly convex Banach space $\mathcal{X}$. Let $\mathcal{T} : \mathcal{C} \cup \mathcal{D} \rightarrow \mathcal{C} \cup \mathcal{D}$ be a noncyclic relatively nonexpansive mapping. suppose $\{x_n\}$ and $\{x'_n\}$ are given by equation (3.1). Then $\lim_{n \rightarrow \infty} \|x_n - p\|$ exists for all $p \in \text{Fix}(\mathcal{T}) \cap \mathcal{D}_0$ and $\lim_{n \rightarrow \infty} \|x'_n - q\|$ exists for all $q \in \text{Fix}(\mathcal{T}) \cap \mathcal{C}_0$, where $\text{Fix}(\mathcal{T})$ denotes the set of all fixed points of the mapping $\mathcal{T}$.*

Proof. From Lemma 2.5, $(\mathcal{C}_0, \mathcal{D}_0)$ is nonempty. Let $(\mathcal{C}_0, \mathcal{D}_0)$ is closed and convex. For any $q \in \text{Fix}(\mathcal{T}) \cap \mathcal{C}_0$, we have

$$\begin{aligned} \|x'_{n+1} - q\| &= \|(1 - \alpha_n)\mathcal{T}x'_n + \alpha_n\mathcal{T}y'_n - q\| \\ &\leq (1 - \alpha_n)\|\mathcal{T}x'_n - q\| + \alpha_n\|\mathcal{T}y'_n - q\| \\ &\leq (1 - \alpha_n)\|x'_n - q\| + \alpha_n\|y'_n - q\| \\ &= (1 - \alpha_n)\|x'_n - q\| + \alpha_n\|(1 - \beta_n)z'_n + \beta_n\mathcal{T}z'_n - q\| \\ &\leq (1 - \alpha_n)\|x'_n - q\| + \alpha_n(1 - \beta_n)\|z'_n - q\| + \alpha_n\beta_n\|\mathcal{T}z'_n - q\| \\ &\leq (1 - \alpha_n)\|x'_n - q\| + \alpha_n(1 - \beta_n)\|z'_n - q\| + \alpha_n\beta_n\|z'_n - q\| \\ &= (1 - \alpha_n)\|x'_n - q\| + \alpha_n\|z'_n - q\| \\ &= (1 - \alpha_n)\|x'_n - q\| + \alpha_n\|(1 - \gamma_n)x'_n + \gamma_n\mathcal{T}x'_n - q\| \\ &\leq (1 - \alpha_n)\|x'_n - q\| + \alpha_n(1 - \gamma_n)\|x'_n - q\| + \alpha_n\gamma_n\|\mathcal{T}x'_n - q\| \\ &\leq (1 - \alpha_n)\|x'_n - q\| + \alpha_n(1 - \gamma_n)\|x'_n - q\| + \alpha_n\gamma_n\|x'_n - q\| \\ &= (1 - \alpha_n)\|x'_n - q\| + \alpha_n\|x'_n - q\| \\ &= \|x'_n - q\|. \end{aligned}$$

This implies that $\{\|x'_n - q\|\}$ is nonincreasing and hence $\lim_{n \rightarrow \infty} \|x'_n - q\|$ exists for all $q \in \text{Fix}(\mathcal{T}) \cap \mathcal{C}_0$. Similarly we can also show that $\lim_{n \rightarrow \infty} \|x_n - p\|$ exists for all $p \in \text{Fix}(\mathcal{T}) \cap \mathcal{D}_0$ and hence the lemma. $\square$

Theorem 3.2. *Let $(\mathcal{C}, \mathcal{D})$ be a nonempty, closed bounded and convex pair in uniformly convex Banach space $\mathcal{X}$ and let $\mathcal{T} : \mathcal{C} \cup \mathcal{D} \rightarrow \mathcal{C} \cup \mathcal{D}$ be a noncyclic relatively nonexpansive mappings. select $x_1 \in \mathcal{C}_0$ and $x'_1 \in \mathcal{D}_0$ be a unique proximal point of x_1 . Let $\{x_n\}$ and $\{x'_n\}$ are given by equation (3.1), where α_n, β_n and γ_n are in $[\epsilon, 1 - \epsilon]$ for all $n \in \mathbb{N}$ and for some $\epsilon \in (0, 1)$. Then*

$$\lim_{n \rightarrow \infty} \|x_n - \mathcal{T}x_n\| = 0 \text{ and } \lim_{n \rightarrow \infty} \|x'_n - \mathcal{T}x'_n\| = 0,$$

further the sequence pair $\{(x_n, x'_n)\}$ converges to a Best proximity pair of $\mathcal{T}$ if $\mathcal{T}(\mathcal{C})$ and $\mathcal{T}(\mathcal{D})$ lies in a compact subset.

Proof. If $\text{dist}(\mathcal{C}, \mathcal{D}) = 0$ then $\mathcal{C}_0 = \mathcal{D}_0 = \mathcal{C} \cap \mathcal{D}$ then by Lemma 2.4 it follows. Now, we can establish the theorem from the fact that $\mathcal{T} : \mathcal{C} \cup \mathcal{D} \rightarrow \mathcal{C} \cup \mathcal{D}$ is noncyclic relatively nonexpansive,

let $\text{dist}(\mathcal{C}, \mathcal{D}) > 0$. by Theorem 2.6, there exists $\rho \in \mathcal{D}_0$ such that $\mathcal{T}\rho = \rho$. Now,

$$\begin{aligned} \|x_{n+1} - \rho\| &= \|(1 - \alpha_n)\mathcal{T}x_n + \alpha_n\mathcal{T}y_n - \rho\| \\ &\leq (1 - \alpha_n)\|\mathcal{T}x_n - \rho\| + \alpha_n\|\mathcal{T}y_n - \rho\| \\ &\leq (1 - \alpha_n)\|x_n - \rho\| + \alpha_n\|y_n - \rho\|. \end{aligned}$$

Therefore,

$$\|x_{n+1} - \rho\| \leq (1 - \alpha_n)\|x_n - \rho\| + \alpha_n\|y_n - \rho\|. \quad (3.2)$$

Again

$$\begin{aligned} \|y_n - \rho\| &= \|(1 - \beta_n)z_n + \beta_n\mathcal{T}z_n - \rho\| \\ &\leq (1 - \beta_n)\|z_n - \rho\| + \beta_n\|\mathcal{T}z_n - \rho\| \\ &\leq (1 - \beta_n)\|z_n - \rho\| + \beta_n\|z_n - \rho\| \\ &= \|z_n - \rho\| \\ &= \|(1 - \gamma_n)x_n + \gamma_n\mathcal{T}x_n - \rho\| \\ &\leq (1 - \gamma_n)\|x_n - \rho\| + \|\mathcal{T}x_n - \rho\| \\ &\leq (1 - \gamma_n)\|x_n - \rho\| + \|x_n - \rho\| \\ &= \|x_n - \rho\|. \end{aligned}$$

$$\|y_n - \rho\| \leq \|z_n - \rho\| \leq \|x_n - \rho\| \quad (3.3)$$

Again from (3.2),

$$\begin{aligned} \|x_{n+1} - \rho\| &\leq (1 - \alpha_n)\|x_n - \rho\| + \alpha_n\|y_n - \rho\| \\ &\leq (1 - \alpha_n)\|x_n - \rho\| + \alpha_n\|x_n - \rho\| \\ &= \|x_n - \rho\|. \end{aligned}$$

Therefore,

$$\|x_{n+1} - \rho\| \leq \|x_n - \rho\|. \quad (3.4)$$

Hence, the sequence $\{\|x_n - \rho\|\}$ is nonincreasing, so one can select a $d > 0$ such that $\lim_{n \rightarrow \infty} \|x_n - \rho\| = d$.

We claim that $\lim_{n \rightarrow \infty} \|x_{n_k} - \mathcal{T}x_{n_k}\| = 0$. Suppose for contradiction $\|x_{n_k} - \mathcal{T}x_{n_k}\| \neq 0$. Then there is a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ and $\epsilon > 0$ such that $\|x_{n_k} - \mathcal{T}x_{n_k}\| \geq \epsilon > 0$ for every k . However the modulus of convexity δ of $\mathcal{X}$ is a continuous and increasing function and we choose $\xi > 0$ as small that $(1 - c\delta(\frac{\epsilon}{d+\xi}))(d + \xi) < d$, where $c > 0$.

Now, we choose k , such that $\|x_{n_k} - \rho\| \leq d + \xi$ and $\|\mathcal{T}x_{n_k} - \rho\| \leq d + \xi$. We have

$$\begin{aligned}
 \|\rho - x_{n_k+1}\| &= \|\rho - ((1 - \alpha_{n_k})\mathcal{T}x_{n_k} + \alpha_{n_k}\mathcal{T}y_{n_k})\| \\
 &= \|\rho - (1 - \alpha_{n_k})\mathcal{T}x_{n_k} - \alpha_{n_k}\mathcal{T}((1 - \beta_{n_k})z_{n_k} + \beta_{n_k}\mathcal{T}z_{n_k})\| \\
 &= \|(1 - \alpha_{n_k})\rho + \alpha_{n_k}\rho - (1 - \alpha_{n_k})\mathcal{T}x_{n_k} - \alpha_{n_k}\mathcal{T}((1 - \beta_{n_k})z_{n_k} + \beta_{n_k}\mathcal{T}z_{n_k})\| \\
 &= (1 - \alpha_{n_k})\|\rho - \mathcal{T}x_{n_k}\| + \alpha_{n_k}\|\rho - \mathcal{T}((1 - \beta_{n_k})z_{n_k} + \beta_{n_k}\mathcal{T}z_{n_k})\| \\
 &= (1 - \alpha_{n_k})\|\mathcal{T}\rho - \mathcal{T}x_{n_k}\| + \alpha_{n_k}\|\mathcal{T}\rho - \mathcal{T}((1 - \beta_{n_k})z_{n_k} + \beta_{n_k}\mathcal{T}z_{n_k})\| \\
 &\leq (1 - \alpha_{n_k})\|\rho - x_{n_k}\| + \alpha_{n_k}\|\rho - (1 - \beta_{n_k})z_{n_k} + \beta_{n_k}\mathcal{T}z_{n_k}\| \\
 &= (1 - \alpha_{n_k})(d + \xi) + \alpha_{n_k}\|(1 - \beta_{n_k})\rho + \beta_{n_k}\rho - (1 - \beta_{n_k})z_{n_k} - \beta_{n_k}\mathcal{T}z_{n_k}\| \\
 &\leq (1 - \alpha_{n_k})(d + \xi) + \alpha_{n_k}\|(1 - \beta_{n_k})(\rho - z_{n_k}) + \beta_{n_k}(\rho - \mathcal{T}z_{n_k})\| \\
 &\leq (1 - \alpha_{n_k})(d + \xi) + \alpha_{n_k}((1 - \beta_{n_k})\|\rho - z_{n_k}\| + \beta_{n_k}\|\rho - \mathcal{T}z_{n_k}\|) \\
 &\leq (1 - \alpha_{n_k})(d + \xi) + \alpha_{n_k}((1 - \beta_{n_k})\|\rho - z_{n_k}\| + \beta_{n_k}\|\rho - z_{n_k}\|) \\
 &= (1 - \alpha_{n_k})(d + \xi) + \alpha_{n_k}((1 - \beta_{n_k})\|(1 - \alpha_{n_k})(\rho - x_{n_k}) + \alpha_{n_k}(\rho - \mathcal{T}x_{n_k})\| \\
 &\quad + \beta_{n_k}\|(1 - \alpha_{n_k})(\rho - x_{n_k}) + \alpha_{n_k}(\rho - \mathcal{T}x_{n_k})\|) \\
 &= (1 - \alpha_{n_k})(d + \xi) + \alpha_{n_k}(1 - \beta_{n_k} + \beta_{n_k})\|(1 - \alpha_{n_k})(\rho - x_{n_k}) + \alpha_{n_k}(\rho - \mathcal{T}x_{n_k})\| \\
 &= (1 - \alpha_{n_k})(d + \xi) + \alpha_{n_k}\|(1 - \alpha_{n_k})(\rho - x_{n_k}) + \alpha_{n_k}(\rho - \mathcal{T}x_{n_k})\|
 \end{aligned}$$

By proposition 2.1,

$$\begin{aligned}
 \|\rho - x_{n_k+1}\| &= (1 - \alpha_{n_k})(d + \xi) + \alpha_{n_k}(1 - 2\delta\left(\frac{\epsilon}{d + \xi}\right)\min\{\alpha_{n_k}, 1 - \alpha_{n_k}\})(d + \xi) \\
 &= (1 - \alpha_{n_k} + \alpha_{n_k} - 2\alpha_{n_k}\delta\left(\frac{\epsilon}{d + \xi}\right)\min\{\alpha_{n_k}, 1 - \alpha_{n_k}\})(d + \xi) \\
 &= (1 - 2\delta\left(\frac{\epsilon}{d + \xi}\right)\min\{\alpha_{n_k}\alpha_{n_k}, \alpha_{n_k}(1 - \alpha_{n_k})\})(d + \xi).
 \end{aligned}$$

Since there exists $h > 0$ such that $2\min\{\alpha_{n_k}\alpha_{n_k}, \alpha_{n_k}(1 - \alpha_{n_k})\} \geq h$,

$$\begin{aligned}
 \|\rho - x_{n_k}\| &= \left(1 - 2\delta\left(\frac{\epsilon}{d + \xi}\right)\min\{\alpha_{n_k}\alpha_{n_k}, \alpha_{n_k}(1 - \alpha_{n_k})\}\right)(d + \xi) \\
 &\leq \left(1 - h\delta\left(\frac{\epsilon}{d + \xi}\right)\right)(d + \xi).
 \end{aligned}$$

Suppose choosing very small $\xi > 0$ we have $(1 - h\delta(\frac{\epsilon}{d+\xi}))(d + \xi) < d$, which is a contradiction.

This implies that $\lim_{n \rightarrow \infty} \|x_n - \mathcal{T}x_n\| = 0$. Similarly, we can prove that $\lim_{n \rightarrow \infty} \|x'_n - \mathcal{T}x'_n\| = 0$.

Now, suppose that $\mathcal{T}(\mathcal{C})$ lies in a compact subset, then $\{\mathcal{T}x_n\}$ has a convergent subsequence $\{\mathcal{T}x_{n_k}\}$ converging to some point $u \in \mathcal{C}_0$. That is $\{\mathcal{T}x_{n_k}\} \rightarrow u$. Since $\lim_{n \rightarrow \infty} \|\mathcal{T}x_{n_k} - x_{n_k}\| = 0$ then $x_{n_k} \rightarrow u$.

From proposition 2.2, $\mathcal{T}(\mathcal{P}u) = \mathcal{P}(\mathcal{T}u)$ we have

$$\|\mathcal{T}x_{n_k} - \mathcal{P}(\mathcal{T}u)\| = \|\mathcal{T}x_{n_k} - \mathcal{T}(\mathcal{P}(u))\| \leq \|x_{n_k} - \mathcal{P}u\| = \text{dist}(\mathcal{C}, \mathcal{D}).$$

Which concludes that $\lim_{n \rightarrow \infty} \|\mathcal{T}x_{n_k} - \mathcal{P}(\mathcal{T}u)\| = \text{dist}(\mathcal{C}, \mathcal{D})$. Let $\mathcal{T}u \in \mathcal{C}_0$ such that $\lim_{n \rightarrow \infty} \|\mathcal{T}u - \mathcal{P}(\mathcal{T}u)\| = \text{dist}(\mathcal{C}, \mathcal{D})$.

From the definition of UC property $\lim_{n \rightarrow \infty} \|\mathcal{T}x_{n_k} - \mathcal{T}u\| = 0$, that is $\mathcal{T}x_{n_k} \rightarrow \mathcal{T}u$. Thus $\mathcal{T}u = u$ and so $\mathcal{T}(\mathcal{P}u) = \mathcal{P}u$. Therefore, by Lemma 3.1, $\lim_{n \rightarrow \infty} \|x_n - \mathcal{P}u\|$ exists.

Also, $\lim_{n \rightarrow \infty} \|x_n - \mathcal{P}u\| = \lim_{k \rightarrow \infty} \|x_{n_k} - \mathcal{P}u\| = \|u - \mathcal{P}u\| = \text{dist}(\mathcal{C}, \mathcal{D})$. Which implies

$x_n \rightarrow u \in \text{Fix}(\mathcal{T}) \cap \mathcal{C}_0$. Beside, by the assumption $\|x_1 - x'_1\| = \text{dist}(\mathcal{C}, \mathcal{D})$. We have

$$\begin{aligned}
 \|x_2 - x'_2\| &= \|(1 - \alpha_1)\mathcal{T}x_1 + \alpha_1\mathcal{T}y_1 - (1 - \alpha_1)\mathcal{T}x'_1 - \alpha_1\mathcal{T}y'_1\| \\
 &= (1 - \alpha_1)\|\mathcal{T}x_1 - \mathcal{T}x'_1\| + \alpha_1\|\mathcal{T}y_1 - \mathcal{T}y'_1\| \\
 &\leq (1 - \alpha_1)\|x_1 - x'_1\| + \alpha_1\|y_1 - y'_1\| \\
 &= (1 - \alpha_1)\|x_1 - x'_1\| + \alpha_1\|(1 - \beta_1)z_1 + \beta_1\mathcal{T}z_1 - (1 - \beta_1)z'_1 + \beta_1\mathcal{T}z'_1\| \\
 &\leq (1 - \alpha_1)\|x_1 - x'_1\| + \alpha_1\left((1 - \beta_1)\|z_1 - z'_1\| + \alpha_1\beta_1\|\mathcal{T}z_1 - \mathcal{T}z'_1\|\right) \\
 &= (1 - \alpha_1)\|x_1 - x'_1\| + \alpha_1\left((1 - \beta_1)\|z_1 - z'_1\| + \alpha_1\beta_1\|z_1 - z'_1\|\right) \\
 &= (1 - \alpha_1)\|x_1 - x'_1\| + \alpha_1\|z_1 - z'_1\| \\
 &= (1 - \alpha_1)\|x_1 - x'_1\| + \alpha_1\left(\|(1 - \gamma_1)x_1 + \gamma_1\mathcal{T}x_1 - (1 - \gamma_1)x'_1 - \gamma_1\mathcal{T}x'_1\|\right) \\
 &= (1 - \alpha_1)\|x_1 - x'_1\| + \alpha_1\left((1 - \gamma_1)\|x_1 - x'_1\| + \alpha_1\gamma_1\|\mathcal{T}x_1 - \mathcal{T}x'_1\|\right) \\
 &= (1 - \alpha_1)\|x_1 - x'_1\| + \alpha_1\|x_1 - x'_1\| \\
 &= \|x_1 - x'_1\| \\
 &= \text{dist}(\mathcal{C}, \mathcal{D}),
 \end{aligned}$$

continuing this process by induction we obtain $\|x_n - x'_n\| = \text{dist}(\mathcal{C}, \mathcal{D})$ for each $n \in \mathbb{N}$. Now, suppose that $\mathcal{T}(\mathcal{D})$ lies in a compact subset then $\{\mathcal{T}x'_n\}$ has a convergent subsequence $\{\mathcal{T}x'_{n_k}\}$ converging to some point $v \in \mathcal{D}_0$. That is $\mathcal{T}x'_{n_k} \rightarrow v$.

Since $\lim_{n \rightarrow \infty} \|\mathcal{T}x'_{n_k} - x'_{n_k}\| = 0$ then $x'_{n_k} \rightarrow v$.

By a similar argument, we conclude that,

$x'_n \rightarrow v \in \text{Fix}(\mathcal{T}) \cap \mathcal{D}_0$. On the other hand, $\|u - v\| = \lim_{n \rightarrow \infty} \|x_n - x'_n\| = \text{dist}(\mathcal{C}, \mathcal{D})$ which conclude that (u, v) is best proximity pair of mapping $\mathcal{T}$. $\square$

Example 1. Let us consider $(\mathcal{X} = \mathbb{R}^2, \|\cdot\|_2)$.

Let $\mathcal{C} = \{(x, y) : -2 \leq x \leq -1, 0 \leq y \leq 1\}$

and $\mathcal{D} = \{(x', y') : 1 \leq x' \leq 2, 0 \leq y' \leq 1\}$.

Define

$$\mathcal{T}(x, y) = \begin{cases} \left(\frac{x-4}{5}, \frac{y}{5}\right), & \text{if } (x, y) \in \mathcal{C}, \\ \left(\frac{x'+4}{5}, \frac{y'}{5}\right), & \text{if } (x', y') \in \mathcal{D}. \end{cases} \quad (3.5)$$

• **Case 1:** when $(x, y) \in \mathcal{C}$ and $(x', y') \in \mathcal{D}$. Then,

$$\begin{aligned}
 \|\mathcal{T}(x, y) - \mathcal{T}(x', y')\| &= \left\| \left(\frac{x-4}{5}, \frac{y}{5}\right) - \left(\frac{x'+4}{5}, \frac{y'}{5}\right) \right\| \\
 &= \left\| \frac{x-x'-8}{5}, \frac{y-y'}{5} \right\| \\
 &= \sqrt{\left(\frac{x-x'-8}{5}\right)^2 + \left(\frac{y-y'}{5}\right)^2} \\
 &\leq \sqrt{(x-x')^2 + (y-y')^2}.
 \end{aligned}$$

$$\text{Since, } x' - x \geq 2 \text{ and } \begin{cases} x - x' > x - x' - 8 \\ \left(\frac{x - x'}{5}\right)^2 > \left(\frac{x - x' - 8}{5}\right)^2 \\ (x - x')^2 > \left(\frac{x - x'}{5}\right)^2 > \left(\frac{x - x' - 8}{5}\right)^2 \\ (x - x')^2 > \left(\frac{x - x' - 8}{5}\right)^2 \end{cases}$$

Therefore, $\|\mathcal{T}(x, y) - \mathcal{T}(x', y')\| = \sqrt{(x - x')^2 + (y - y')^2} = \|(x, y) - (x', y')\|$. Hence $\mathcal{T}$ is relatively nonexpansive mapping.

- **Case 2:** When $(x', y') \in \mathcal{C}$ and $(x, y) \in \mathcal{D}$. Then,

$$\begin{aligned} \|\mathcal{T}(x', y') - \mathcal{T}(x, y)\| &= \left\| \left(\frac{x' - 4}{5}, \frac{y'}{5}\right) - \left(\frac{x + 4}{5}, \frac{y}{5}\right) \right\| \\ &= \left\| \frac{x' - x - 8}{5}, \frac{y - y'}{5} \right\| \\ &= \sqrt{\left(\frac{x' - x - 8}{5}\right)^2 + \left(\frac{y - y'}{5}\right)^2} \\ &\leq \sqrt{(x' - x)^2 + (y - y')^2}. \end{aligned}$$

$$\text{Since, } x - x' \geq 2 \text{ and } \begin{cases} x' - x > x' - x - 8 \\ \left(\frac{x' - x}{5}\right)^2 > \left(\frac{x' - x - 8}{5}\right)^2 \\ (x' - x)^2 > \left(\frac{x' - x}{5}\right)^2 > \left(\frac{x' - x - 8}{5}\right)^2 \\ (x' - x)^2 > \left(\frac{x' - x - 8}{5}\right)^2 \end{cases}$$

Then $\|\mathcal{T}(x', y') - \mathcal{T}(x, y)\| \leq \|(x', y') - (x, y)\|$. Hence, $\mathcal{T}$ is a relatively nonexpansive mapping.

- **Case 3:** When $(x, y) \in \mathcal{C}$ and $(x', y') \in \mathcal{C}$. Then,

$$\begin{aligned} \|\mathcal{T}(x, y) - \mathcal{T}(x', y')\| &= \left\| \left(\frac{x - 4}{5}, \frac{y}{5}\right) - \left(\frac{x' - 4}{5}, \frac{y'}{5}\right) \right\| \\ &= \left\| \frac{x - x'}{5}, \frac{y - y'}{5} \right\| \\ &= \sqrt{\left(\frac{x - x'}{5}\right)^2 + \left(\frac{y - y'}{5}\right)^2} \\ &\leq \sqrt{(x - x')^2 + (y - y')^2}. \end{aligned}$$

Then $\|\mathcal{T}(x, y) - \mathcal{T}(x', y')\| \leq \|(x, y) - (x', y')\|$. So the mapping is nonexpansive, hence by the theorem, it gives the fixed point.

- **Case 4:** When $(x, y) \in \mathcal{D}$ and $(x', y') \in \mathcal{D}$. Then,

$$\begin{aligned}
 \|\mathcal{T}(x, y) - \mathcal{T}(x', y')\| &= \left\| \left(\frac{x+4}{5}, \frac{y}{5} \right) - \left(\frac{x'+4}{5}, \frac{y'}{5} \right) \right\| \\
 &= \left\| \frac{x-x'}{5}, \frac{y-y'}{5} \right\| \\
 &= \sqrt{\left(\frac{x-x'}{5} \right)^2 + \left(\frac{y-y'}{5} \right)^2} \\
 &\leq \sqrt{(x-x')^2 + (y-y')^2}.
 \end{aligned}$$

Then $\|\mathcal{T}(x, y) - \mathcal{T}(x', y')\| \leq \|(x, y) - (x', y')\|$. So the mapping is nonexpansive, hence by the theorem, it gives the fixed point.

Now set $\alpha_n = \frac{1}{3+n}$, $\beta_n = \frac{1}{5+n}$, $\gamma_n = \frac{1}{7+n}$, using MATLAB coding, we give the following Table 1 and Table 2. These shows that for $(x, y) \in \mathcal{C}$, iteration (3.1) converges to a point $u = (-1, 0)$ and for $(x', y') \in \mathcal{D}$ iteration (3.1) converges to a point $v = (1, 0)$. Therefore, we get the best proximity pair (u, v) such that $\mathcal{T}u = u$ and $\mathcal{T}v = v$.

Table 1: This table shows that the values for $(x, y) \in \mathcal{C}$

1	(-1.1000000000, 0.1000000000)	(-1.5000000000, 0.5000000000)	(-1.9000000000, 0.9000000000)
2	(-1.0158666667, 0.0158666667)	(-1.0793333333, 0.0793333333)	(-1.1428000000, 0.1428000000)
3	(-1.0028263822, 0.0028263822)	(-1.0141319111, 0.0141319111)	(-1.0254374400, 0.0254374400)
4	(-1.0005268376, 0.0005268376)	(-1.0026341882, 0.0026341882)	(-1.0047415388, 0.0047415388)
5	(-1.0001004704, 0.0001004704)	(-1.0005023522, 0.0005023522)	(-1.0009042340, 0.0009042340)
6	(-1.0000194147, 0.0000194147)	(-1.0000970736, 0.0000970736)	(-1.0001747325, 0.0001747325)
7	(-1.0000037831, 0.0000037831)	(-1.0000189155, 0.0000189155)	(-1.0000340479, 0.0000340479)
8	(-1.0000007413, 0.0000007413)	(-1.0000037066, 0.0000037066)	(-1.0000066719, 0.0000066719)
9	(-1.0000001458, 0.0000001458)	(-1.0000007292, 0.0000007292)	(-1.0000013126, 0.0000013126)
10	(-1.0000000288, 0.0000000288)	(-1.0000001439, 0.0000001439)	(-1.0000002590, 0.0000002590)
11	(-1.0000000057, 0.0000000057)	(-1.0000000285, 0.0000000285)	(-1.0000000512, 0.0000000512)
12	(-1.0000000011, 0.0000000011)	(-1.0000000056, 0.0000000056)	(-1.0000000101, 0.0000000101)
13	(-1.0000000002, 0.0000000002)	(-1.0000000011, 0.0000000011)	(-1.0000000020, 0.0000000020)
14	(-1.0000000000, 0.0000000000)	(-1.0000000002, 0.0000000002)	(-1.0000000004, 0.0000000004)
15	(-1.0000000000, 0.0000000000)	(-1.0000000000, 0.0000000000)	(-1.0000000001, 0.0000000001)
16	(-1.0000000000, 0.0000000000)	(-1.0000000000, 0.0000000000)	(-1.0000000000, 0.0000000000)
17	(-1.0000000000, 0.0000000000)	(-1.0000000000, 0.0000000000)	(-1.0000000000, 0.0000000000)
18	(-1.0000000000, 0.0000000000)	(-1.0000000000, 0.0000000000)	(-1.0000000000, 0.0000000000)
19	(-1.0000000000, 0.0000000000)	(-1.0000000000, 0.0000000000)	(-1.0000000000, 0.0000000000)
20	(-1.0000000000, 0.0000000000)	(-1.0000000000, 0.0000000000)	(-1.0000000000, 0.0000000000)

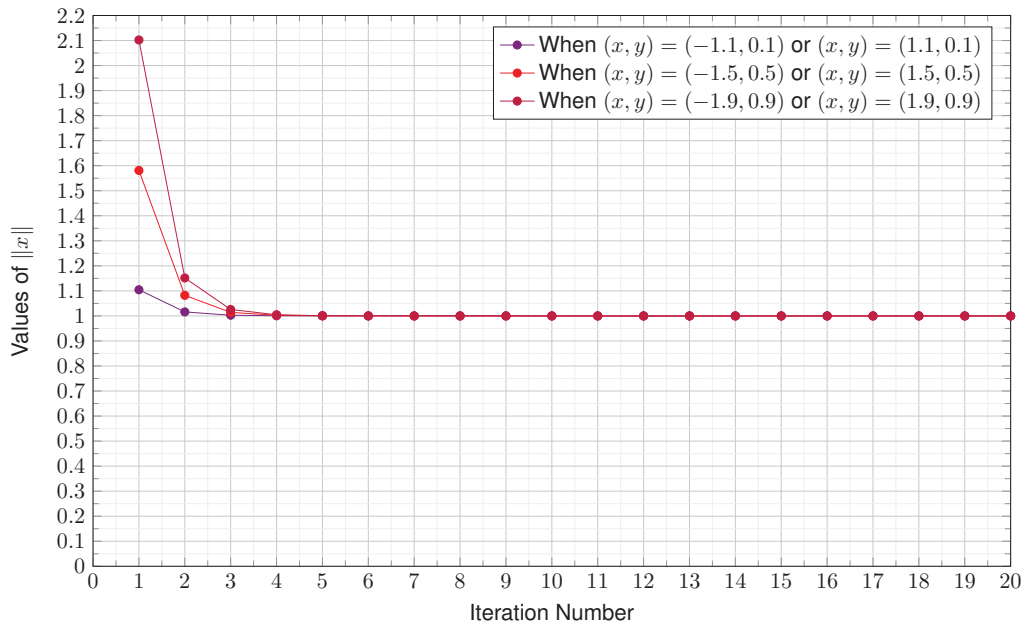
Table 2: This table shows that the values for $(x, y) \in \mathcal{D}$

1	(1.1000000000 , 0.1000000000)	(1.5000000000 , 0.5000000000)	(1.9000000000 , 0.9000000000)
2	(1.0158666667 , 0.0158666667)	(1.0793333333 , 0.0793333333)	(1.1428000000 , 0.1428000000)
3	(1.0028263822 , 0.0028263822)	(1.0141319111 , 0.0141319111)	(1.0254374400 , 0.0254374400)
4	(1.0005268376 , 0.0005268376)	(1.0026341882 , 0.0026341882)	(1.0047415388 , 0.0047415388)
5	(1.0001004704 , 0.0001004704)	(1.0005023522 , 0.0005023522)	(1.0009042340 , 0.0009042340)
6	(1.0000194147 , 0.0000194147)	(1.0000970736 , 0.0000970736)	(1.0001747325 , 0.0001747325)
7	(1.0000037831 , 0.0000037831)	(1.0000189155 , 0.0000189155)	(1.0000340479 , 0.0000340479)
8	(1.0000007413 , 0.0000007413)	(1.0000037066 , 0.0000037066)	(1.0000066719 , 0.0000066719)
9	(1.0000001458 , 0.0000001458)	(1.0000007292 , 0.0000007292)	(1.0000013126 , 0.0000013126)
10	(1.0000000288 , 0.0000000288)	(1.0000001439 , 0.0000001439)	(1.0000002590 , 0.0000002590)
11	(1.0000000057 , 0.0000000057)	(1.0000000285 , 0.0000000285)	(1.0000000512 , 0.0000000512)
12	(1.0000000011 , 0.0000000011)	(1.0000000056 , 0.0000000056)	(1.0000000101 , 0.0000000101)
13	(1.0000000002 , 0.0000000002)	(1.0000000011 , 0.0000000011)	(1.0000000020 , 0.0000000020)
14	(1.0000000000 , 0.0000000000)	(1.0000000002 , 0.0000000002)	(1.0000000004 , 0.0000000004)
15	(1.0000000000 , 0.0000000000)	(1.0000000000 , 0.0000000000)	(1.0000000001 , 0.0000000001)
16	(1.0000000000 , 0.0000000000)	(1.0000000000 , 0.0000000000)	(1.0000000000 , 0.0000000000)
17	(1.0000000000 , 0.0000000000)	(1.0000000000 , 0.0000000000)	(1.0000000000 , 0.0000000000)
18	(1.0000000000 , 0.0000000000)	(1.0000000000 , 0.0000000000)	(1.0000000000 , 0.0000000000)
19	(1.0000000000 , 0.0000000000)	(1.0000000000 , 0.0000000000)	(1.0000000000 , 0.0000000000)
20	(1.0000000000 , 0.0000000000)	(1.0000000000 , 0.0000000000)	(1.0000000000 , 0.0000000000)

Table 3: This table shows that the influence of initial points

S.no.	Parameters	$(-1.1, 0.1) \in \mathcal{C}$ or $(1.1, 0.1) \in \mathcal{D}$	$(-1.5, 0.5) \in \mathcal{C}$ or $(1.5, 0.5) \in \mathcal{D}$	$(-1.9, 0.9) \in \mathcal{C}$ or $(1.9, 0.9) \in \mathcal{D}$
1	$\alpha_n = \frac{1}{3+n}, \beta_n = \frac{1}{2+n}, \gamma_n = \frac{1}{1+n}$	14	15	16
2	$\alpha_n = \frac{1}{3n+1}, \beta_n = \frac{1}{2n+1}, \gamma_n = \frac{1}{n+1}$	14	15	15
3	$\alpha_n = \frac{1}{n}, \beta_n = \frac{1}{n}, \gamma_n = \frac{1}{n}$	11	12	13

Graph Corresponding to Table 3.4



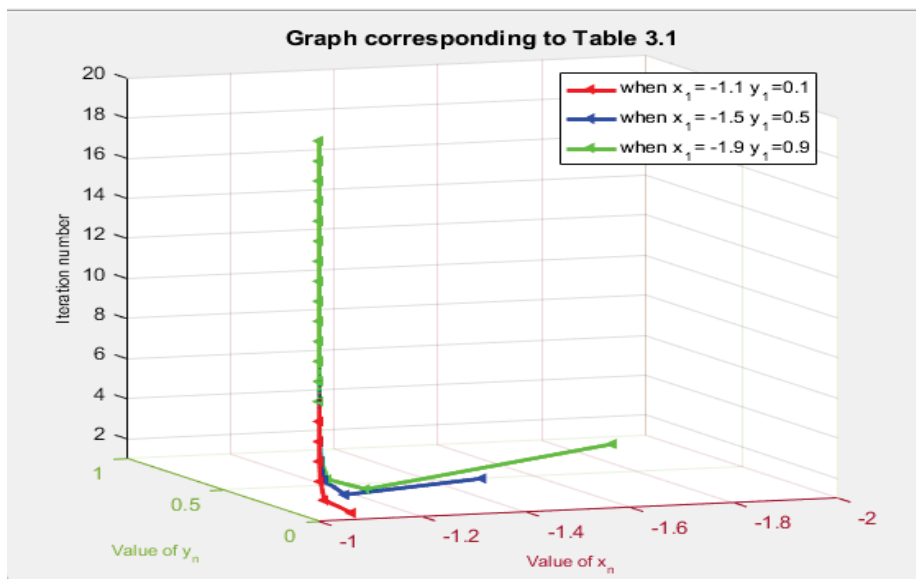


Figure 1: The graph represent the value of $(x, y) \in \mathcal{C}$

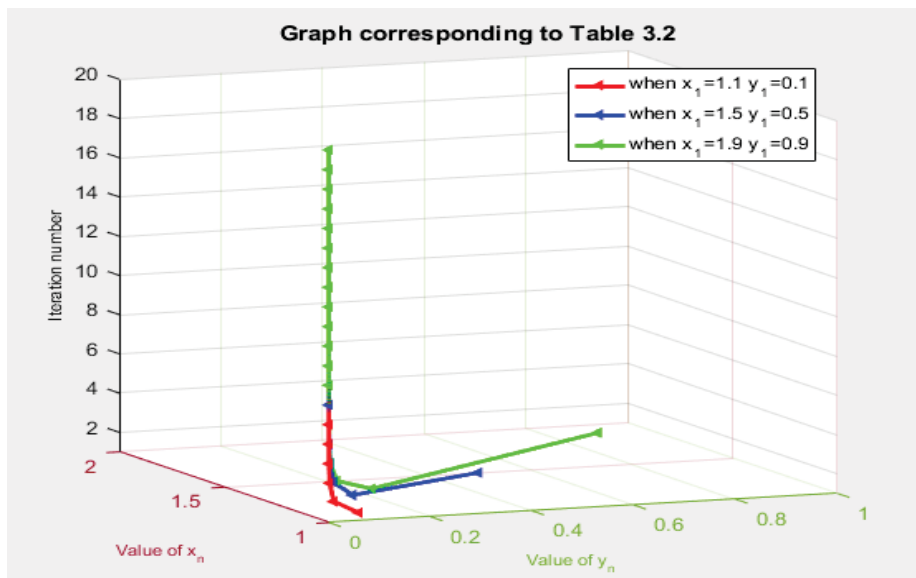


Figure 2: The graph represent the value of $(x, y) \in \mathcal{D}$

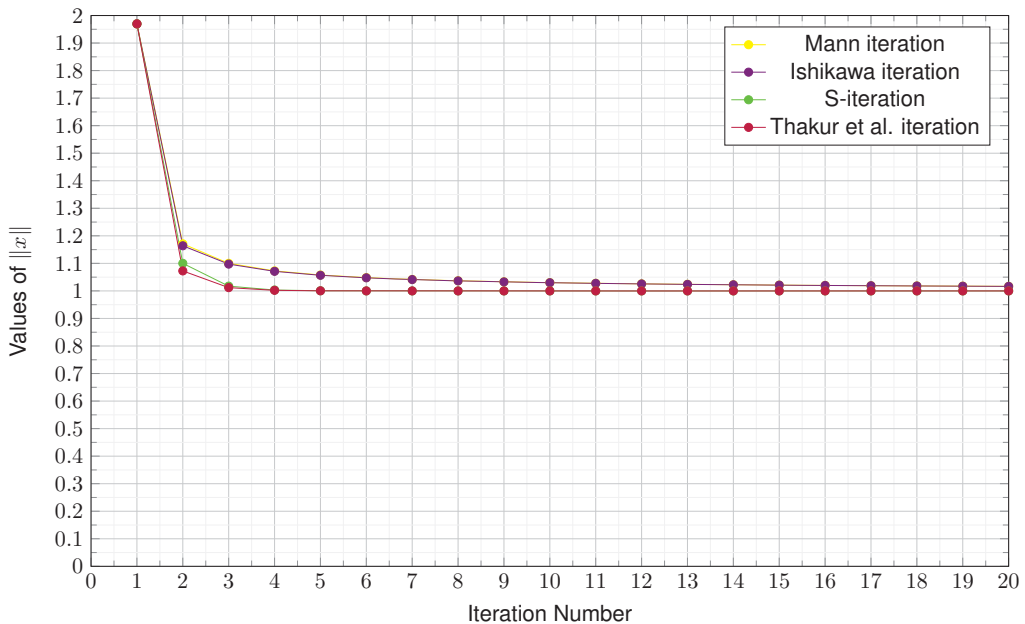
Table 4: If $\mathfrak{z} = (x, y)$ then values of the $\|\mathfrak{z}\|$ corresponding to different initial values.

step	when $(x, y) = (-1.1, 0.1)$	when $(x, y) = (-1.5, 0.5)$	when $(x, y) = (-1.9, 0.9)$
	or when $(x, y) = (1.1, 0.1)$	or when $(x, y) = (1.5, 0.5)$	or when $(x, y) = (1.9, 0.9)$
1	1.1045361017	1.5811388301	2.1023796042
2	1.0159905687	1.0822449917	1.1516873187
3	1.0028303652	1.0142303703	1.0257528975
4	1.0005685374	1.0026376486	1.0047303507
5	1.0001004521	1.0005024783	1.0009046425
6	1.0000194149	1.0000970783	1.0001747478
7	1.0000037831	1.0000189157	1.0000340485
8	1.0000007413	1.0000037066	1.0000066719
9	1.0000001458	1.0000007292	1.0000013126
10	1.0000000288	1.0000001439	1.0000002590
11	1.0000000057	1.0000000285	1.0000000512
12	1.0000000011	1.0000000056	1.0000000101
13	1.0000000002	1.0000000011	1.0000000020
14	1.0000000000	1.0000000002	1.0000000004
15	1.0000000000	1.0000000000	1.0000000001
16	1.0000000000	1.0000000000	1.0000000000
17	1.0000000000	1.0000000000	1.0000000000
18	1.0000000000	1.0000000000	1.0000000000
19	1.0000000000	1.0000000000	1.0000000000
20	1.0000000000	1.0000000000	1.0000000000

Table 5: Values of the $\|\mathfrak{z}\|$ corresponding to different iteration process, where $\alpha = \frac{1}{n}, \beta = \frac{1}{2n}, \gamma = \frac{1}{3n}$ and initial value of $\mathfrak{z}$ is $(-1.8, 0.8)$ and $(1.8, 0.8)$ for $\mathcal{C}$ and $\mathcal{D}$ respectively.

Step	Mann	Ishikawa	S-iteration	Thakur et al.
1	1.9697715604	1.96977156	1.9697715604	1.9697715604
2	1.1709824935	1.163965635	1.1001963461	1.0727125990
3	1.1001963461	1.097413442	1.0174267525	1.0119912831
4	1.0727125990	1.071159764	1.0033078350	1.0022194762
5	1.0578203462	1.056806651	1.0006441752	1.0004257592
6	1.0483767668	1.047652524	1.0001267409	1.0000829396
7	1.0418080809	1.04125938	1.0000250653	1.0000162884
8	1.0369512285	1.036518035	1.0000049721	1.0000032143
9	1.0332006984	1.032848094	1.0000009882	1.0000006363
10	1.0302088772	1.029915034	1.0000001967	1.0000001262
11	1.0277613410	1.027511843	1.0000000392	1.0000000251
12	1.0257183236	1.025503231	1.0000000078	1.0000000050
13	1.0239846504	1.023796863	1.0000000016	1.0000000010
14	1.0224931658	1.022327463	1.0000000003	1.0000000002
15	1.0211950672	1.021047516	1.0000000001	1.0000000000
16	1.0200539959	1.019921571	1.0000000000	1.0000000000
17	1.0190422830	1.018922615	1.0000000000	1.0000000000
18	1.0181384817	1.018029687	1.0000000000	1.0000000000
19	1.0173256991	1.017226257	1.0000000000	1.0000000000
20	1.0165904399	1.016499111	1.0000000000	1.0000000000

Graph Corresponding to Table 3.6



4 Conclusions

In this paper, we extend the applicability of Thakur et al. iterative process (3.1) for best proximity pair for noncyclic relatively nonexpansive mappings in the setting of uniformly convex Banach spaces. We established strong convergence using this iterative process under appropriate conditions. To validate our theoretical findings, we provided a numerical example that illustrates the convergence properties of iterative process. Furthermore, we show that the Thakur et al. iteration process (3.1) is faster than the Mann, Ishikawa and S-iteration processes for noncyclic relatively nonexpansive mappings.

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6 Declaration

- Conflicts of interest: The authors declare that they have no conflict of interest.
- Human participants: This article does not contain any studies with human participants or animals performed by the authors.

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