

On Single Server Feedback Retrial Queues with Batch Departures

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ABSTRACT

The present paper discusses a two-state single server retrial queueing model with feedback and batch departures. If an arriving customer finds the server free, it is served immediately else it joins the orbit. Both type of customers follow the Poisson arrival process. The customers depart from the system in batches of random size. Also, if a departing batch has unsatisfied customers, it may join the orbit with a certain probability to receive a satisfied service. The transient state probabilities are obtained by solving the difference-differential equations recursively. Numerical and graphical illustrations are also given.

Keywords: queueing, retrial, feedback, batch departures.

2010 Mathematics Subject Classification: 60K25, 90B22, 60M20.

1 Introduction

In day-to-day life, queueing is a prevalent phenomenon. Retrial queueing models form a tiny subset of the huge variety of models of queueing theory that has been studied during the past century. These models are used as a powerful tool for modeling communication networks, operating systems, etc. In retrial queueing systems, if a customer at arrival epoch finds the server free, it is served immediately. Else, it is obliged to join the orbit or pool. The secondary customers retry for service after a random amount of time. In order to have a brief knowledge regarding retrial queueing theory (Falín, 1990), (Falín and Templeton, 1997), (Artalejo and Gómez-Corral, 1999) and (Artalejo and Gómez-Corral, 2008) can be referred.

Extensive work has been done on queueing models with batch arrivals, batch departures, and both. In industrial problems, bulk or batch queues occur frequently. For example, raw material arrives in bulk or in batches of different sizes. The finished products also depart in batches for further quality checks.

The pioneering work on bulk queueing models was done by (Bailey, 1954) in his paper 'On queueing processes with bulk service' where the equilibrium distribution of queue length has

been studied by the imbedded Markov chain method, and ergodicity is established when the average demand is less than the average supply available. Further, it was followed by (Downton, 1955) in the paper 'Waiting time in bulk service queues' where the Laplace transform of the waiting time distribution is derived from the probability generating function of the queue length distribution. (Choudhury, Tadj and Deka, 2010) worked on 'A batch arrival retrial queueing system with two phases of service and service interruption.' (Sasikala and Indhira, 2016) published 'Bulk service queueing models- a survey' where an overview about bulk service queueing models is provided.

In some circumstances customers after receiving service, feel unsatisfied. These units may join the orbit again in order to receive satisfactory service. This phenomenon of rejoining orbit to get a satisfied service is called feedback. (Atencia, Fortes and Sánchez, 2009) worked on 'A discrete-time retrial queueing system with starting failures, Bernoulli feedback and general retrial times' generalization of the previous works are done. (Upadhyaya, 2014) analyzed 'Performance analysis of a batch arrival retrial queue with Bernoulli feedback'. (Pegden and Rosenshine, 1982) analyzed 'Some new results for the M/M/1 queue' in which solution is obtained for the probability that exactly i number of arrivals, j number of services occur over a time interval t . (Kalra et al., 2019) studied 'A Two-State Retrial Queueing Model with Batch Departures.'

The present paper is divided in various sections as follows:

Section 2 gives the mathematical description of the model along with the difference-differential equations. In section 3, the transient state probabilities are obtained. Verification of results is done in section 4. Numerical and graphical representations are provided in section 5. In section 6, the busy period probabilities are given. Finally, the paper is concluded in section 7.

2 Model Description

We considered a two-state retrial queueing model with feedback and batch departures. The primary or fresh customers arrive at the system following the Poisson process. On arrival, if the entering customer encounters the server free, it is served immediately else joins the orbit and retries for service after a random amount of time as a repeated call or secondary customer. The secondary customers also follow the Poisson process. Service times follow Exponential distribution. The departures take place in batches of irregular size. Also, after service, the unsatisfied units may join the orbit again in order to receive satisfying service.

- Arrival Process: The primary calls arrive at the system following Poisson process with mean arrival rate λ .
- Retrial Process: The secondary calls from the orbit also arrive at the system according to Poisson process with mean retrial rate θ .
- Service Process: In the present model the arrivals occur singly whereas the departures take place in batches of irregular size with rate μ .

The server's capacity is a random variable and is determined at the beginning of each service. The batch size is either equal to the server's capacity that is decided afresh

before each service or equal to the total number of units, whichever is less. The server can serve ν units with probability b_ν such that $\sum_{\nu=1}^k b_\nu=1$, where k is the maximum capacity of the server.

- **Feedback Rule:** After receiving service, the customer joins the orbit with probability γ (i.e., when unsatisfied) and departs from the system with probability $1-\gamma$.

The input flow of primary calls, intervals between repetitions, service times are statistically independent.

Laplace Transformation of $\bar{f}(s)$ of $f(t)$ is given by:

$$\bar{f}(s) = \int_0^\infty e^{-st} f(t) dt; \quad \operatorname{Re}(s) > 0$$

The Laplace Inverse of

$$\frac{Q(p)}{P(p)} = \sum_{k=1}^n \sum_{l=1}^{m_k} \frac{t^{m_k-l} e^{a_k t}}{(m_k-l)!(l-1)!} \times \left(\frac{d}{dp} \right)^{l-1} \frac{Q(p)}{P(p)} \quad \forall p = a_k, a_i \neq a_k \text{ for } i \neq k$$

where

$$P(p) = (p - a_1)^{m_1} (p - a_2)^{m_2} \dots (p - a_n)^{m_n}$$

$$Q(p) \text{ is a polynomial of degree } < m_1 + m_2 + m_3 + \dots + m_n - 1$$

The Laplace Inverse of

$$\begin{aligned} \bar{N}_{n_1, n_2, n_3}^{a, b, c}(s) &= \frac{1}{(s+a)^{n_1} + (s+b)^{n_2} + (s+c)^{n_3}} \text{ is} \\ N_{n_1, n_2, n_3}^{a, b, c}(t) &= \sum_{l=1}^{n_3} \sum_{m=1}^l \frac{e^{-at} t^{n_3-l} (-1)^{m+1} \binom{l-1}{m-1} \left(\prod_{g_1=0}^{l-m-1} (n_1 + g_1) \right) \left(\prod_{g_2=0}^{m-2} (n_2 + g_2) \right)}{(n_3-l)!(m-1)!(b-a)^{n_2+m-1} (c-a)^{n_1+l-m}} \\ &\quad + \sum_{l=1}^{n_2} \sum_{m=1}^l \frac{e^{-bt} t^{n_2-l} (-1)^{m+1} \binom{l-1}{m-1} \left(\prod_{g_1=0}^{l-m-1} (n_1 + g_1) \right) \left(\prod_{g_2=0}^{m-2} (n_3 + g_2) \right)}{(n_2-l)!(m-1)!(a-b)^{n_3+m-1} (c-b)^{n_1+l-m}} \\ &\quad + \sum_{l=1}^{n_1} \sum_{m=1}^l \frac{e^{-ct} t^{n_1-l} (-1)^{m+1} \binom{l-1}{m-1} \left(\prod_{g_1=0}^{l-m-1} (n_2 + g_1) \right) \left((n_3 + g_2) \right)}{(n_1-l)!(m-1)!(a-c)^{n_3+m-1} (b-c)^{n_2+l-m}} \end{aligned}$$

If $L^{-1}\{f(s)\} = F(t)$ and $L^{-1}\{g(s)\} = G(t)$, then

$$L^{-1}\{f(s)g(s)\} = \int_0^1 F(u)G(t-u)du = F * G,$$

$F * G$ is called the convolution of F and G .

2.1 The Difference-Differential Equations Governing the System are:

$$\frac{d}{dt} P_{i,i,0}(t) = -\lambda P_{i,i,0}(t) + \mu(1-\gamma) \sum_{\nu=1}^k \left(\sum_{l=\nu}^k b_l \right) P_{i,i-\nu,1}(t); \quad i \geq 0, \quad i \geq k \quad (2.1)$$

$$\frac{d}{dt} P_{i,j,0}(t) = -(\lambda + (i-j)\theta) P_{i,j,0}(t) + \mu(1-\gamma) \sum_{\nu=1}^k b_\nu P_{i,j-\nu,1}(t) + \mu\gamma \sum_{\nu=1}^k b_\nu P_{i,j,1}(t);$$

$$i > j, i > 0, j \geq k \quad (2.2)$$

$$\frac{d}{dt}P_{1,0,1}(t) = -(\lambda + \mu)P_{1,0,1}(t) + \lambda P_{0,0,0}(t) + \theta P_{1,0,0}(t) \quad (2.3)$$

$$\begin{aligned} \frac{d}{dt}P_{i,j,1}(t) &= -(\lambda + \mu)P_{i,j,1}(t) + \lambda P_{i-1,j,0}(t) + \lambda(1 - \delta_{i-1,j})P_{i-1,j,1}(t) + (i - j)\theta P_{i,j,0}(t); \\ i > 1, i > j \geq 0 \end{aligned} \quad (2.4)$$

where

$$\delta_{i-1,j} = \begin{cases} 1; & i - 1 = j \\ 0 & \text{otherwise} \end{cases}$$

Using Laplace Transform $\bar{f}(s)$ of $f(t)$ is given by

$$\bar{f}(s) = \int_0^\infty e^{-st} f(t) dt; \quad \operatorname{Re}(s) > 0$$

and using initial condition in equations (1) to (4), we have

$$(s + \lambda)\bar{P}_{i,0}(s) = \mu(1 - \gamma) \sum_{\nu=1}^k \left(\sum_{l=\nu}^k b_l \right) \bar{P}_{i,i-\nu,1}(s); \quad i \geq 0, i \geq k \quad (2.5)$$

$$\begin{aligned} (s + \lambda + (i - j)\theta)\bar{P}_{i,j,0}(s) &= \mu(1 - \gamma) \sum_{\nu=1}^k b_\nu \bar{P}_{i,j-\nu,1}(s) + \mu\gamma \sum_{\nu=1}^k b_\nu \bar{P}_{i,j,1}(s); \\ i > j, i > 0, j \geq k \end{aligned} \quad (2.6)$$

$$(s + \lambda + \mu)\bar{P}_{1,0,1}(s) = \lambda\bar{P}_{0,0,0}(s) + \theta\bar{P}_{1,0,0}(s) \quad (2.7)$$

$$\begin{aligned} (s + \lambda + \mu)\bar{P}_{i,j,1}(s) &= \lambda\bar{P}_{i-1,j,0}(s) + \lambda(1 - \delta_{i-1,j})\bar{P}_{i-1,j,1}(s) + (i - j)\theta\bar{P}_{i,j,0}(s); \\ i > 1, i > j \geq 0 \end{aligned} \quad (2.8)$$

3 Solution of the Problem

Solving equations (5) to (8) recursively, we have

$$\bar{P}_{0,0,0}(s) = \frac{1}{s + \lambda} \quad (3.1)$$

$$\bar{P}_{1,0,1}(s) = \frac{\lambda}{s + \lambda + \mu} \left(\frac{1}{s + \lambda} \right) + \frac{\theta}{s + \lambda + \mu} \bar{P}_{1,0,0}(s) \quad (3.2)$$

$$\bar{P}_{1,1,0}(s) = \frac{\mu(1 - \gamma)b_1}{s + \lambda} \left[\frac{\lambda}{s + \lambda + \mu} \left(\frac{1}{s + \lambda} \right) + \frac{\theta}{s + \lambda + \mu} \bar{P}_{1,0,0}(s) \right] \quad (3.3)$$

$$\bar{P}_{i,i,0}(s) = \frac{\mu(1 - \gamma)}{s + \lambda} \sum_{\nu=1}^k \left(\sum_{l=\nu}^k b_l \right) \bar{P}_{i,i-\nu,1}(s); \quad i \geq 2 \quad (3.4)$$

$$\bar{P}_{i,1,0}(s) = \frac{\mu(1 - \gamma)b_1}{s + \lambda + (i - 1)\theta} \bar{P}_{i,0,1}(s) + \frac{\mu\gamma b_1}{s + \lambda + (i - 1)\theta} \bar{P}_{i,1,1}(s); \quad i \geq 2 \quad (3.5)$$

$$\begin{aligned} \bar{P}_{i,i-1,0}(s) &= \frac{\mu(1 - \gamma)}{s + \lambda + \theta} \sum_{\nu=1}^k b_\nu \bar{P}_{i,i-2,1}(s) + \frac{\mu\gamma}{s + \lambda + \theta} \sum_{\nu=1}^k b_\nu \bar{P}_{i,i-1,1}(s); \\ i &\geq 3, i - 1 \geq k \end{aligned} \quad (3.6)$$

$$\bar{P}_{i,0,1}(s) = \frac{\lambda}{s + \lambda + \mu} \bar{P}_{i-1,0,0}(s) + \frac{\lambda}{s + \lambda + \mu} \bar{P}_{i-1,0,1}(s) + \frac{i\theta}{s + \lambda + \mu} \bar{P}_{i,0,0}(s);$$

$$i \geq 2 \quad (3.7)$$

$$\bar{P}_{i,i-1,1}(s) = \frac{\lambda}{s + \lambda + \mu} \bar{P}_{i-1,i-1,0}(s) + \frac{\theta}{s + \lambda + \mu} \bar{P}_{i,i-1,0}(s); \quad i \geq 2 \quad (3.8)$$

$$\bar{P}_{i,j,1}(s) = \sum_{k=1}^{i-j} \left(\frac{\lambda}{s + \lambda + \mu} \right)^{i-j-k} \eta'_k(s) \bar{P}_{j+k,j,0}(s) + \left(\frac{\lambda}{s + \lambda + \mu} \right)^{i-j-1} \bar{P}_{j+1,j,1}(s);$$

$$i \geq j + 2, j \geq 1 \quad (3.9)$$

where

$$\eta'_k = \begin{cases} 1; & k = 1 \\ 1 + \frac{k\theta}{s + \lambda + \mu}; & k = 2 \text{ to } i - j - 1 \\ \frac{k\theta}{s + \lambda + \mu}; & k = i - j \end{cases}$$

$$\bar{P}_{i,j,0}(s) = \frac{\mu(1-\gamma)}{s + \lambda + (i-j)\theta} \sum_{\nu=1}^k b_{\nu} \bar{P}_{i,j-\nu,1}(s) + \frac{\mu\gamma}{s + \lambda + (i-j)\theta} \sum_{\nu=1}^k b_{\nu} \bar{P}_{i,j,1}(s);$$

$$i > j > 1 \quad (3.10)$$

Taking the Laplace inverse of (9)-(19), we get the transient state probabilities as:

$$P_{0,0,0}(t) = e^{-\lambda t} \quad (3.11)$$

$$P_{1,0,1}(t) = \lambda e^{-\lambda t} \left\{ \frac{1}{\mu} - \frac{e^{-\mu t}}{\mu} \right\} + \theta e^{-(\lambda+\mu)t} * P_{1,0,0}(t) \quad (3.12)$$

$$P_{1,1,0}(t) = \mu(1-\gamma)b_1\lambda t e^{-\lambda t} \left\{ \frac{1}{\mu} - \frac{e^{-\mu t}}{\mu} \right\} + \mu(1-\gamma)b_1\theta e^{-\lambda t} \left\{ \frac{1}{\mu} - \frac{e^{-\mu t}}{\mu} \right\} * P_{1,0,0}(t) \quad (3.13)$$

$$P_{i,i,0}(t) = \mu(1-\gamma) \sum_{\nu=1}^k \left(\sum_{l=\nu}^k b_l \right) e^{-\lambda t} * P_{i,i-\nu,1}(t); \quad i \geq 2 \quad (3.14)$$

$$P_{i,1,0}(t) = \mu(1-\gamma)b_1 e^{-(\lambda+(i-1)\theta)t} * P_{i,0,1}(t) + \mu\gamma b_1 e^{-(\lambda+(i-1)\theta)t} * P_{i,1,1}(t);$$

$$i \geq 2 \quad (3.15)$$

$$P_{i,i-1,0}(t) = \mu(1-\gamma) \sum_{\nu=1}^k b_{\nu} e^{-(\lambda+\theta)t} * P_{i,i-2,1}(t) + \mu\gamma \sum_{\nu=1}^k b_{\nu} e^{-(\lambda+\theta)t} * P_{i,i-1,1}(t);$$

$$i \geq 3, i-1 \geq k \quad (3.16)$$

$$P_{i,0,1}(t) = \lambda e^{-(\lambda+\mu)t} * P_{i-1,0,0}(t) + \lambda e^{-(\lambda+\mu)t} * P_{i-1,0,1}(t) + i\theta e^{-(\lambda+\mu)t} * P_{i,0,0}(t);$$

$$i \geq 2 \quad (3.17)$$

$$P_{i,i-1,1}(t) = \lambda e^{-(\lambda+\mu)t} * P_{i-1,i-1,0}(t) + \theta e^{-(\lambda+\mu)t} * P_{i,i-1,0}(t); \quad i \geq 2 \quad (3.18)$$

$$P_{i,j,1}(t) = \lambda^{i-j-1} \frac{t^{i-j-2}}{(i-j-2)!} e^{-(\lambda+\mu)t} * P_{j,j,0}(t) +$$

$$\sum_{k=2}^{i-j-1} \lambda^{i-j-k} \frac{t^{i-j-k-1}}{(i-j-k-1)!} e^{-(\lambda+\mu)t} * P_{j+k,j,0}(t)$$

$$\begin{aligned}
 & + \sum_{k=2}^{i-j-1} \lambda^{i-j-k} k \theta \frac{t^{i-j-k}}{(i-j-k)!} e^{-(\lambda+\mu)t} * P_{j+k,j,0}(t) + \\
 & (i-j)\theta e^{-(\lambda+\mu)t} * P_{i,j,0}(t) + \lambda^{i-j-1} \frac{t^{i-j-2}}{(i-j-2)!} e^{-(\lambda+\mu)t} * P_{j+1,j,1}(t); \\
 & i \geq j+2, j \geq 1
 \end{aligned} \tag{3.19}$$

$$\begin{aligned}
 P_{i,j,0}(t) &= \mu(1-\gamma) \left(\sum_{\nu=1}^k b_{\nu} \right) e^{-(\lambda+(i-j)\theta)t} * P_{i,j-\nu,1}(t) + \mu\gamma \left(\sum_{\nu=1}^k b_{\nu} \right) \\
 & e^{-(\lambda+(i-j)\theta)t} * P_{i,j,1}(t); \quad i > j > 1
 \end{aligned} \tag{3.20}$$

4 Verification of Results

- The Laplace transform of $\bar{P}_i(s)$ is given as:

$$\bar{P}_i(s) = \sum_{j=0}^i \bar{P}_{i,j}(s) = \frac{\lambda^i}{(s+\lambda)^{i+1}}; \quad i > 0$$

and its Inverse Laplace is given by:

$$P_i(t) = \frac{e^{-\lambda t} (\lambda t)^i}{i!}$$

which proves the basic assumption that primary arrivals follow Poisson process.

- The probability that exactly j customers depart from the system by time t is given as:

$$P_{.j}(t) = \sum_{i=j}^{\infty} P_{i,j}(t)$$

- Summing equations (9)-(18) over i and j we get:

$$\sum_{i=0}^{\infty} \sum_{j=0}^i \{ \bar{P}_{i,j,0}(s) + \bar{P}_{i,j,1}(s) \} = \frac{1}{s}$$

and hence

$$\sum_{i=0}^{\infty} \sum_{j=0}^i \{ P_{i,j,0}(t) + P_{i,j,1}(t) \} = 1$$

- Define $Q_{n,m}(t)$ = Probability that there are n customers in the orbit where $m = (0, 1)$ i.e., either the server is idle or busy at time t .

For idle server we represent it by probability $Q_{n,0}(t)$.

$$Q_{n,0}(t) = \sum_{j=0}^{\infty} P_{j+n,j,0}(t)$$

The number of customers in the orbit in this case are calculated with the following formula:

$n = (\text{number of arrivals} - \text{number of departures})$

When the server is busy, it is represented by the probability $Q_{n,1}(t)$

$$Q_{n,1}(t) = \sum_{j=0}^{\infty} P_{j+n+1,j,1}(t)$$

The number of customers in the orbit in this case is calculated by the following formula:

$n = (\text{number of arrivals} - \text{number of departures} - 1)$

Using above definitions in (1)-(4) and letting $\gamma=0$, the equations we get under statistical equilibrium are:

$$\lambda Q_{0,0} = \mu \sum_{\nu=1}^k \left(\sum_{l=\nu}^k b_l \right) Q_{\nu,1} \quad (4.1)$$

$$(\lambda + n\theta)Q_{n,0} = \mu \sum_{\nu=1}^k b_{\nu} Q_{n+\nu,1}; \quad n > 0 \quad (4.2)$$

$$(\lambda + \mu)Q_{n,1} = \lambda Q_{n,0} + \lambda Q_{n-1,1} + (n+1)\theta Q_{n+1,0}; \quad n > 1 \quad (4.3)$$

Let $k = 1$, $b_1 = 1$, $b_2=b_3=b_4=\dots=b_k=0$ in (29)-(31), we get

$$(\lambda + n\theta)Q_{n,0} = \mu Q_{n,1}; \quad n \geq 0 \quad (4.4)$$

$$(\lambda + \mu)Q_{n,1} = \lambda Q_{n,0} + \lambda Q_{n-1,1} + (n+1)\theta Q_{n+1,0}; \quad n > 1 \quad (4.5)$$

which coincides with the result (1.5) and (1.6) of (Falin and Templeton, 1997).

5 Numerical Solution and Graphical Representation

Using MATLAB programming the numerical results are generated for the case $\rho=0.8$, $\eta=0.7$, $\gamma=0.5$, $b_1=0.4$, $b_2=0.3$ and $b_3=0.3$. It is evident from the tables below that the sum of probabilities approaches 1 for various time instants.

The probabilities against time are plotted in following figures. The probability $P_{4,4,0}$ is evaluated for batch departures with $\rho=0.8$, $\eta=0.7$ and $\gamma=0.5$ taking $k=3$ ($b_1=0.4$, $b_2=0.3$ and $b_3=0.3$). The above probability is plotted against time t and compared with the corresponding probability for the case where departures occur singly in figure 5.1. It is evident that the probability values increase initially and after reaching their peaks the values decrease. Also, the probability for batch departures is higher than that of departures occurring singly. The probability $P_{5,4,1}$ is plotted against time t for for batch departures with $\rho=0.8$, $\eta=0.7$ and $\gamma=0.5$ taking $k=3$ ($b_1=0.4$, $b_2=0.3$ and $b_3=0.3$). The above probability is compared with the corresponding probability for the case where departures occur singly in figure 5.2. Initially, the probability values for both the cases start increasing and after attaining the highest values these decrease. The probabilities values for batch departures are higher than that of without batch departures.

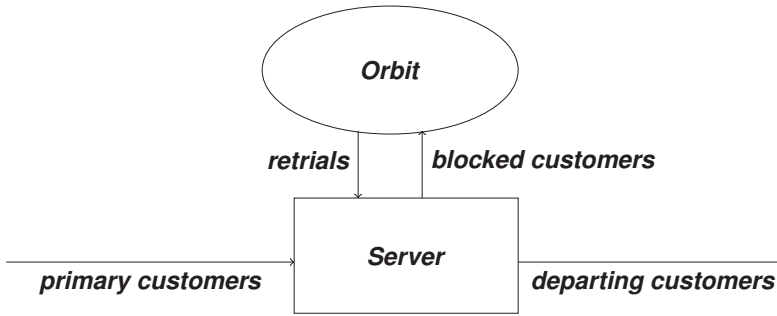


Figure 1: Basic Structure of a Retrial Queueing System

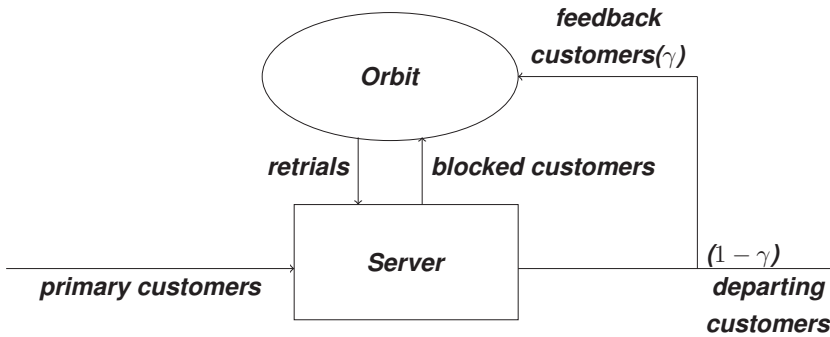


Figure 2: Diagrammatic Representation of a Retrial Queueing System with Feedback

Table 1: At t=1

$P_{0,0,0}$	$P_{1,0,0}$	$P_{1,1,0}$	$P_{2,0,0}$	$P_{2,2,0}$	$P_{1,0,1}$	$P_{2,0,1}$	$P_{2,1,1}$	$P_{3,0,1}$	<i>Sum</i>
0.4493	0.0535	0.0677	0.0124	0.0126	0.2383	0.0947	0.0163	0.025	0.9698

Table 2: At t=10

$P_{2,2,0}$	$P_{3,3,0}$	$P_{4,3,0}$	$P_{4,4,0}$	$P_{5,4,0}$	$P_{5,5,0}$	$P_{6,4,0}$	$P_{6,5,0}$	$P_{6,6,0}$	$P_{7,2,0}$
0.0056	0.0113	0.071	0.017	0.0099	0.0203	0.0078	0.011	0.02	0.0093
$P_{7,3,0}$	$P_{7,4,0}$	$P_{7,5,0}$	$P_{7,6,0}$	$P_{7,7,0}$	$P_{4,2,1}$	$P_{4,3,1}$	$P_{5,2,1}$	$P_{5,3,1}$	$P_{5,4,1}$
0.018	0.0294	0.0447	0.0664	0.1536	0.0091	0.0098	0.0117	0.0148	0.0131
$P_{6,2,1}$	$P_{6,3,1}$	$P_{6,4,1}$	$P_{6,5,1}$	$P_{7,0,1}$	$P_{7,1,1}$	$P_{7,2,1}$	$P_{7,3,1}$	$P_{7,4,1}$	
0.0128	0.0175	0.018	0.0142	0.0142	0.0291	0.0477	0.0686	0.0766	
$P_{7,5,1}$	$P_{7,6,1}$	<i>Sum</i>							
0.0722	0.051	0.9757							

Table 3: At t=20

$P_{7,3,0}$	$P_{7,4,0}$	$P_{7,5,0}$	$P_{7,6,0}$	$P_{7,7,0}$	$P_{7,1,1}$	$P_{7,2,1}$	$P_{7,3,1}$	$P_{7,4,1}$
0.0024	0.0068	0.0194	0.0658	0.7778	0.0014	0.0041	0.0098	0.0195

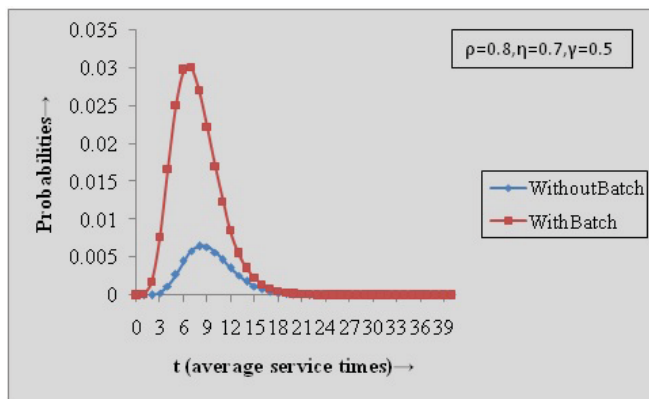
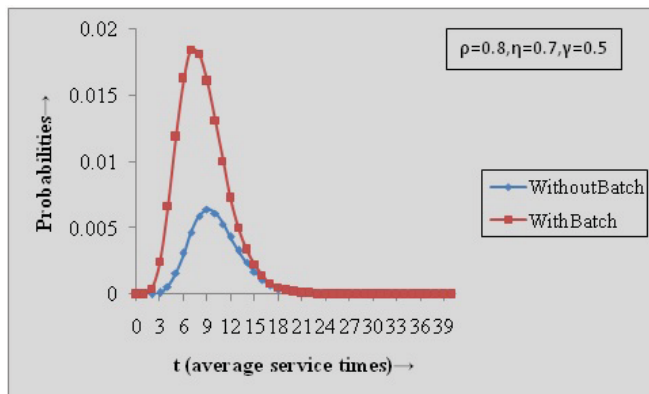
$P_{7,5,1}$	$P_{7,6,1}$	Sum
0.0345	0.053	0.9945

 Table 4: At $t=30$

$P_{3,3,0}$	$P_{6,4,0}$	$P_{7,3,0}$	$P_{7,4,0}$	$P_{7,7,0}$	$P_{7,2,1}$	$P_{7,3,1}$	Sum
0	0	0.0001	0.0004	0.97	0.0001	0.0004	0.971

 Table 5: At $t=40$

$P_{4,4,0}$	$P_{7,4,0}$	$P_{7,6,0}$	$P_{1,0,1}$	$P_{5,4,1}$	$P_{7,6,1}$	Sum
0	0.0001	0.9968	0	0	0.0013	0.9982


 Figure 3: **Probability $P_{4,4,0}$ without batch departure and with batch departure against t (average service times)**

 Figure 4: **Probability $P_{5,4,1}$ without batch departure and with batch departure against t (average service times)**

6 Busy Period Probabilities

The busy period probabilities of the system as well as the server is discussed in this section. The probability that server is busy is given by:

$$P(\text{Server busy}) = \sum_{i>j \geq 0} P_{i,j,1}(t) \quad (6.1)$$

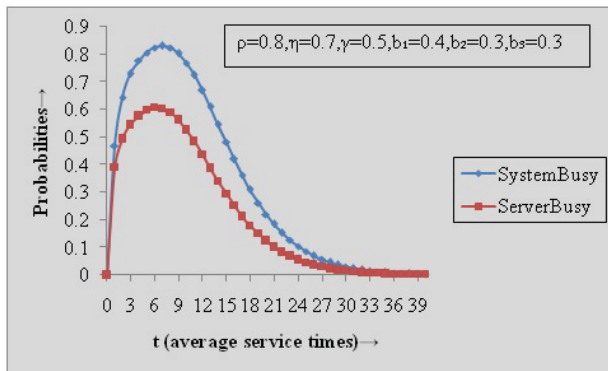
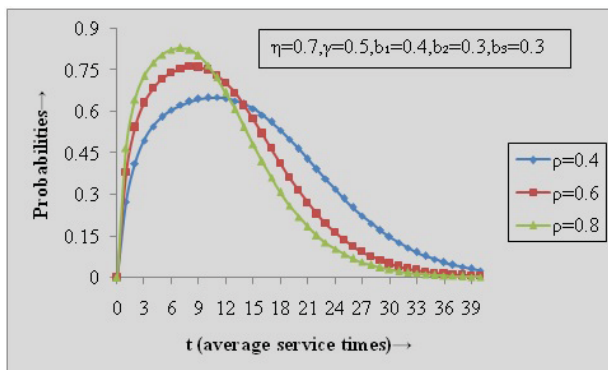
The probability that system is busy is given by:

$$P(\text{System busy}) = \sum_{i>j \geq 0} (P_{i,j,0}(t) + P_{i,j,1}(t)) \quad (6.2)$$

6.1 Numerical and Graphical Representation of Busy Period

Using MATLAB programming and following the work of (Bunday, 1986) the numerical results are generated. The probability of system busy and server busy are studied by varying the value of ρ and keeping other parameters as constant ($\eta=0.7$, $\gamma=0.5$, $b_1=0.4$, $b_2=0.3$ and $b_3=0.3$). In figure 5 the probabilities of System busy and Server busy are compared at various time instants t for the case $\rho=0.8$, $\eta=0.7$, $\gamma=0.5$, $b_1=0.4$, $b_2=0.3$ and $b_3=0.3$. Both the probabilities start increasing initially and then decrease after achieving their highest points. It can also be seen that the probability of System busy is always higher than that of Server busy as expected. The probabilities of System busy and Server busy for various values of ρ are plotted against time t in figures 6 and 7 respectively keeping other parameters as $\eta=0.7$, $\gamma=0.5$, $b_1=0.4$, $b_2=0.3$ and $b_3=0.3$. It is observed that the probabilities are higher for larger values of ρ , but after reaching their respective peaks, the trend gets reversed, i.e., the probabilities are larger for smaller ρ . Also, the System busy probabilities are larger than the Server busy. In order to study the effect of γ (feedback factor), the System busy and Server busy probabilities are plotted for various values of γ keeping other parameters constant at $\rho=0.8$, $\eta=0.7$, $b_1=0.4$, $b_2=0.3$ and $b_3=0.3$ in figures 8 and 9 respectively. It is seen that the probabilities are larger for the higher values of γ . It is also observed that the probabilities for System busy are higher than that of Server busy.

t	Probability(System Busy)			Probability(Server Busy)		
	$\rho=0.4$	$\rho=0.6$	$\rho=0.8$	$\rho=0.4$	$\rho=0.6$	$\rho=0.8$
0	0	0	0	0	0	0
1	0.2742	0.3797	0.4687	0.2257	0.3138	0.389
2	0.4141	0.5452	0.6442	0.3091	0.4129	0.4946
3	0.496	0.6323	0.7287	0.3543	0.4631	0.5454
4	0.5477	0.6843	0.7769	0.3839	0.4953	0.5772
5	0.5823	0.7181	0.807	0.4049	0.5176	0.5971
6	0.6066	0.741	0.8246	0.4203	0.5328	0.606
7	0.6243	0.7559	0.8305	0.4318	0.5414	0.6034
8	0.637	0.7631	0.8238	0.4399	0.543	0.5887
9	0.6458	0.7622	0.8036	0.445	0.5372	0.5627
10	0.6507	0.7523	0.77	0.447	0.5236	0.527


 Figure 5: Probabilities of System busy and Server busy against t (average service times)

 Figure 6: Effect of ρ on the probability of System busy against t (average service times)

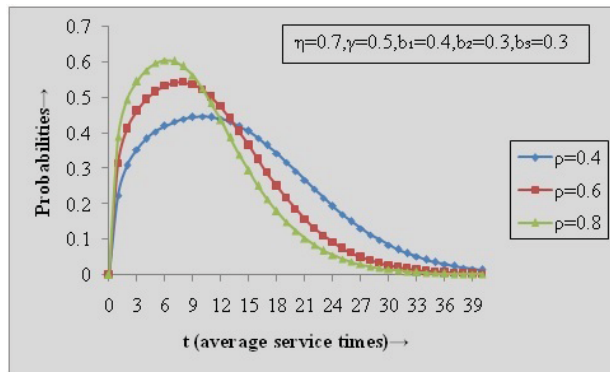


Figure 7: Effect of ρ on the probability of Server busy against t (average service times)

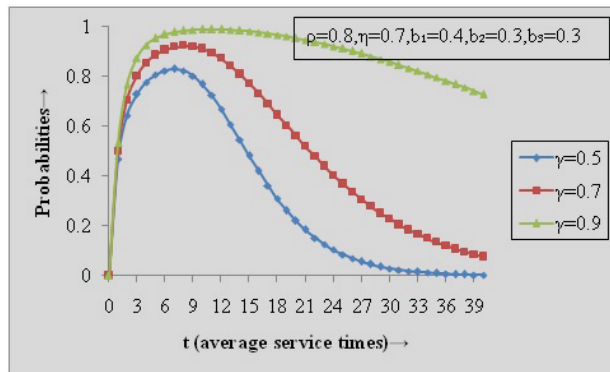


Figure 8: Effect of γ on the probability of System busy against t (average service times)

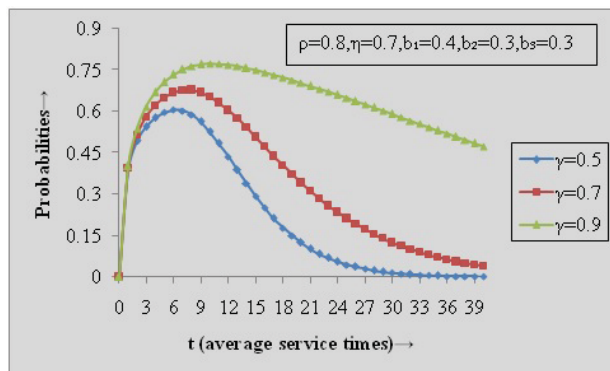


Figure 9: Effect of γ on the probability of Server busy against t (average service times)

7 Conclusion

In this paper, we analyzed a two-state retrial queueing model with feedback and batch departures. The results are well-quantified from a two-state model. Bulk or batch queueing systems are applicable in actual life processes such as loading and unloading cargoes, manufacturing process, etc. The transient state probabilities are obtained for the exact number of arrivals and departures from the system when the server is busy or idle by time t . MATLAB programming is used to generate various numerical results using which various graphical representations are given to study the effect of various parameters. Busy period probabilities along with their numerical and graphical solutions are also provided.

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