

Bayesian predictive analysis for Lindley distribution with various priors

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ABSTRACT

Bayesian analysis provides a distinctive approach to forecasting by allowing the integration of current data with existing predictions. This method improves the reliability of predictions by effectively addressing uncertainties, offering an edge over traditional forecasting techniques. In this study, we derive the predictive density using the Lindley distribution under various prior assumptions, including conjugate, quasi, uniform, and non-informative priors. The results underscore the effectiveness of Bayesian predictive methods in generating more accurate and credible predictions.

Keywords: likelihood function, posterior distribution and predictive density.

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Journal of Economic Literature (JEL) Classification : C11 Bayésien Analysais : General

1. INTRODUCTION

Lindley distribution was originally developed by Lindley (1958). The Lindley distribution is a continuous probability distribution named after the British statistician Dennis V. Lindley. It is primarily used in reliability analysis, survival studies, and modelling waiting times. The Lindley distribution is a mixture of the exponential and gamma distributions, characterized by a single parameter, θ , which controls the shape and scale of the distribution. In the year 2008 M.E. Ghitany et.al have studied the statistical properties of Lindley distribution.

In Bayesian analysis, a conjugate prior refers to a prior distribution that, when combined with a likelihood function from a particular statistical model, results in a posterior distribution of the same family as the prior. This property makes the mathematical computation of the posterior distribution more straightforward, as it avoids the need for complex integration. For instance, if the likelihood is binomial and the prior is a beta distribution, the resulting posterior will also be a beta distribution. Conjugate priors are particularly useful in Bayesian analysis because they simplify the process of updating beliefs with new data. A quasi-prior is a more flexible concept compared to the conjugate prior. It refers to a prior distribution that may not belong to the same family as the likelihood but is

chosen based on specific modelling needs or prior knowledge. Quasi-priors can be used when the exact conjugate prior is either unknown or does not exist for a given likelihood function. They allow for more flexibility in modelling but often require more complex calculations to determine the posterior distribution. The posterior distribution was derived by combining likelihood function and prior distribution and dividing by normalizing constant. The normalizing can be obtained by integrating the likelihood function and prior distribution with respect to unknown parameter. In this research the unknown parameter was θ

$$p(\theta/x) = \frac{1}{k} l(x_i) \cdot p(\theta)$$

Where

$$k = \int_0^{\infty} l(x_i) \cdot p(\theta) d\theta$$

$l(x_i)$ - likelihood function

$p(\theta)$ - prior distribution

Predictive modelling is essential in many disciplines, where accurately forecasting future events can significantly impact decision-making processes. Conventional predictive methods often struggle to adequately manage the uncertainty inherent in data, leading to less reliable forecasts. Bayesian analysis, by contrast, provides a powerful approach by allowing current data to update and refine predictions, thereby addressing uncertainties more effectively. The predictive density was obtained by integrating the posterior distribution with respect to unknown parameter since it is difficult to derive several mathematical techniques are used. In this study it is denoted as $g(x)$. The $g(x)$ is given by

$$g(x) = \int_0^{\infty} p(\theta/x) \cdot d\theta$$

This article delves into the use of the Lindley distribution within a Bayesian framework, exploring how the selection of different priors influences the resulting predictive density (Ashok K. Bansal 2007).

2. Review of literature

Hahn, Martin, and Walker's (2003) Recent Contribution to Bayesian Statistical Models for Predictive Distribution of Hydroclimatic Variables: This literature review explores a Bayesian statistical model designed by Hahn, Martin, and Walker, aimed at deriving predictive distributions for hydroclimatic variables. The model encompasses predictive densities and survival functions of generalized order statistics within a versatile class of distributions suggested by AL-Hussaini (1999), covering various distributions found in prior literature. Specializations to ordinary order statistics and records are presented and compared with existing results, demonstrating the model's efficacy. Applications to the Weibull(α, β) model showcase its adaptability in scenarios where either α or both (α, β) are unknown, emphasizing its practical utility in hydroclimatic modeling. **Ashok K. Bansal 2007** have provided the

information of predictive inference in statistical analysis in his book Bayesian parametric inference , emphasizing its dual purpose of understanding observed data and making predictions about future outcomes. Aitchison, Dunsmore (1975), and Geisser (1993), who argue that inferences about unobservable parameters have limited practical relevance. Instead, they advocate for focusing on observable outcomes that can be directly evaluated and are meaningful for decision-making. he further elaborates on the complexity of defining predictive distributions, particularly the challenge of specifying a practical distribution $g(y|x)g(y|x)g(y|x)$ without parameters due to the dependency between observed and future data. To address this, the Bayesian approach is presented as a robust solution, where a parametric model for observables is assumed, and prior distributions for unknown parameters are incorporated to derive the posterior predictive distribution $g(y|x)g(y|x)g(y|x)$. This method integrates uncertainty and prior knowledge, offering a practical tool for making informed predictions and bridging the gap between theoretical analysis and real-world applications. The author's discussion highlights the significance of focusing on observable outcomes and provides valuable insights into the practical utility of predictive inference in statistical decision-making.

M.E. Ghitany et.al 2008 studied a one parameter distribution by name Lindley distribution and discussed its statistical properties. The properties studied include: moments, cumulants, characteristic function, failure rate function, mean residual life function, mean deviations, Lorenz curve, stochastic ordering, entropies, asymptotic distribution of the extreme order statistics, distributions of sums, products and ratios, maximum likelihood estimation and simulation schemes. An application to waiting time data at a bank was described. This review highlights Hahn, Martin, and Walker's contributions, underscoring the model's versatility and its potential to advance predictive modelling in hydroclimatology.

Sanku Dey and Tanujit Dey's (2014) Contribution to Bayesian Estimation and Prediction Intervals for Rayleigh Distribution: This literature review explores the research efforts of Sanku Dey and Tanujit Dey, who present Bayesian estimators for the Rayleigh parameter and associated risk under a conjugate prior, specifically utilizing a square root inverted gamma prior. Considering both symmetric and asymmetric loss functions, the study derives highest posterior density (HPD) intervals for the Rayleigh parameter and HPD prediction intervals for future observations. Two distinct Bayes estimators are introduced and validated through simulation studies and real data analysis, demonstrating their effectiveness. The paper also explores the interchangeability of Bayes and frequentist predictive intervals for large sample sizes, and intriguingly notes the limited impact of hyper-parameter choices on estimation results in both simulated and real-life data analyses. This comprehensive review underscores the practical relevance and robustness of the proposed Bayesian techniques in the context of Rayleigh distribution parameter estimation.

Meenakshi G. and Maheswari P. R. (2019) have introduced a Multinomial distribution by Bayesian Methodology. Bayesian methodology is used to estimate the HIV replication for the future period by using multinomial distribution. $CD_4^+Tcount$ is classified into four categories. A joint density function of multinomial distribution along with a prior distribution as Dirichelt. A predictive density function obtained and simulations were done on prior and predictive distribution

2 Bayesian analysis for Lindley distribution

Let x be a random variable follow Lindley distribution with parameter θ then the pdf of Lindley distribution is given by

$$f(x) = \frac{\theta^2}{\theta+1} [1+x] e^{-\theta x} \quad x > 0, \theta > 0 \quad \dots (1)$$

The likelihood function is given by

$$\begin{aligned} l(x_i) &= \prod_{i=1}^n \frac{\theta^2}{\theta+1} [1+x_i] e^{-\theta x_i} \\ &= \frac{\theta^{2n}}{(\theta+1)^n} e^{-\theta \sum_{i=1}^n x_i} \prod_{i=1}^n [1+x_i] \quad \dots (2) \end{aligned}$$

2.1 The predictive density using gamma prior (conjugate prior)

The prior distribution is given by

$$p(\theta) = \frac{e^{-\theta} \cdot \theta^{r-1}}{\Upsilon r} \quad r > 0, \theta > 0$$

The posterior distribution for Lindley distribution is given by

$$\begin{aligned} p(\theta/x) &\propto \frac{e^{-\theta} \cdot \theta^{r-1}}{\Upsilon r} \prod_{i=1}^n \frac{\theta^2}{\theta+1} [1+x_i] e^{-\theta x_i} \quad \dots (3) \\ &= \frac{1}{k} \frac{e^{-\theta} \cdot \theta^{r-1}}{\Upsilon r} \prod_{i=1}^n \frac{\theta^2}{\theta+1} [1+x_i] e^{-\theta x_i} \end{aligned}$$

Where

$$\begin{aligned} k &= \int_0^{\infty} \frac{e^{-\theta} \cdot \theta^{r-1}}{\Upsilon r} \prod_{i=1}^n \frac{\theta^2}{\theta+1} [1+x_i] e^{-\theta x_i} d\theta \\ k &= \sum_{j=0}^{\infty} (-1)^j c(n+j-1, j) \frac{\prod_{i=1}^n [1+x_i]}{\Upsilon r} \left[\frac{\Gamma(2n+r+j)}{(1+\sum_{i=1}^n x_i)^{2n+r+j}} \right] \quad \dots (4) \end{aligned}$$

The predictive density is given by

$$g(x) = \int_0^{\infty} \frac{(1 + \sum_{i=1}^n x_i)^{2n+r+j}}{\sum_{j=0}^{\infty} (-1)^j c(n+j-1, j) [\Gamma(2n+r+j)]} \cdot \frac{\theta^{2n+r-1}}{(\theta+1)^n} e^{-\theta(1+\sum_{i=1}^n x_i)} \cdot \frac{\theta^2}{\theta+1} [1 + x] e^{-\theta x} d\theta$$

$$(1+\theta)^{-m} = \sum_{l=0}^{\infty} (-1)^l c(m+l-1, j) (\theta)^l$$

$n+1=m$

$$\begin{aligned} &= \sum_{l=0}^{\infty} (-1)^l c(m+l-1, j) \\ &- 1, l) \frac{(1 + \sum_{i=1}^n x_i)^{2n+r+j} \cdot [1+x]}{\sum_{j=0}^{\infty} (-1)^j c(n+j-1, j) [\Gamma(2n+r+j)]} \int_0^{\infty} \theta^{2n+r+l+2-1} e^{-\theta(x+1+\sum_{i=1}^n x_i)} \cdot d\theta \\ &g(x) \\ &= \sum_{l=0}^{\infty} (-1)^l c(m+l-1, j) \\ &- 1, l) \frac{(1 + \sum_{i=1}^n x_i)^{2n+r+j} \cdot [1+x]}{\sum_{j=0}^{\infty} (-1)^j c(n+j-1, j) [\Gamma(2n+r+j)]} \left[\frac{\Gamma(2n+r+l+2)}{(x+1+\sum_{i=1}^n x_i)^{2n+r+l+2}} \right] \quad \dots (5) \end{aligned}$$

2.2 The predictive density using quasi prior

The quasi-prior distribution is given by

$$p(\theta) = \frac{1}{\theta^d} \quad \dots (6)$$

The posterior distribution with quasi is given by

$$p(\theta/x) \propto \frac{1}{\theta^d} \prod_{i=1}^n \frac{\theta^2}{\theta+1} [1+x_i] e^{-\theta x_i}$$

$$p(\theta/x) = \frac{1}{k \theta^d} \frac{\theta^{2n}}{(\theta+1)^n} e^{-\theta \sum_{i=1}^n x_i} \prod_{i=1}^n [1+x_i] d\theta \quad \dots (7)$$

$$k = \sum_{j=0}^{\infty} (-1)^j c(n+j-1, j) \prod_{i=1}^n [1+x_i] \left\{ \frac{\Gamma(2n+j-d+1)}{(\sum_{i=1}^n x_i)^{2n+j-d+1}} \right\} \quad \dots (8)$$

The posterior distribution is given by

$$p(\theta/x) = \frac{1}{\sum_{j=0}^{\infty} (-1)^j c(n+j-1, j) \left\{ \frac{\Gamma(2n+j-d+1)}{(\sum_{i=1}^n x_i)^{2n+j-d+1}} \right\} (\theta+1)^n} \theta^{2n-d} e^{-\theta \sum_{i=1}^n x_i}$$

$$p(\theta/x) = \frac{(\sum_{i=1}^n x_i)^{2n+j-d+1}}{\sum_{j=0}^{\infty} (-1)^j c(n+j-1, j) \left\{ \Gamma(2n+j-d+1) \right\} (\theta+1)^n} \theta^{2n-d} e^{-\theta \sum_{i=1}^n x_i} \quad \dots (9)$$

The predictive density is given by

$$g(x) = \int_0^{\infty} \frac{(\sum_{i=1}^n x_i)^{2n+j-d+1}}{\sum_{j=0}^{\infty} (-1)^j c(n+j-1, j) \left\{ \Gamma(2n+j-d+1) \right\} (\theta+1)^{n+1}} \theta^{2n-d+2} e^{-\theta(x+\sum_{i=1}^n x_i)} \cdot [1+x] d\theta$$

The predictive density with quasi for Lindley distribution is given by

$$g(x) = \frac{(\sum_{i=1}^n x_i)^{2n+j-d+1} \sum_{l=0}^{\infty} (-1)^l c(m+l-1, l) [1+x]}{\sum_{j=0}^{\infty} (-1)^j c(n+j-1, j) \left\{ \Gamma(2n+j-d+1) \right\} (x+\sum_{i=1}^n x_i)^{2n-d+3+l} \cdot [\Gamma(2n-d+3+l)]} \dots (10)$$

2.3 The predictive Density using diffuse Prior and non-informative Prior

The predictive density with diffuse prior is given by $r=0$ in equation 5

$$g(x) = \sum_{l=0}^{\infty} (-1)^l c(m+l-1, l) \frac{(1+\sum_{i=1}^n x_i)^{2n+j} \cdot [1+x]}{\sum_{j=0}^{\infty} (-1)^j c(n+j-1, j) [\Gamma(2n+j)]} \left[\frac{\Gamma(2n+l+2)}{(x+1+\sum_{i=1}^n x_i)^{2n+r+l+2}} \right]$$

The predictive density with non-infective prior is given by $r=1$

$$g(x) = \sum_{l=0}^{\infty} (-1)^l c(m+l-1, l) \frac{(1+\sum_{i=1}^n x_i)^{2n+1+j} \cdot [1+x]}{\sum_{j=0}^{\infty} (-1)^j c(n+j-1, j) [\Gamma(2n+1+j)]} \left[\frac{\Gamma(2n+l+3)}{(x+1+\sum_{i=1}^n x_i)^{2n+l+3}} \right] \quad \dots (11)$$

3. RESULTS

3.1. The value of predictive density for various values of x with conjugate prior which is gamma prior

x	g(x)
1	0.700733
2	0.121007
3	0.019273
4	0.002982
5	0.000459
6	7.09E-05
7	1.11E-05
8	7.6E-07
9	1.28E-07
10	4.69E-08

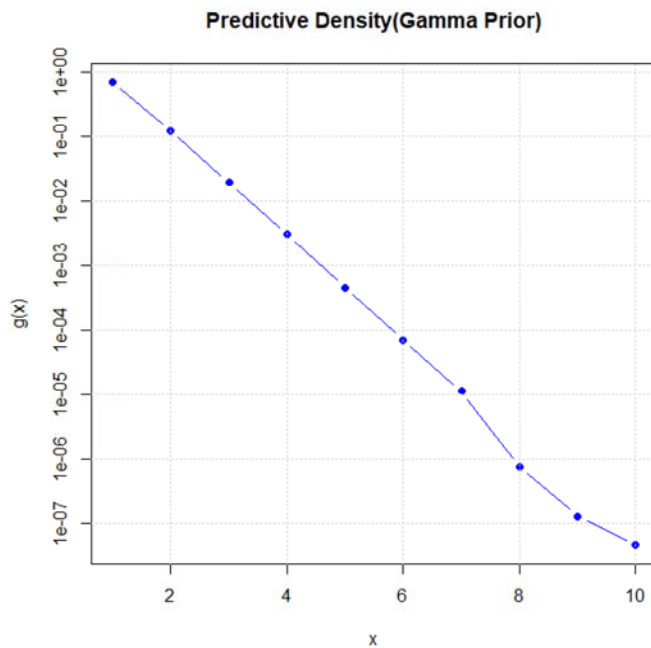


Figure 1 graph for predictive density with conjugate prior

The above figure shows that the conjugate prior gives higher initial density values, especially at $x=1$, and gradually decreases as x increases. The density drops rapidly, but the values remain relatively higher than some other priors for larger x .

3.2. The value of predictive density for various values of x with quasi prior

x	g(x)
1	0.305812
2	1.87E-09
3	1.71E-16
4	4.58E-23
5	1.91E-29
6	9.35E-36
7	4.67E-42
8	2.21E-48
9	9.49E-55
10	3.61E-61

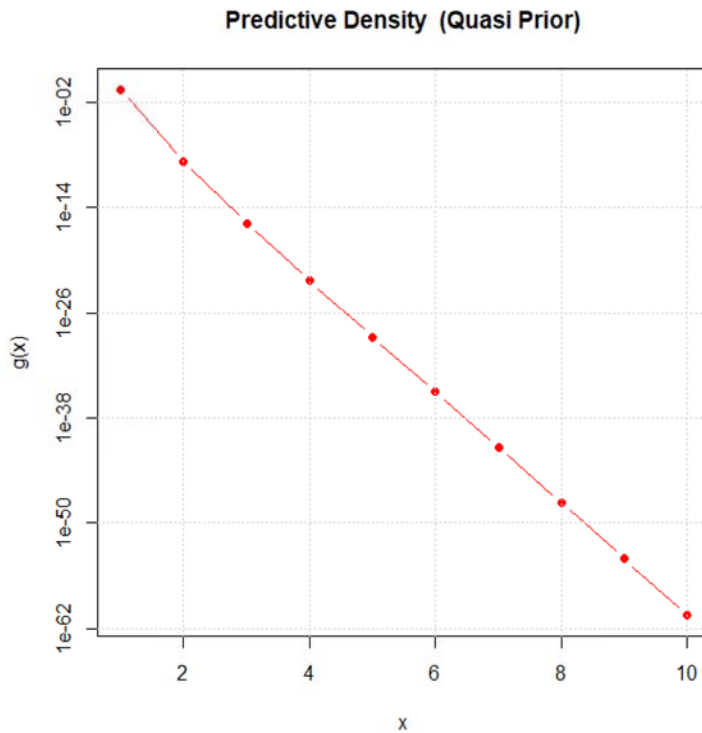


Figure 2 graph for predictive density with Quasi prior

The above figure shows that the quasi-prior starts with a lower initial density at $x=1$ and shows an extremely steep decline as x increases

3.3. The value of predictive density for various values of x with diffuse Prior

x	$g(x)$
1	0.698189
2	0.127014
3	0.021292
4	0.003465
5	0.00056
6	9.08E-05
7	1.49E-05
8	2.48E-06
9	4.2E-07
10	7.25E-08

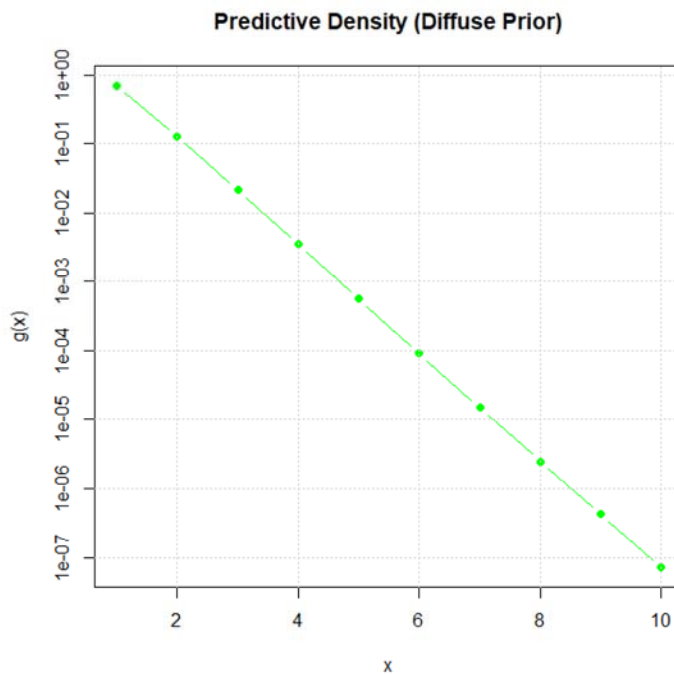


Figure 3 graph for predictive density with diffuse prior

The above figure shows the diffuse prior gives density values similar to the Gamma prior, with slightly lower values at each x . The decline is less steep than the quasi prior, maintaining moderate density for large.

3.4. The value of predictive density for various values of x with non-informative Prior

x	g(x)
1	0.699155
2	0.125001
3	0.0206
4	0.003296
5	0.000524
6	8.36E-05
7	1.35E-05
8	2.21E-06
9	3.69E-07
10	6.27E-08

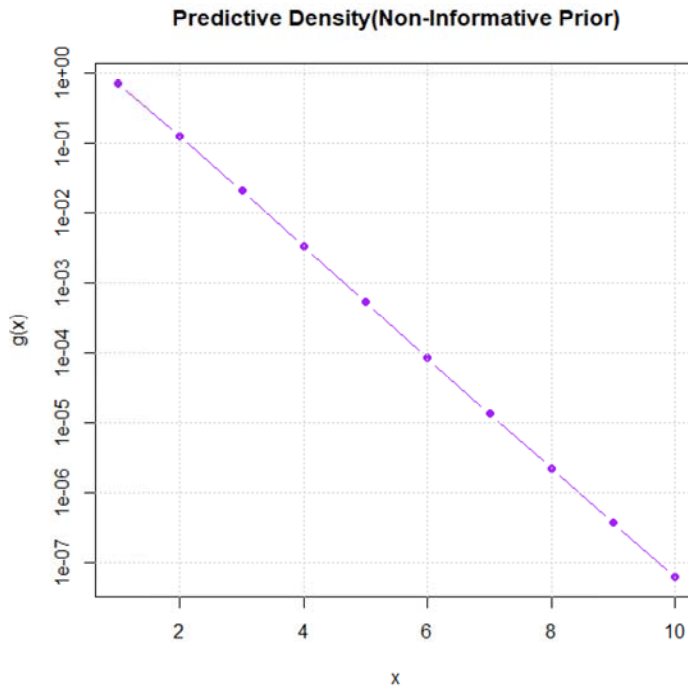


Figure 4 graph for predictive density with non-informative Prior

The above figure shows the non-informative prior produces density values close to those of the Gamma and diffuse priors, with a gradual decline as x increases. It provides balanced predictions with no extreme shifts in density

4. DISCUSSION AND CONCLUSION

The study shows how different choices of prior impact the predictions and results in Bayesian analysis using the Lindley distribution:

Conjugate Prior: This prior gives higher predictions for smaller values and decreases slowly as the values get larger. It has a significant effect on lower values, providing strong predictions for them and a gradual decline as values increase. Quasi-Bayesian Prior: With this prior, predictions drop sharply for larger values, resulting in very small predictions. This leads to a rapid decrease in the impact of larger values, which can affect the reliability of predictions for higher numbers. Diffuse Prior: This prior results in predictions that decrease at a moderate rate. It provides a balanced approach, with values that decrease steadily but not as sharply as with the quasi-Bayesian prior. Non-informative Prior: The predictions with this prior are similar to the conjugate prior but decrease more gradually. It offers a middle ground, balancing between the higher values of the conjugate prior and the more rapid decrease seen with the quasi-Bayesian prior.

In summary, the choice of prior has a significant impact on predictions and results. The conjugate prior leads to high values that decline slowly, the quasi-Bayesian prior results in very low values for larger numbers, the diffuse prior provides moderate predictions with a steady decrease, and the non-informative prior offers a balanced approach. These differences highlight why selecting the right prior is important for accurate and reliable predictions.

5. REFERENCES

- Askok K. Bansal 2007 "Bayesian Parametric Inference" Narosa Publishing House, ISBN : 978-81-7319-9800-7.
- G Meenakshi and Balachandar "BAYESIAN PREDICTIVE ANALYSIS FOR THE CHRIS-JERRY DISTRIBUTION WITH VARIOUS PRIORS" Indian Journal of Natural Sciences, Vol. Issue 83 / Apr- 2024, ISSN: 0976 – 0997.
- G Meenakshi, P. Janakiraman and Balachandar "A Bayesian Approach for Modeling the Viral Replication in HIV Dynamic using Truncated Cauchy Distribution" Indian Journal of Natural Sciences, Vol.14 / Issue 79 / Aug / 2023, ISSN: 0976 – 0997.
- G. MEENAKSHI and P.R. MAHESWARI et.al 2021." A Bayesian Approach For Modeling Of Viral Prediction (Replication) In The Immune System " The International journal of analytical and experimental modal analysis Volume XII, Issue XII, PP 2433
- Gelman, A., Carlin, J. B., Stern, H. S., Dunson, D. B., Vehtari, A., & Rubin, D. B. (2013). Bayesian Data Analysis (3rd ed.). Chapman and Hall/CRC.
- George, E. I., & Xu, X. (2010). Bayesian predictive density estimation. *Frontiers of statistical decision making and Bayesian analysis in honor of James O Berger*. Springer, New York, 83-95.
- Hoff, P. D. (2009). A First Course in Bayesian Statistical Methods. Journal of Bayesian Analysis, 14(3), 201-215.
- Ioannis Andrianakis et.al 2015 " Bayesian History Matching of Complex Infectious Disease Models Using Emulation : A Tutorial and a Case Study on HIV in Uganda" PLOS Computational Biology Volume 11 Issue 1.

Kruschke, J. K. (2015). Doing Bayesian Data Analysis: A Tutorial with R, JAGS, and Stan (2nd ed.). Academic Press..

Lee, P. M. (2012). Bayesian Statistics: An Introduction (4th ed.). International Journal of Statistics, 28(2), 89-104.

Lindley. D. V, Fiducial distributions and Bayes. theorem, Journal of the Royal statistical Society, Series A 20 (1958) 102. 107.) A.Y.J., 2005, *Impact de la structure des données sur les prédictions en régression linéaire multiple*. PhD Thesis, Fac. Univ. Sci. Agron., Gembloux, Belgium, 215 p.

Meenakshi, G., & Balachandar, B. (2024). A BAYESIAN APPROACH FOR CHRIS-JERRY DISTRIBUTION USING VARIOUS LOSS FUNCTIONS. *Reliability: Theory & Applications*, 19(2 (78)), 324-332.

Meenakshi, G., S. Lakshmi Priya. et.al 2018. "Bayesian Methods of Estimation of HIV Replication in a Using Rayleigh Distribution under the Various Loss Function Approach" Int. J. Sci. Res. in Mathematical and Statistical Sciences Vol. 5(6) ISSN: PP 2348-4519.

Meenakshi. G and Maheswari, P.R 2019. Prediction of HIV Replication in the Human Immune System using Multinomial Distribution by Bayesian Methodology. Int. J. Sci. Res. in Mathematical and Statistical Sciences Vol. 6(1) ISSN: PP 2348-4519.