

Numerical solution for Fredholm and Volterra integro-differential equations

Mohamed Raid NADIR¹ and Adel JAWAHDOU²

¹Department of Mathematics, Faculty of Sciences of Bizerte. University of Carthage.
 Jarzouna. Tunisia
 nadir.mohamedraid@yahoo.com

²Department of Mathematics, Institute of Engineering of Bizerte. University of
 Carthage. Jarzouna. Tunisia
 adeljaw2002@yahoo.com

ABSTRACT

In this work we describe new technical approximations for numerical solution of Volterra and Fredholm integro-differential equations using a modified Taylor series method and modified integration method with Lagrange polynomials. For each method we present some numerical examples compared with other known methods in order to illustrate accuracy and efficiency of our two methods.

Keywords: Fredholm and Volterra integro-differential equations, Taylor method, Integral method, Lagrange polynomials.

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1 Introduction

Integro-differential equations of Volterra type and of Fredholm type are at the heart of many mathematical descriptions of physical systems, used by applied mathematicians, physicists and engineers. As it is known, most of those integro-differential equations cannot be solved analytically by a simple formula. For this reason researchers used different approximate analytical methods. A variational form with Bernoulli and Legendre series for continuous or weakly continuous kernel (Nadir, 2014) (M. Nadir, 2018) (Nadir, 2015), optimal auxiliary function method gives a series solution after a few iteration (R. Saadati, 2008), in (S. Mathankumar, 2021) the authors represent an approximate exact solution in the form of convergent series based on the fuzzy concept with the variational iteration method. The author in (El-gendi, 1969) develops method based on quadrature scheme and gives the solution in terms of a finite Chebyshev expansion. The Lagrange polynomials with Finite element method to obtain an approximate solution of the Integro-differential equations of Volterra type (M. Gachpazan, 2014). Given the following integro-differential equation

$$u'(x) + a_0(x)u(x) - \int_{\Omega} k(x,t)u(t)dt = f(x), \quad (1.1)$$

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with its initial condition

$$u(x_0) = \alpha. \quad (1.2)$$

where, the functions $f(x)$ and $k(x, t)$ are given and continuous functions in Ω and $\Omega \times \Omega$, respectively, the function $u(t)$ is to be determined as continuous. Depending on $\Omega = [a, s]$ or $\Omega = [a, b]$ the equation (1.1) is a Volterra Integro-differential equation or Fredholm Integro-differential equation, respectively.

The goal of this problem is to finding the solution $u(x)$ of the equation (1.1) that satisfies an initial condition (1.2), where x_0 is some fixed value of point and $u(x_0)$ is a fixed state. We write this initial value problem with integral equation (IVPIE) concisely as

$$\begin{cases} u'(x) + a_0(x)u(x) - \int_{\Omega} k(x, t)u(t)dt = f(x), \\ u(x_0) = \alpha. \end{cases} \quad (1.3)$$

The initial conditions usually picks out a specific value of the arbitrary constant C that appears in the general solution of the equation. So, it selects one of the many possible states that satisfy the differential equation.

2 Numerical Methods of IVPIE

Suppose we solve the following initial value problem with integral equation (1.3) on the interval Ω

$$\begin{cases} u'(x) + a_0(x)u(x) - \int_{\Omega} k(x, t)u(t)dt = f(x), \\ u(x_0) = \alpha. \end{cases}$$

If the function $a_0(x)$ is continuous on the interval $\Omega = [a, b]$ or $\Omega = [a, x]$ containing the point $x = x_0$. Then there exists a unique solution $u(x)$ of the problem (1.3) if its integral is of Volterra, whereas if its integral is of Fredholm, we must add appropriate conditions for the existence of the solution. Conversely, neither existence nor uniqueness of a solution is guaranteed at a discontinuity of $a_0(x)$.

Seeking a continuous solution defined at each point x , we develop a strategy of discretization the problem to determine an approximation at discrete points in the interval of interest. Therefore, the idea is to replace the continuous function model (1.3) with an approximate discrete function model that is amenable to computer solution.

We divide the interval $a \leq x \leq b$ into N segments of constant length h , called the stepsize. Thus the stepsize is $h = \frac{b-a}{N}$. This defines a set of equally spaced discrete points $a = x_0 \leq x_1 \leq x_2 \leq \dots \leq x_N = b$, where $x_{i+1} = x_i + h$, $i = 0, 1, 2, \dots, N$. Now, suppose we know the solution $u(x_i)$ of the initial value problem at the point x_i . How could we estimate the solution at the point x_{i+1} ?

$$u(x_0) = \alpha$$

$$u'(x) = f(x) - a_0(x)u(x) + \int_{\Omega} k(x, t)u(t)dt$$

$$u''(x) = f'(x) - (a'_0(x)u(x) + a_0(x)u'(x)) + \left(\int_{\Omega} k(x,t)u(t)dt\right)' \quad (2.1)$$

where the derivative $\left(\int_{\Omega} k(x,t)u(t)dt\right)'$ if $\Omega = [a, x]$ is given by

$$\left(\int_a^x k(x,t)u(t)dt\right)' = k(x,x)u(x) + \int_a^x \frac{\partial}{\partial x} k(x,t)u(t)dt \quad (2.2)$$

and the derivative $\left(\int_{\Omega} k(x,t)u(t)dt\right)'$ if $\Omega = [a, b]$ is given by

$$\left(\int_a^b k(x,t)u(t)dt\right)' = \int_a^b \frac{\partial}{\partial x} k(x,t)u(t)dt \quad (2.3)$$

2.1 Numerical Methods of IVPIE

Suppose we solve the following initial value problem with integral equation (2.3) on the interval Ω

$$\begin{cases} u'(x) + a_0(x)u(x) - \int_{\Omega} k(x,t)u(t)dt = f(x), \\ u(x_0) = \alpha. \end{cases}$$

If the function $a_0(x)$ is continuous on the interval $\Omega = [a, b]$ or $\Omega = [a, x]$ containing the point $x = x_0$. Then there exists a unique solution $u(x)$ of the problem (2.3) if its integral is of Volterra, whereas if its integral is of Fredholm, we must add appropriate conditions for the existence of the solution. Conversely, neither existence nor uniqueness of a solution is guaranteed at a discontinuity of $a_0(x)$.

Seeking a continuous solution defined at each point x , we develop a strategy of discretization the problem to determine an approximation at discrete points in the interval of interest. Therefore, the idea is to replace the continuous function model (2.3) with an approximate discrete function model that is amenable to computer solution.

We divide the interval $a \leq x \leq b$ into N segments of constant length h , called the stepsize. Thus the stepsize is $h = \frac{b-a}{N}$. This defines a set of equally spaced discrete points $a = x_0 \leq x_1 \leq x_2 \leq x_3 \dots, x_N = b$, where $x_{i+1} = x_i + h$, $i = 0, 1, 2, \dots, N$. Now, suppose we know the solution $u(x_i)$ of the initial value problem at the point x_i . How could we estimate the solution at the point x_{i+1} ?

$$u(x_0) = \alpha$$

$$u'(x) = f(x) - a_0(x)u(x) + \int_{\Omega} k(x,t)u(t)dt$$

$$u''(x) = f'(x) - (a'_0(x)u(x) + a_0(x)u'(x)) + \left(\int_{\Omega} k(x,t)u(t)dt\right)' \quad (2.4)$$

where the derivative $\left(\int_{\Omega} k(x,t)u(t)dt\right)'$ if $\Omega = [a, x]$ is given by

$$\left(\int_a^x k(x,t)u(t)dt\right)' = k(x,x)u(x) + \int_a^x \frac{\partial}{\partial x} k(x,t)u(t)dt \quad (2.5)$$

and the derivative $(\int_{\Omega} k(x, t)u(t)dt)'$ if $\Omega = [a, b]$ is given by

$$\left(\int_a^b k(x, t)u(t)dt\right)' = \int_a^b \frac{\partial}{\partial x} k(x, t)u(t)dt \quad (2.6)$$

Taylor series Method of IVPVI

$$u(x_{i+1}) = u(x_i) + hu'(x_i) + \frac{h^2}{2}u''(x_i) + \frac{h^3}{3!}u'''(\omega_i); \quad x_i < \omega_i < x_{i+1}. \quad (2.7)$$

For the expression (2.7) the truncation error, due to the terms omitted from the series, is

$$E = \frac{h^3}{3!}u'''(\omega_i); \quad x_i < \omega_i < x_{i+1}.$$

Integral Method using Lagrange polynomials

$$\int_a^x u'(t)dt = \int_a^x f(t)dt - \int_a^x a_0(t)u(t)dt + \int_a^x \int_{\Omega} k(t, z)u(z)dzdt \quad (2.8)$$

$$u(x) = \int_a^x f(t)dt - \int_a^x a_0(t)u(t)dt + \int_a^x \left(\int_{\Omega} k(t, z)u(z)dz\right)dt + \alpha \quad (2.9)$$

For the integral equation (2.9) using Lagrange polynomials, the error represents the difference between the value $u(x)$ of the integral and its numerical result is

$$E = \frac{\varphi(x)}{N!} \sup \left| f^{(N)}(\omega) \right|$$

where $\varphi(x) = \prod_{i=1}^N (x - x_i)$ is the monic polynomial of degree N with roots at $x_1, x_2, \dots, x_N$ and

h is the "step length", given by $h = \frac{b-a}{N}$.

Denoting by u_i the approximation of the solution $u(x_i)$ at $x = x_i$. By algorithm and the given value $u_0 = u(x_0) = \alpha$ of the solution at the point x_0 we can calculate the approximations solutions $u_i, i = 1, 2, \dots, N$.

Example 1

Solve the initial value problem with Volterra integral

$$u'(x) + 3u(x) - \int_0^x \sin(x+t)u(t)dt = f(x), \quad u(0) = 0.$$

$$f(x) = (x^2 - x - 2) \cos 2x + (1 - 2x) \sin 2x + 2 \cos x - \sin x + 3x^2 - x - 1,$$

from $x = 0$ to 1 with Taylor series of order two where the analytical solution of the The problem is

$$u(x) = x^2 - x$$

Taylor series Method of IVPVI

$$u'(x) + 3u(x) - \int_0^x \sin(x+t)u(t)dt = f(x), \quad u(0) = 0,$$

$$u'(x) = f(x) - 3u(x) + \int_0^x \sin(x+t)u(t)dt, \quad u'(0) = -1,$$

$$u''(x) = f'(x) - 3u'(x) + u(x) \sin 2x + \int_0^x \cos(x+t)u(t)dt, \quad u''(0) = 2,$$

which is approximated by

$$u(x_{i+1}) \simeq u(x_i) + hu'(x_i) + \frac{h^2}{2}u''(x_i)$$

values of x	Exact solution	Approx solution	Error
0.0000e+00	0.0000e+00	0.0000e+00	0.0000e+00
2.5000e-01	-1.8750e-01	-1.8754e-01	4.2945e-05
5.0000e-01	-2.5000e-01	-2.5002e-01	2.5391e-05
7.5000e-01	-1.8750e-01	-1.8748e-01	1.9692e-05
1.0000e+00	0.0000e+00	0.0000e+00	0.0000e+00

Taylor series Method of order two,

Integral Method using Lagrange polynomials

$$u'(x) + 3u(x) - \int_0^x \sin(x+t) u(t) dt = f(x), \quad u(0) = 0,$$

$$\int_0^x u'(x) dx + \int_0^x 3u(x) dx - \int_0^x \int_0^x \sin(x+t) u(t) dt dx = \int_0^x f(x) dx,$$

which is approximated by

$$u(x) - \int_0^x \left(\int_t^x \sin(x+t) dx - 3 \right) u(t) dt = \int_0^x f(x) dx + u(0)$$

values of x	Exact solution	Approx solution	Error
0.0000e+00	0.0000e+00	4.4180e-16	4.4180e-16
2.5000e-01	-1.8750e-01	-1.8750e-01	1.4913e-10
5.0000e-01	-2.5000e-01	-2.5000e-01	1.9722e-10
7.5000e-01	-1.8750e-01	-1.8750e-01	1.4029e-10
1.0000e+00	0.0000e+00	-1.7695e-11	1.7695e-11

Integral Method using Lagrange polynomials,

Example 2

Solve the initial value problem with Volterra integral

$$u'(x) + u(x) - \int_0^x e^{t-x} u(t) dt = f(x), \quad u(0) = 1,$$

$$f(x) = 0,$$

from $x = 0$ to 1 with Taylor series of order two where the analytical solution of the The problem is

$$u(x) = e^{-x} \cosh x$$

Taylor series Method of IVPVI

$$u'(x) + u(x) - \int_0^x e^{t-x} u(t) dt = f(x), \quad u(0) = 1,$$

$$u'(x) = -u(x) + \int_0^x e^{t-x} u(t) dt, \quad u'(0) = -1,$$

$$u''(x) = -u'(x) + u(x) - \int_0^x e^{t-x} u(t) dt, \quad u''(0) = 2,$$

which is approximated by

$$u(x_{i+1}) \simeq u(x_i) + hu'(x_i) + \frac{h^2}{2}u''(x_i)$$

values of x	Exact solution	Approx solution	Error
0.0000e+00	1.0000e+00	1.0000e+00	0.0000e+00
2.5000e-01	8.0326e-01	8.0391e-01	6.4855e-04
5.0000e-01	6.8393e-01	6.8472e-01	7.8970e-04
7.5000e-01	6.1156e-01	6.1211e-01	5.5264e-04
1.0000e+00	5.6766e-01	5.6766e-01	0.0000e+00

Taylor series Method of order two,

Integral Method using Lagrange polynomials

$$u'(x) + u(x) - \int_0^x e^{t-x}u(t)dt = f(x), \quad u(0) = 1.$$

$$\int_0^x u'(x)dx + \int_0^x u(x)dx - \int_0^x \int_0^x e^{t-x}u(t)dx = \int_0^x f(x)dx,$$

which is approximated by

$$u(x) - \int_0^x \left(\int_t^x e^{t-x}dx - 1 \right) u(t)dt = \int_0^x f(x)dx + u(0)$$

values of x	Exact solution	Approx solution	Error	Error (D. Elmaci, 2021)
0.0000e+00	1.0000e+00	1.0000e+00	4.3387e-09	0.0000e+00
2.5000e-01	8.0326e-01	8.0326e-01	1.6131e-09	
5.0000e-01	6.8393e-01	6.8393e-01	6.8064e-11	6.699e-08
7.5000e-01	6.1156e-01	6.1156e-01	1.5189e-09	2.042e-07
1.0000e+00	5.6766e-01	5.6766e-01	3.9189e-09	5.014e-06

Integral Method using Lagrange polynomials,

Example 3

Solve the initial value problem with Volterra integral

$$u'(x) - \int_0^x u(t)dt = f(x), \quad u(0) = -1,$$

$$f(x) = 1 + \sin x,$$

from $x = 0$ to 1 with Taylor series of order two where the analytical solution of the The problem is

$$u(x) = \frac{1}{4}e^x - \frac{3}{4}e^{-x} - \frac{1}{2}\cos x$$

Taylor series Method of IVPVI

$$u'(x) - \int_0^x u(t)dt = f(x), \quad u(0) = -1,$$

$$u'(x) = 1 + \sin x + \int_0^x u(t)dt, \quad u'(0) = 1$$

$$u''(x) = \cos x + u(x), \quad u''(0) = 0,$$

$$u(x_{i+1}) \simeq u(x_i) + hu'(x_i) + \frac{h^2}{2}u''(x_i)$$

which is approximated by

$$u(x_{i+1}) \simeq u(x_i) + hu'(x_i) + \frac{h^2}{2}u''(x_i)$$

values of x	Exact solution	Approx solution	Error
0.0000e+00	-1.0000e+00	-1.0000e+00	0.0000e+00
2.5000e-01	-7.4755e-01	-7.4719e-01	3.5123e-04
5.0000e-01	-4.8150e-01	-4.8102e-01	4.8476e-04
7.5000e-01	-1.9086e-01	-1.9048e-01	3.7940e-04
1.0000e+00	1.3350e-01	1.3350e-01	0.0000e+00

Taylor series Method of order two,

Integral Method using Lagrange polynomials

$$u'(x) - \int_0^x u(t) dt = f(x), \quad u(0) = -1,$$

$$\int_0^x u'(x) dx - \int_0^x \int_0^x u(t) dx = \int_0^x f(x) dx,$$

which is approximated by

$$u(x) - \int_0^x \left(\int_t^x dx - 1 \right) u(t) dt = \int_0^x f(x) dx + u(0)$$

values of x	Exact solution	App solution	Error	Error (R. Saadati, 2008)
0.0000e+00	-1.0000e+00	1.0000e+00	4.6897e-11	
2.5000e-01	-7.4755e-01	8.0326e-01	1.9226e-11	3.9000e-09
5.0000e-01	-4.8150e-01	6.8393e-01	2.2381e-12	2.0000e-10
7.5000e-01	-1.9086e-01	6.1156e-01	1.4634e-11	1.4300e-08
1.0000e+00	1.3350e-01	5.6766e-01	5.2644e-11	2.5050e-07

Integral Method using Lagrange polynomials,

Example 4

Solve the initial value problem with Fredholm integral

$$u'(x) - \int_0^{\frac{\pi}{2}} -\frac{1}{4}xtu(t) dt = f(x), \quad u(0) = 0,$$

$$f(x) = \cos x + \frac{1}{4}x,$$

from $x = 0$ to 1 with Taylor series of order two where the analytical solution of the The problem is

$$u(x) = \sin x$$

Taylor series Method of IVPFI

$$u'(x) - \int_0^{\frac{\pi}{2}} \frac{1}{4} xtu(t) dt = f(x), \quad u(0) = 0,$$

$$u'(x) = \cos x + \frac{1}{4}x - \int_0^{\frac{\pi}{2}} \frac{1}{4} xtu(t) dt,$$

$$\frac{u(x_{i+1}) - u(x_{i-1}))}{2h} \simeq \sin x + \frac{1}{4} - \int_0^{\frac{\pi}{2}} \frac{1}{4} tu(t) dt,$$

which is approximated by

values of x	Exact solution	Approx solution	Error
0.0000e+00	0.0000e+00	0.0000e+00	0.0000e+00
2.5000e-01	2.0129e-01	2.0131e-01	1.9755e-05
5.0000e-01	3.9435e-01	3.9439e-01	3.5066e-05
7.5000e-01	5.7126e-01	5.7131e-01	4.5087e-05
1.0000e+00	7.2479e-01	7.2484e-01	4.9043e-05

Taylor series Method of order two,

Integral Method using Lagrange polynomials

$$u'(x) - \int_0^{\frac{\pi}{2}} \frac{1}{4} xtu(t) dt = f(x), \quad u(0) = 0,$$

$$\int_0^x u'(x) dx - \int_0^x \int_0^{\frac{\pi}{2}} \frac{1}{4} xtu(t) dt = \int_0^x f(x) dx,$$

which is approximated by

$$u(x) - \int_0^{\frac{\pi}{2}} \left(\int_0^x \frac{1}{4} xtdx \right) u(t) dt = \int_0^x f(x) dx + u(0)$$

values of x	Exact solution	App solution	Error	Error (A. Ansari, n.d.)
0.0000e+00	0.0000e+00	0.0000e+00	2.9570e-12	0.0
2.0000e-01	2.0711e-01	2.0711e-01	1.0186e-13	2.0E-10
4.0000e-01	3.9399e-01	3.9399e-01	5.6732e-14	4.0E-10
6.0000e-01	5.6573e-01	5.6573e-01	1.2936e-12	6.0E-10
8.0000e-01	7.2424e-01	7.2424e-01	1.2961e-12	

Integral Method using Lagrange polynomials,

Example 5

Solve the initial value problem with Fredholm integral

$$u'(x) - \int_0^1 u(t) dt = f(x), \quad u(0) = 1,$$

$$f(x) = -e^{-x} + e^{-1} - 1,$$

from $x = 0$ to 1 with Taylor series of order two where the analytical solution of the The problem is

$$u(x) = e^{-x}$$

Taylor series Method of IVPFI

$$u'(x) - \int_0^1 u(t) dt = f(x), \quad u(0) = 1,$$

$$u'(x) = -e^{-x} + e^{-1} - 1 + \int_0^1 u(t) dt,$$

$$\frac{u(x_{i+1}) - u(x_{i-1}))}{2h} \simeq -e^{-x} + e^{-1} - 1 + \int_0^1 u(t) dt,$$

which is approximating by

values of x	Exact solution	Approx solution	Error
0.0000e+00	1.0000e+00	1.0000e+00	0.0000e+00
2.5000e-01	7.7880e-01	7.7881e-01	1.0280e-05
5.0000e-01	6.0653e-01	6.0654e-01	1.2597e-05
7.5000e-01	4.7236e-01	4.7237e-01	8.7136e-06
1.0000e+00	3.6787e-01	3.6787e-01	0.0000e+00

Taylor series Method of order two,

Integral Method using Lagrange polynomials

$$u'(x) - \int_0^1 u(t) dt = f(x), \quad u(0) = 1,$$

$$\int_0^x u'(x) dx - \int_0^x \int_0^1 u(t) dt dx = \int_0^x (-e^{-x} + e^{-1} - 1) dx,$$

$$u(x) - \int_0^1 \left(\int_0^x dx \right) u(t) dt = \int_0^x f(x) dx + u(0)$$

which is approximating by

values of x	Exact solution	App solution	Error	Error (L. Rahmani, 2011)
0.0000e+00	1.0000e+00	1.0000e+00	2.6587e-11	7.752e-03
2.5000e-01	7.7880e-01	7.7880e-01	1.6671e-11	3.877e-03
5.0000e-01	6.0653e-01	6.0653e-01	1.3327e-11	4.688e-03
7.5000e-01	4.7236e-01	4.7236e-01	9.8560e-12	2.366e-03
1.0000e+00	3.6787e-01	3.6787e-01	5.1495e-11	6.700e-04

Integral Method using Lagrange polynomials,

Example 6

Solve the initial value problem with Fredholm integral

$$u'(x) - \int_0^1 xu(t) dt = f(x), \quad u(0) = 0,$$

$$f(x) = xe^x + e^x - x,$$

from $x = 0$ to 1 with Taylor series of order two where the analytical solution of the The problem is

$$u(x) = xe^x$$

Taylor series Method of IVPFI

$$u'(x) - \int_0^1 xu(t) = f(x), \quad u(0) = 0,$$

$$u'(x) = xe^x + e^x - x + \int_0^1 xu(t),$$

$$\frac{u(x_{i+1}) - u(x_{i-1}))}{2h} \simeq xe^x + e^x - x + \int_0^1 xu(t),$$

which is approximating by

values of x	Exact solution	Approx solution	Error
0.0000e+00	0.0000e+00	0.0000e+00	0.0000e+00
2.5000e-01	3.2100e-01	3.2092e-01	8.2077e-05
5.0000e-01	8.2436e-01	8.2426e-01	9.4885e-05
7.5000e-01	1.5877e+00	1.5876e+00	5.8524e-05
1.0000e+00	2.7182e+00	2.7182e+00	0.0000e+00

Taylor series Method of order two,

Integral Method Lagrange polynomials

$$u'(x) - \int_0^1 xu(t) = f(x), \quad u(0) = 0,$$

$$\int_0^x u'(x)dx - \int_0^x \int_0^1 xu(t) dx = \int_0^x f(x) dx,$$

$$u(x) - \int_0^1 \left(\int_0^x x dx \right) u(t) dt = \int_0^x f(x) dx + u(0)$$

which is approximating by

values of x	Exact solution	App solution	Error	Error (L. Zada, 2021)
0.0000e+00	0.0000e+00	0.0000e+00	6.5299e-10	5.448e-06
2.5000e-01	3.2100e-01	3.2100e-01	2.4617e-10	2.804e-06
5.0000e-01	8.2436e-01	8.2436e-01	3.1308e-11	5.448e-06
7.5000e-01	1.5877e+00	1.5877e+00	3.1737e-10	2.574e-06
1.0000e+00	2.7182e+00	2.7182e+00	5.9139e-10	5.448e-06

Integral Method using Lagrange polynomials,

3 Conclusion

Using the modified Taylor series method and the modified integration method to initial value problem with integral equation (IVPIE) can be transformed into a Taylor series and integral

equation of the second kind. The numerical solution of those equations using the two methods have been proved those convergence. A some numerical examples illustrated the performance of this new methods.

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