

Analysis of Three-State Retrial Queueing System with Catastrophe having Two Non-Identical Parallel Servers

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ABSTRACT

This research discusses a retrial queueing system with catastrophe having two non-identical parallel servers. Primary and secondary arrivals follow Poisson processes. Service times are exponentially distributed with different service rates. Catastrophe occurs on busy servers following Poisson process. Servers are sent for repair after failure. The repair times are also exponentially distributed. Time dependent probabilities for exact number of arrivals (in the system), number of departed units (after taking service from the system) and number of customers (in the orbit) with both, one or none servers busy are obtained recursively. The probabilities of the servers being under repair are also obtained. Verification of results is given. Numerical results are obtained and represented graphically.

Keywords: Queueing, Retrial, Catastrophe, Repair.

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1 Introduction

Whenever the system is full to its capacity, the customers are said to be blocked, i.e., customers are not allowed to enter the system. But there exist a phenomenon in which a customer instead of leaving the system forever, retries later. This phenomenon is known as retrial. Retrial queues are the queueing systems in which customers who do not receive service immediately on arrival, wait for some time in a virtual queue called orbit and try again with a hope of getting free server. Such kind of queueing systems play an important role to model many practical situations like telecommunication systems, data transferring, computer networks, etc. We face some situations in real life also where we have to repeat the request for service.

For example, at the time of submitting an online application form, we might receive an error message that the server is busy, please try again. It usually happens when we fill form on the closing date of submission. Retrial queueing systems have recently received significant attention of many researchers. An overview of retrial queues can be found in (Cohen, 1957)

and (Falin and Templeton, 1997).

In the classical transient results for $M/M/1$, $M/M/c$ and $M/G/1$ queue, the probability $P_n(t)$ gives the distribution for number of customers in the system at time t , but practically provides no information on how the system has regulated up until time t . (Pegden and Rosenshine, 1982) was the first who introduced the concept of two-state in 'Some new results for the $M/M/1$ queue'. In this paper they obtained a closed form solution for the probability that exactly i arrivals and j services occur over a time interval of length t . (Singla and Kalra, 2018) studied the two-state single server retrial queueing model in which the time dependent probabilities for exact number of arrivals and departures from the system when the server is free or busy are obtained. (Neelam Singla, 2022) studied 'An Analysis of Two-State Retrial Queueing Model with Feedback having Impatient Customers'.

Queues with two servers having different service rate can be widely used in modeling various situations. The quality of service has an important impact on system's performance. The servers with different service rate are called heterogeneous servers or non-identical servers. Queueing systems with heterogeneous servers can be seen in various situations in which customers are being served by humans rather than machines. For example in banks, post-offices, supermarkets, etc. (Singla and Kaur, 2021) studied 'A Two-State Retrial Queueing Model with Feedback having Two Non-Identical Parallel Servers' in which transient state probabilities for the system with both, one or no busy server are obtained. Using the following diagram, we can understand how retrial queues work.

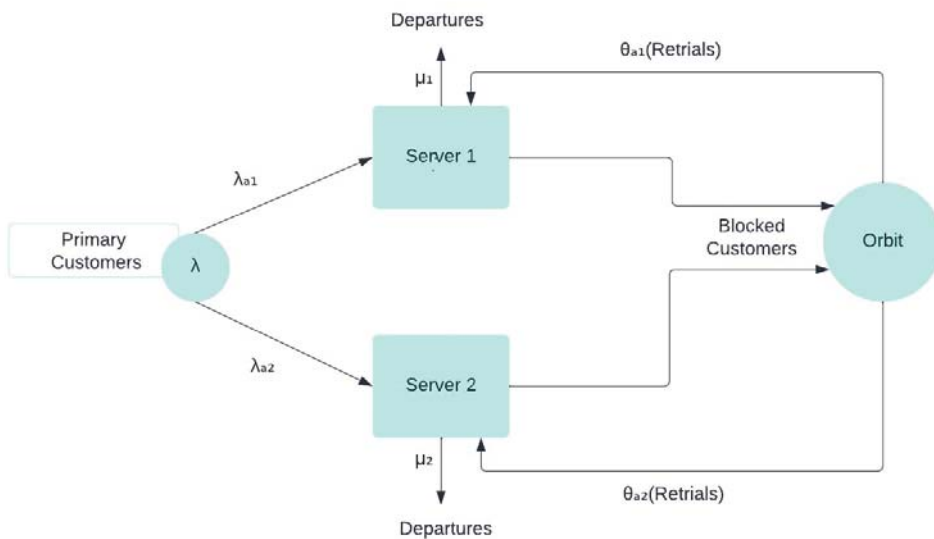


Figure 1: Structure of a Retrial Queueing System.

In recent years researchers have studied retrial queueing systems with catastrophe due to their wide range of applications in computer and communication systems. Catastrophe is a sudden,

unexpected failure in a machine, computer network, electronic system, communication system, etc. Catastrophe occurs at random, deletes all the customers present in the system and inactivates the service facilities for a short period of time. Catastrophe resets the system from current state to zero state at random time intervals. Catastrophe may come from outside the system or from another service station. Retrial queueing system with catastrophe can be seen in call centers, computer networks and in telecommunication networks.

For example: In call centers with the occurrence of catastrophic events like power failure, virus attacks will result in loss of all the calls present at that time and breakdown of the network.

The loss of customers due to these breakdowns (also referred to as a kind of negative arrivals) was first introduced by (Gelenbe, 1991). (Di Crescenzo, Giorno, Nobile and Ricciardi, 2003), (Chao, 1995) are the works done on Catastrophe occurring in a simple Markovian queue. (Kumar and Arivudainambi, 2000) discussed the asymptotic behavior of the probability of server being free. (Kumar, Krishnamoorthy, Madheswari and Basha, 2007) worked on M/M/1 queueing system with catastrophes. Transient solution is obtained for system with server failure and non-zero repair time.

Furthermore, the server is sent for repair immediately when the failure occurs. After getting repaired, the server comes back to its working position and the system becomes ready to serve new customers.

(Artalejo and Gómez-Corral, 1999) analyzed 'Computation of the limiting distribution in queueing systems with repeated attempts and disasters' in which the repeated attempts with the possibility of catastrophe are discussed and state probabilities are derived recursively. (Wang, Liu and Li, 2008) studied M/G/1 retrial queueing system with disasters and obtained Laplace transforms of both transient and steady state solutions. (Bura, 2022) proposed an infinite server Markovian queueing system with randomly occurring breakdowns and non zero exponentially distributed repair time wherein time dependent probabilities are obtained by using continued fraction technique.

This paper is divided into the following sections:

In section 2 the mathematical description of the model is given. Also, the difference-differential equations are derived in this section. Solution of the model is given in section 3 in which the transient state probabilities and the probabilities of server being under repair are obtained. In section 4 verification of results is done. The numerical results are obtained and represented graphically in section 5. Section 6 discusses the conclusion. At the end of the paper, references are provided.

2 Model Description

The detailed description of the model is given as follows.

- **Arrival Process:** The primary customers arrive according to Poisson process with mean arrival rate λ and they join the first server with probability a_1 , and second server with probability a_2 .
- **Retrial Process:** At the time of a customer's arrival if any of the servers is free it is served immediately. Otherwise, it joins the orbit and tries again following the Poisson process

with mean retrial rate θa_1 for first server and θa_2 for second server.

- Service Process: Service times are distributed exponentially with parameter μ_1 for first server and μ_2 for second server.
- Catastrophe: Catastrophe occurs at the system following Poisson process with rate ξ .
- Repair: When the system failure occurs, it is sent for repair and the repair times are distributed exponentially with parameter τ . Further it is assumed that no arrival can take place during repair time.

Also, the primary and secondary arrivals, inter-arrival times, service times, departures and catastrophes are mutually independent.

Laplace transformation $\bar{f}(s)$ of $f(t)$ is given by:

$$\bar{f}(s) = \int_0^{\infty} e^{-st} f(t) dt, \quad \operatorname{Re}(s) > 0$$

$$\text{If } L^{-1}\{\bar{f}(s)\} = F(t) \text{ and } L^{-1}\{\bar{g}(s)\} = G(t) \text{ then}$$

$$L^{-1}\{\bar{f}(s)\bar{g}(s)\} = \int_0^t F(u)G(t-u)du = F * G$$

$F * G$ is called the convolution of F and G .

2.1 Notations

$P_{i,j,k}^{(0)}(t)$ = Probability that there are exactly i number of arrivals in the system, j number of units departed from the system after taking the service, k number of units in the orbit by time t and both the servers are free.

$P_{i,j,k}^{(1,l)}(t)$ = Probability that there are exactly i number of arrivals in the system, j number of units departed from the system after taking the service, k number of units in the orbit by time t and l^{th} ($l = 1$ or 2) server is busy.

$P_{i,j,k}^{(2)}(t)$ = Probability that there are exactly i number of arrivals in the system, j number of units departed from the system after taking the service, k number of units in the orbit by time t and both the servers are busy.

$Q_{i,j}(t)$ = Probability that there are exactly i number of arrivals, j number of units departed from the system after taking service and the servers are under repair by time t .

$P_{i,j,k}(t)$ = Probability that there are exactly i arrivals in the system, j number of units departed from the system after taking the service, k number of units in the orbit by time t .

$$P_{i,j,k}(t) = P_{i,j,k}^{(0)}(t) + P_{i,j,k}^{(1,1)}(t) + P_{i,j,k}^{(1,2)}(t) + P_{i,j,k}^{(2)}(t) \quad \forall i, j, k \quad i \geq j, k$$

$$P_{i,j,k}^{(1)}(t) = P_{i,j,k}^{(1,1)}(t) + P_{i,j,k}^{(1,2)}(t)$$

$$\text{and } P_{i,j,k}^{(m)}(t) = 0 (m = 1 \text{ or } 2), \quad i \leq j, k; \quad P_{i,j,k}^{(0)}(t) = 0, \quad i < j, k;$$

Initially

$$P_{0,0,0}^{(0)}(0) = 1; \quad P_{i,j,k}^{(0)}(0) = 0, \quad P_{i,j,k}^{(l)}(0) = 0 (l = 1 \text{ or } 2), \quad i, j, k \neq 0; \quad Q_{i,j}(0) = 0 \quad \forall i, j;$$

2.2 The Difference-Differential equations governing the system are:

$$\begin{aligned} \frac{d}{dt} P_{i,j,k}^{(0)}(t) = & -(\lambda + \xi + k\theta)P_{i,j,k}^{(0)}(t) + \mu_1(1 - \delta_{0,j})P_{i,j-1,k}^{(1,1)}(t) + \mu_2(1 - \delta_{0,j})P_{i,j-1,k}^{(1,2)}(t) \\ & + \tau\delta_{0,k}Q_{i,j}(t) \quad i \geq j, k \geq 0 \end{aligned} \quad (2.1)$$

$$\frac{d}{dt} P_{i,0,k}^{(1,1)}(t) = -(\lambda + \mu_1 + \xi + k\theta)P_{i,0,k}^{(1,1)}(t) + \lambda a_1 \delta_{0,k} P_{i-1,0,k}^{(0)}(t) \quad i > k \geq 0 \quad (2.2)$$

$$\begin{aligned} \frac{d}{dt} P_{i,j,k}^{(1,1)}(t) = & -(\lambda + \mu_1 + \xi + k\theta)P_{i,j,k}^{(1,1)}(t) + \lambda a_1 P_{i-1,j,k}^{(0)}(t) + \mu_2 P_{i,j-1,k}^{(2)}(t) \\ & + (k+1)\theta a_1 P_{i,j,k+1}^{(0)}(t) \quad i > j > k \geq 0 \end{aligned} \quad (2.3)$$

$$\frac{d}{dt} P_{i,0,k}^{(1,2)}(t) = -(\lambda + \mu_2 + \xi + k\theta)P_{i,0,k}^{(1,2)}(t) + \lambda a_2 \delta_{0,k} P_{i-1,0,k}^{(0)}(t) \quad i > k \geq 0 \quad (2.4)$$

$$\begin{aligned} \frac{d}{dt} P_{i,j,k}^{(1,2)}(t) = & -(\lambda + \mu_2 + \xi + k\theta)P_{i,j,k}^{(1,2)}(t) + \lambda a_2 P_{i-1,j,k}^{(0)}(t) + \mu_1 P_{i,j-1,k}^{(2)}(t) \\ & + (k+1)\theta a_2 P_{i,j,k+1}^{(0)}(t) \quad i > j > k \geq 0 \end{aligned} \quad (2.5)$$

$$\begin{aligned} \frac{d}{dt} P_{i,j,k}^{(2)}(t) = & -(\lambda + \mu_1 + \mu_2 + \xi)P_{i,j,k}^{(2)}(t) + \lambda \{P_{i-1,j,k}^{(1,1)}(t) + P_{i-1,j,k}^{(1,2)}(t)\} \\ & + \lambda(1 - \delta_{0,k})P_{i-1,j,k-1}^{(2)}(t) + (k+1)\theta \{P_{i,j,k+1}^{(1,1)}(t) + P_{i,j,k+1}^{(1,2)}(t)\} \quad i \geq 2 \quad i > j, k \geq 0 \end{aligned} \quad (2.6)$$

$$\begin{aligned} \frac{d}{dt} Q_{i,j}(t) = & -\tau Q_{i,j}(t) + \xi \left(P_{i,j,0}^{(0)}(t) + \sum_{k=1}^{i-j} (1 - \delta_{0,j}) P_{i,j,k}^{(0)}(t) + \sum_{k=0}^{i-j-1} \left\{ P_{i,j,k}^{(1,1)}(t) + P_{i,j,k}^{(1,2)}(t) \right\} \right. \\ & \left. + \sum_{k=0}^{i-j-2} P_{i,j,k}^{(2)}(t) \right) \quad i \geq j \geq 0 \end{aligned} \quad (2.7)$$

where

$$\delta_{0,j} = \begin{cases} 1 & \text{when } j = 0 \\ 0 & \text{otherwise} \end{cases}$$

and

$$\delta_{0,k} = \begin{cases} 1 & \text{when } k = 0 \\ 0 & \text{otherwise} \end{cases}$$

Using the Laplace transformation $\bar{f}(s)$ of $f(t)$ given by

$$\bar{f}(s) = \int_0^\infty e^{-st} f(t) dt, \quad \operatorname{Re}(s) > 0$$

in the equations (2.1)-(2.7) along with the initial conditions, we have

$$\begin{aligned} (s + \lambda + \xi + k\theta)\bar{P}_{i,j,k}^{(0)}(s) = & \mu_1(1 - \delta_{0,j})\bar{P}_{i,j-1,k}^{(1,1)}(s) + \mu_2(1 - \delta_{0,j})\bar{P}_{i,j-1,k}^{(1,2)}(s) \\ & + \tau\delta_{0,k}\bar{Q}_{i,j}(s) \quad i \geq j, k \geq 0 \end{aligned} \quad (2.8)$$

$$(s + \lambda + \mu_1 + \xi + k\theta)\bar{P}_{i,0,k}^{(1,1)}(s) = \lambda a_1 \delta_{0,k} \bar{P}_{i-1,0,k}^{(0)}(s) \quad i > k \geq 0 \quad (2.9)$$

$$(s + \lambda + \mu_1 + \xi + k\theta)\bar{P}_{i,j,k}^{(1,1)}(s) = \lambda a_1 \bar{P}_{i-1,j,k}^{(0)}(s) + \mu_2 \bar{P}_{i,j-1,k}^{(2)}(s) + (k+1)\theta a_1 \bar{P}_{i,j,k+1}^{(0)}(s) \quad i > j > k \geq 0 \quad (2.10)$$

$$(s + \lambda + \mu_2 + \xi + k\theta)\bar{P}_{i,0,k}^{(1,2)}(s) = \lambda a_2 \delta_{0,k} \bar{P}_{i-1,0,k}^{(0)}(s) \quad i > k \geq 0 \quad (2.11)$$

$$(s + \lambda + \mu_2 + \xi + k\theta)\bar{P}_{i,j,k}^{(1,2)}(s) = \lambda a_2 \bar{P}_{i-1,j,k}^{(0)}(s) + \mu_1 \bar{P}_{i,j-1,k}^{(2)}(s) + (k+1)\theta a_2 \bar{P}_{i,j,k+1}^{(0)}(s) \quad i > j > k \geq 0 \quad (2.12)$$

$$(s + \lambda + \mu_1 + \mu_2 + \xi)\bar{P}_{i,j,k}^{(2)}(s) = \lambda \{ \bar{P}_{i-1,j,k}^{(1,1)}(s) + \bar{P}_{i-1,j,k}^{(1,2)}(s) \} + \lambda(1 - \delta_{0,k})\bar{P}_{i-1,j,k-1}^{(2)}(s) + (k+1)\theta \{ \bar{P}_{i,j,k+1}^{(1,1)}(s) + \bar{P}_{i,j,k+1}^{(1,2)}(s) \} \quad i \geq 2 \quad j, k \geq 0 \quad (2.13)$$

$$(s + \tau)\bar{Q}_{i,j}(s) = \xi \left(\bar{P}_{i,j,0}^{(0)}(s) + \sum_{k=1}^{i-j} (1 - \delta_{0,j}) \bar{P}_{i,j,k}^{(0)}(s) + \sum_{k=0}^{i-j-1} \left\{ \bar{P}_{i,j,k}^{(1,1)}(s) + \bar{P}_{i,j,k}^{(1,2)}(s) \right\} + \sum_{k=0}^{i-j-2} \bar{P}_{i,j,k}^{(2)}(s) \right) \quad i \geq j \geq 0 \quad (2.14)$$

3 Solution of the Problem

Solving equations (2.8)-(2.14) recursively, we have obtained the following recursive probabilities:

$$\bar{Q}_{0,0}(s) = \frac{\xi}{(s + \tau)(s + \lambda + \xi) - \xi\tau} \quad (3.1)$$

$$\bar{P}_{0,0,0}^{(0)}(s) = \frac{1}{s + \lambda + \xi} + \frac{\tau}{s + \lambda + \xi} \bar{Q}_{0,0}(s) \quad (3.2)$$

$$\bar{P}_{i,0,0}^{(0)}(s) = \frac{\tau}{s + \lambda + \xi} \bar{Q}_{i,0}(s) \quad i \geq 1 \quad (3.3)$$

$$\bar{P}_{i,1,0}^{(0)}(s) = \frac{\mu_1}{s + \lambda + \xi} \bar{P}_{i,0,0}^{(1,1)}(s) + \frac{\mu_2}{s + \lambda + \xi} \bar{P}_{i,0,0}^{(1,2)}(s) + \frac{\tau}{s + \lambda + \xi} \bar{Q}_{i,1}(s) \quad i \geq 1 \quad (3.4)$$

$$\begin{aligned} \bar{P}_{i,2,k}^{(0)}(s) &= \frac{\lambda}{s + \lambda + \xi + k\theta} \left[\frac{\mu_1 a_1}{s + \lambda + \mu_1 + \xi + k\theta} + \frac{\mu_2 a_2}{s + \lambda + \mu_2 + \xi + k\theta} \right] \bar{P}_{i-1,1,k}^{(0)}(s) \\ &+ \frac{\mu_1 \mu_2}{s + \lambda + \xi + k\theta} \left[\frac{1}{s + \lambda + \mu_1 + \xi + k\theta} + \frac{1}{s + \lambda + \mu_2 + \xi + k\theta} \right] \bar{P}_{i,0,k}^{(2)}(s) \\ &+ \delta_{0,k} \frac{\tau}{s + \lambda + \xi} \bar{Q}_{i,2}(s) \quad i \geq 2, k \geq 0 \end{aligned} \quad (3.5)$$

$$\bar{P}_{i,0,0}^{(1,1)}(s) = \frac{\lambda a_1}{s + \lambda + \mu_1 + \xi} \bar{P}_{i-1,0,0}^{(0)}(s) \quad i \geq 1 \quad (3.6)$$

$$\bar{P}_{i,0,0}^{(1,2)}(s) = \frac{\lambda a_2}{s + \lambda + \mu_2 + \xi} \bar{P}_{i-1,0,0}^{(0)}(s) \quad i \geq 1 \quad (3.7)$$

$$\begin{aligned} \bar{P}_{i,j,k}^{(0)}(s) &= \frac{\lambda}{s + \lambda + \xi + k\theta} \left[\frac{\mu_1 a_1}{s + \lambda + \mu_1 + \xi + k\theta} + \frac{\mu_2 a_2}{s + \lambda + \mu_2 + \xi + k\theta} \right] \bar{P}_{i-1,j-1,k}^{(0)}(s) \\ &+ \frac{\mu_1 \mu_2}{s + \lambda + \xi + k\theta} \left[\frac{1}{s + \lambda + \mu_1 + \xi + k\theta} + \frac{1}{s + \lambda + \mu_2 + \xi + k\theta} \right] \bar{P}_{i,j-2,k}^{(2)}(s) \\ &+ \frac{(k+1)\theta}{s + \lambda + \xi + k\theta} \left[\frac{\mu_1 a_1}{s + \lambda + \mu_1 + \xi + k\theta} + \frac{\mu_2 a_2}{s + \lambda + \mu_2 + \xi + k\theta} \right] \bar{P}_{i,j-1,k+1}^{(0)}(s) \\ &+ \delta_{0,k} \frac{\tau}{s + \lambda + \xi} \bar{Q}_{i,j}(s) \quad i \geq j \geq 3, k \geq 0 \end{aligned} \quad (3.8)$$

$$\begin{aligned} \bar{P}_{i,j,0}^{(1,1)}(s) &= \frac{\lambda a_1}{s + \lambda + \mu_1 + \xi} \bar{P}_{i-1,j,0}^{(0)}(s) + \frac{\mu_2}{s + \lambda + \mu_1 + \xi} \left(\frac{\lambda}{s + \lambda + \xi + \mu_1 + \mu_2} \right) \\ &\quad \left\{ \bar{P}_{i-1,j-1,0}^{(1,1)}(s) + \bar{P}_{i-1,j-1,0}^{(1,2)}(s) \right\} + \frac{\mu_2}{s + \lambda + \mu_1 + \xi} \left(\frac{\theta}{s + \lambda + \mu_1 + \mu_2 + \xi} \right) \\ &\quad \left\{ \bar{P}_{i,j-1,0}^{(1,1)}(s) + \bar{P}_{i,j-1,0}^{(1,2)}(s) \right\} + \frac{\theta a_1}{s + \lambda + \mu_1 + \xi} \bar{P}_{i,j,1}^{(0)}(s) \quad i > j > 0 \end{aligned} \quad (3.9)$$

$$\begin{aligned} \bar{P}_{i,j,0}^{(1,2)}(s) &= \frac{\lambda a_2}{s + \lambda + \mu_2 + \xi} \bar{P}_{i-1,j,0}^{(0)}(s) + \frac{\mu_1}{s + \lambda + \mu_2 + \xi} \left(\frac{\lambda}{s + \lambda + \xi + \mu_1 + \mu_2} \right) \\ &\quad \left\{ \bar{P}_{i-1,j-1,0}^{(1,1)}(s) + \bar{P}_{i-1,j-1,0}^{(1,2)}(s) \right\} + \frac{\mu_1}{s + \lambda + \mu_2 + \xi} \left(\frac{\theta}{s + \lambda + \mu_1 + \mu_2 + \xi} \right) \\ &\quad \left\{ \bar{P}_{i,j-1,0}^{(1,1)}(s) + \bar{P}_{i,j-1,0}^{(1,2)}(s) \right\} + \frac{\theta a_2}{s + \lambda + \mu_2 + \xi} \bar{P}_{i,j,1}^{(0)}(s) \quad i > j > 0 \end{aligned} \quad (3.10)$$

$$\begin{aligned} \bar{P}_{i,1,k}^{(1,1)}(s) &= \frac{\mu_2}{s + \lambda + \mu_1 + \xi + k\theta} \left[\sum_{p=0}^{k+1} \left(\frac{\lambda}{s + \lambda + \mu_1 + \mu_2 + \xi} \right)^{k+1-p} \eta'_p(s) \left\{ \bar{P}_{k+p,0,p}^{(1,1)}(s) \right. \right. \\ &\quad \left. \left. + \bar{P}_{k+p,0,p}^{(1,2)}(s) \right\} \right] \quad i > 2, k \geq 1 \end{aligned} \quad (3.11)$$

where

$$\eta'_p = \begin{cases} 1 & \text{if } p = 0 \\ 1 + \frac{p\theta}{s + \lambda + \mu_1 + \mu_2 + \xi} & \text{if } p = 1 \text{ to } k \\ \frac{p\theta}{s + \lambda + \mu_1 + \mu_2 + \xi} & \text{if } p = k + 1 \end{cases}$$

$$\begin{aligned} \bar{P}_{i,1,k}^{(1,2)}(s) &= \frac{\mu_1}{s + \lambda + \mu_2 + \xi + k\theta} \left[\sum_{p=0}^{k+1} \left(\frac{\lambda}{s + \lambda + \mu_1 + \mu_2 + \xi} \right)^{k+1-p} \eta'_p(s) \left\{ \bar{P}_{k+p,0,p}^{(1,1)}(s) \right. \right. \\ &\quad \left. \left. + \bar{P}_{k+p,0,p}^{(1,2)}(s) \right\} \right] \quad i > 2, k \geq 1 \end{aligned} \quad (3.12)$$

where

$$\eta'_p = \begin{cases} 1 & \text{if } p = 0 \\ 1 + \frac{p\theta}{s + \lambda + \mu_1 + \mu_2 + \xi} & \text{if } p = 1 \text{ to } k \\ \frac{p\theta}{s + \lambda + \mu_1 + \mu_2 + \xi} & \text{if } p = k + 1 \end{cases}$$

$$\begin{aligned} \bar{P}_{i,j,k}^{(1,1)}(s) &= \frac{\lambda a_1}{s + \lambda + \mu_1 + \xi + k\theta} \bar{P}_{i-1,j,k}^{(0)}(s) + \frac{\mu_2}{s + \lambda + \xi + \mu_1 + k\theta} \\ &\quad \left[\sum_{p=0}^{k+1} \left(\frac{\lambda}{s + \lambda + \mu_1 + \mu_2 + \xi} \right)^{k+1-p} \eta'_p(s) \left\{ \bar{P}_{i-k-1+p,j-1,p}^{(1,1)}(s) \right. \right. \\ &\quad \left. \left. + \bar{P}_{i-k-1+p,j-1,p}^{(1,2)}(s) \right\} \right] + \frac{(k+1)\theta a_1}{s + \lambda + \mu_1 + \xi + k\theta} \bar{P}_{i,j,k+1}^{(0)}(s) \quad i \geq j + k + 1, j > k \geq 1 \end{aligned} \quad (3.13)$$

where

$$\eta'_p = \begin{cases} 1 & \text{if } p = 0 \\ 1 + \frac{p\theta}{s + \lambda + \mu_1 + \mu_2 + \xi} & \text{if } p = 1 \text{ to } k \\ \frac{p\theta}{s + \lambda + \mu_1 + \mu_2 + \xi} & \text{if } p = k + 1 \end{cases}$$

$$\begin{aligned} \bar{P}_{i,j,k}^{(1,2)}(s) &= \frac{\lambda a_2}{s + \lambda + \mu_2 + \xi + k\theta} \bar{P}_{i-1,j,k}^{(0)}(s) + \frac{\mu_1}{s + \lambda + \xi + \mu_2 + k\theta} \\ &\left[\sum_{p=0}^{k+1} \left(\frac{\lambda}{s + \lambda + \mu_1 + \mu_2 + \xi} \right)^{k+1-p} \eta'_p(s) \left\{ \bar{P}_{i-k-1+p,j-1,p}^{(1,1)}(s) \right. \right. \\ &\left. \left. + \bar{P}_{i-k-1+p,j-1,p}^{(1,2)}(s) \right\} \right] + \frac{(k+1)\theta a_2}{s + \lambda + \mu_2 + \xi + k\theta} \bar{P}_{i,j,k+1}^{(0)}(s) \quad i \geq j + k + 1, j > k \geq 1 \end{aligned} \quad (3.14)$$

where

$$\eta'_p = \begin{cases} 1 & \text{if } p = 0 \\ 1 + \frac{p\theta}{s + \lambda + \mu_1 + \mu_2 + \xi} & \text{if } p = 1 \text{ to } k \\ \frac{p\theta}{s + \lambda + \mu_1 + \mu_2 + \xi} & \text{if } p = k + 1 \end{cases}$$

$$\begin{aligned} \bar{P}_{i,j,0}^{(2)}(s) &= \frac{\lambda}{s + \lambda + \xi + \mu_1 + \mu_2} \left\{ \bar{P}_{i-1,j,0}^{(1,1)}(s) + \bar{P}_{i-1,j,0}^{(1,2)}(s) \right\} + \frac{\theta}{s + \lambda + \xi + \mu_1 + \mu_2} \\ &\left\{ \bar{P}_{i,j,1}^{(1,1)}(s) + \bar{P}_{i,j,1}^{(1,2)}(s) \right\} \quad i > j + 1, j \geq 0 \end{aligned} \quad (3.15)$$

$$\begin{aligned} \bar{P}_{i,j,k}^{(2)}(s) &= \sum_{p=0}^{k+1} \left(\frac{\lambda}{s + \lambda + \xi + \mu_1 + \mu_2} \right)^{k+1-p} \eta'_p(s) \left\{ \bar{P}_{i-k-1+p,j,p}^{(1,1)}(s) + \bar{P}_{i-k-1+p,j,p}^{(1,2)}(s) \right\} \\ &\quad i \geq j + k + 2, j \geq 0, k \geq 1 \end{aligned} \quad (3.16)$$

where

$$\eta'_p = \begin{cases} 1 & \text{if } p = 0 \\ 1 + \frac{p\theta}{s + \lambda + \mu_1 + \mu_2 + \xi} & \text{if } p = 1 \text{ to } k \\ \frac{p\theta}{s + \lambda + \mu_1 + \mu_2 + \xi} & \text{if } p = k + 1 \end{cases}$$

$$\bar{Q}_{i,0}(s) = \frac{\xi}{s + \tau} \left[\bar{P}_{i,0,0}^{(0)}(s) + \sum_{k=0}^{i-1} \left\{ \bar{P}_{i,0,k}^{(1,1)}(s) + \bar{P}_{i,0,k}^{(1,2)}(s) \right\} + \sum_{k=0}^{i-2} \bar{P}_{i,0,k}^{(2)}(s) \right] \quad i \geq 1 \quad (3.17)$$

$$\bar{Q}_{i,j}(s) = \frac{\xi}{s + \tau} \left[\sum_{k=0}^{i-j} \bar{P}_{i,j,k}^{(0)}(s) + \sum_{k=0}^{i-j-1} \left\{ \bar{P}_{i,j,k}^{(1,1)}(s) + \bar{P}_{i,j,k}^{(1,2)}(s) \right\} + \sum_{k=0}^{i-j-2} \bar{P}_{i,j,k}^{(2)}(s) \right] \quad i \geq j \geq 1 \quad (3.18)$$

Taking the Inverse Laplace transform of equations (3.1) – (3.18), we obtained the following *transient state probabilities*.

$$Q_{0,0}(t) = 2\xi \left[\left(\lambda^2 + \tau^2 + \xi^2 + 2 \left(\tau\xi + \lambda\xi - \lambda\tau \right) \right)^{1/2} e^{-1/2(\lambda+\xi+\tau)t} \sinh \left(\frac{1}{2}t \left\{ \lambda^2 + 2\lambda(\xi - \tau) + (\xi + \tau)^2 \right\}^{1/2} \right) \right]^{-1} \quad (3.19)$$

$$P_{0,0,0}^{(0)}(t) = e^{-(\lambda+\xi)t} + \tau e^{-(\lambda+\xi)t} * Q_{0,0}(t) \quad (3.20)$$

$$P_{i,0,0}^{(0)}(t) = \tau e^{-(\lambda+\xi)t} * Q_{i,0}(t) \quad i \geq 1 \quad (3.21)$$

$$P_{i,1,0}^{(0)}(t) = \mu_1 e^{-(\lambda+\xi)t} * P_{i,0,0}^{(1,1)}(t) + \mu_2 e^{-(\lambda+\xi)t} * P_{i,0,0}^{(1,2)}(t) + \tau e^{-(\lambda+\xi)t} * Q_{i,1}(t) \quad i \geq 1 \quad (3.22)$$

$$\begin{aligned} P_{i,2,k}^{(0)} &= \lambda\mu_1 a_1 e^{-(\lambda+\xi+k\theta)t} \left[\frac{1}{\mu_1 + \xi + k\theta} - \frac{e^{-(\mu_1+\xi+k\theta)t}}{\mu_1 + \xi + k\theta} \right] * P_{i-1,1,k}^{(0)}(t) \\ &+ \lambda\mu_2 a_2 e^{-(\lambda+\xi+\theta)t} \left[\frac{1}{\mu_2 + \xi + k\theta} - \frac{e^{-(\mu_2+\xi+k\theta)t}}{\mu_2 + \xi + k\theta} \right] * P_{i-1,1,k}^{(0)}(t) \\ &+ \mu_1 \mu_2 e^{-(\lambda+\xi+k\theta)t} \left[\frac{1}{\mu_1 + \xi + k\theta} - \frac{e^{-(\mu_1+\xi+k\theta)t}}{\mu_1 + \xi + k\theta} \right] * P_{i,0,k}^{(2)}(t) \\ &+ \mu_1 \mu_2 e^{-(\lambda+\xi+k\theta)t} \left[\frac{1}{\mu_2 + \xi + k\theta} - \frac{e^{-(\mu_2+\xi+k\theta)t}}{\mu_2 + \xi + k\theta} \right] * P_{i,0,k}^{(2)}(t) \\ &+ \delta_{0,k} \tau e^{-(\lambda+\xi)t} * Q_{i,2}(t) \quad i \geq 2, k \geq 0 \end{aligned} \quad (3.23)$$

$$P_{i,0,0}^{(1,1)}(t) = \lambda a_1 e^{-(\lambda+\mu_1+\xi)t} * P_{i-1,0,0}^{(0)}(t) \quad i \geq 1 \quad (3.24)$$

$$P_{i,0,0}^{(1,2)}(t) = \lambda a_2 e^{-(\lambda+\mu_2+\xi)t} * P_{i-1,0,0}^{(0)}(t) \quad i \geq 1 \quad (3.25)$$

$$\begin{aligned} P_{i,j,k}^{(0)}(t) &= \lambda\mu_1 a_1 e^{-(\lambda+\xi+k\theta)t} \left[\frac{1}{\mu_1 + \xi + k\theta} - \frac{e^{-(\mu_1+\xi+k\theta)t}}{\mu_1 + \xi + k\theta} \right] * P_{i-1,j-1,k}^{(0)}(t) \\ &+ \lambda\mu_2 a_2 e^{-(\lambda+\xi+k\theta)t} \left[\frac{1}{\mu_2 + \xi + k\theta} - \frac{e^{-(\mu_2+\xi+k\theta)t}}{\mu_2 + \xi + k\theta} \right] * P_{i-1,j-1,k}^{(0)}(t) \\ &+ \mu_1 \mu_2 e^{-(\lambda+\xi+k\theta)t} \left[\frac{1}{\mu_1 + \xi + k\theta} - \frac{e^{-(\mu_1+\xi+k\theta)t}}{\mu_1 + \xi + k\theta} \right] * P_{i,j-2,k}^{(2)}(t) \\ &+ \mu_1 \mu_2 e^{-(\lambda+\xi+k\theta)t} \left[\frac{1}{\mu_2 + \xi + k\theta} - \frac{e^{-(\mu_2+\xi+k\theta)t}}{\mu_2 + \xi + k\theta} \right] * P_{i,j-2,k}^{(2)}(t) \\ &+ (k+1)\theta\mu_1 a_1 e^{-(\lambda+\xi+k\theta)t} \left[\frac{1}{\mu_1 + \xi + k\theta} - \frac{e^{-(\mu_1+\xi+k\theta)t}}{\mu_1 + \xi + k\theta} \right] * P_{i,j-1,k+1}^{(0)}(t) \\ &+ (k+1)\theta\mu_2 a_2 e^{-(\lambda+\xi+k\theta)t} \left[\frac{1}{\mu_2 + \xi + k\theta} - \frac{e^{-(\mu_2+\xi+k\theta)t}}{\mu_2 + \xi + k\theta} \right] * P_{i,j-1,k+1}^{(0)}(t) \\ &+ \delta_{0,k} \tau e^{-(\lambda+\xi)t} * Q_{i,j}(t) \quad i \geq j \geq 3, k \geq 0 \end{aligned} \quad (3.26)$$

$$\begin{aligned}
 P_{i,j,0}^{(1,1)}(t) &= \lambda a_1 e^{-(\lambda+\mu_1+\xi)t} * P_{i-1,j,0}^{(0)}(t) + \mu_2 \lambda e^{-(\lambda+\mu_1+\xi)t} \left[\frac{1}{\mu_1 + \mu_2 + \xi} - \frac{e^{-(\mu_1+\mu_2+\xi)t}}{\mu_1 + \mu_2 + \xi} \right] \\
 &* \left\{ P_{i-1,j-1,0}^{(1,1)}(t) + P_{i-1,j-1,0}^{(1,2)}(t) \right\} + \mu_2 \theta e^{-(\lambda+\mu_1+\xi)t} \left[\frac{1}{\mu_1 + \mu_2 + \xi} - \frac{e^{-(\mu_1+\mu_2+\xi)t}}{\mu_1 + \mu_2 + \xi} \right] \\
 &* \left\{ P_{i,j-1,0}^{(1,1)}(t) + P_{i,j-1,0}^{(1,2)}(t) \right\} + \theta a_1 e^{-(\lambda+\mu_1+\xi)t} * P_{i,j,1}^{(0)}(t) \quad i > j > 0 \quad (3.27)
 \end{aligned}$$

$$\begin{aligned}
 P_{i,j,0}^{(1,2)}(t) &= \lambda a_2 e^{-(\lambda+\mu_2+\xi)t} * P_{i-1,j,0}^{(0)}(t) + \mu_1 \lambda e^{-(\lambda+\mu_2+\xi)t} \left[\frac{1}{\mu_1 + \mu_2 + \xi} - \frac{e^{-(\mu_1+\mu_2+\xi)t}}{\mu_1 + \mu_2 + \xi} \right] \\
 &* \left\{ P_{i-1,j-1,0}^{(1,1)}(t) + P_{i-1,j-1,0}^{(1,2)}(t) \right\} + \mu_1 \theta e^{-(\lambda+\mu_2+\xi)t} \left[\frac{1}{\mu_1 + \mu_2 + \xi} - \frac{e^{-(\mu_1+\mu_2+\xi)t}}{\mu_1 + \mu_2 + \xi} \right] \\
 &* \left\{ P_{i,j-1,0}^{(1,1)}(t) + P_{i,j-1,0}^{(1,2)}(t) \right\} + \theta a_2 e^{-(\lambda+\mu_2+\xi)t} * P_{i,j,1}^{(0)}(t) \quad i > j > 0 \quad (3.28)
 \end{aligned}$$

$$\begin{aligned}
 P_{i,1,k}^{(1,1)}(t) &= \mu_2 e^{-(\lambda+\mu_1+\xi+k\theta)t} \sum_{p=0}^k \left(\lambda^{k+1-p} \left[\frac{1}{(\mu_1 + \mu_2 + \xi)^{k+1-p}} - e^{-(\mu_1+\mu_2+\xi)t} \right. \right. \\
 &\left. \left. \sum_{r=0}^{k-p} \frac{t^r}{r!} \frac{1}{(\mu_1 + \mu_2 + \xi)^{k+1-r-p}} \right] * \left\{ P_{p+1,0,p}^{(1,1)}(t) + P_{p+1,0,p}^{(1,2)}(t) \right\} \right) + \mu_2 e^{-(\lambda+\mu_1+\xi+k\theta)t} \\
 &\sum_{p=0}^{k-1} \left(\lambda^{k-p} (p+1) \theta \left[\frac{1}{(\mu_1 + \mu_2 + \xi)^{k+1-p}} - e^{-(\mu_1+\mu_2+\xi)t} \right. \right. \\
 &\left. \left. \sum_{r=0}^{k-p} \frac{t^r}{r!} \frac{1}{(\mu_1 + \mu_2 + \xi)^{k+1-r-p}} \right] * \left\{ P_{p+2,0,p+1}^{(1,1)}(t) + P_{p+2,0,p+1}^{(1,2)}(t) \right\} \right) \\
 &+ (k+1) \theta \mu_2 e^{-(\lambda+\mu_1+\xi+k\theta)t} \left[\frac{1}{\mu_1 + \mu_2 + \xi} - \frac{e^{-(\mu_1+\mu_2+\xi)t}}{\mu_1 + \mu_2 + \xi} \right] \\
 &* \left\{ P_{i,0,k+1}^{(1,1)}(t) + P_{i,0,k+1}^{(1,2)}(t) \right\} \quad i > 2, k \geq 1 \quad (3.29)
 \end{aligned}$$

$$\begin{aligned}
 P_{i,1,k}^{(1,2)}(t) &= \mu_1 e^{-(\lambda+\mu_2+\xi+k\theta)t} \sum_{p=0}^k \left(\lambda^{k+1-p} \left[\frac{1}{(\mu_1 + \mu_2 + \xi)^{k+1-p}} - e^{-(\mu_1+\mu_2+\xi)t} \right. \right. \\
 &\left. \left. \sum_{r=0}^{k-p} \frac{t^r}{r!} \frac{1}{(\mu_1 + \mu_2 + \xi)^{k+1-r-p}} \right] * \left\{ P_{p+1,0,p}^{(1,1)}(t) + P_{p+1,0,p}^{(1,2)}(t) \right\} \right) + \mu_1 e^{-(\lambda+\mu_2+\xi+k\theta)t} \\
 &\sum_{p=0}^{k-1} \left(\lambda^{k-p} (p+1) \theta \left[\frac{1}{(\mu_1 + \mu_2 + \xi)^{k+1-p}} - e^{-(\mu_1+\mu_2+\xi)t} \right. \right. \\
 &\left. \left. \sum_{r=0}^{k-p} \frac{t^r}{r!} \frac{1}{(\mu_1 + \mu_2 + \xi)^{k+1-r-p}} \right] * \left\{ P_{p+2,0,p+1}^{(1,1)}(t) + P_{p+2,0,p+1}^{(1,2)}(t) \right\} \right) \\
 &+ (k+1) \theta \mu_1 e^{-(\lambda+\mu_2+\xi+k\theta)t} \left[\frac{1}{\mu_1 + \mu_2 + \xi} - \frac{e^{-(\mu_1+\mu_2+\xi)t}}{\mu_1 + \mu_2 + \xi} \right] \\
 &* \left\{ P_{i,0,k+1}^{(1,1)}(t) + P_{i,0,k+1}^{(1,2)}(t) \right\} \quad i > 2, k \geq 1 \quad (3.30)
 \end{aligned}$$

$$\begin{aligned}
 P_{i,j,0}^{(2)}(t) &= \lambda e^{-(\lambda+\xi+\mu_1+\mu_2)t} * \left\{ P_{i-1,j,0}^{(1,1)}(t) + P_{i-1,j,0}^{(1,2)}(t) \right\} + \theta e^{-(\lambda+\xi+\mu_1+\mu_2)t} \\
 &* \left\{ P_{i,j,1}^{(1,1)}(t) + P_{i,j,1}^{(1,2)}(t) \right\} \quad i > j + 1, j \geq 0 \quad (3.31)
 \end{aligned}$$

$$\begin{aligned}
 P_{i,j,k}^{(1,1)}(t) &= \lambda a_1 e^{-(\lambda+\mu_1+\xi+k\theta)t} * P_{i-1,j,k}^{(0)}(t) + \mu_2 e^{-(\lambda+\mu_1+\xi+k\theta)t} \sum_{p=0}^k \left(\lambda^{k+1-p} \right. \\
 &\quad \left[\frac{1}{(\mu_1 + \mu_2 + \xi)^{k+1-p}} - e^{-(\mu_1+\mu_2+\xi)t} \sum_{r=0}^{k-p} \frac{t^r}{r!} \frac{1}{(\mu_1 + \mu_2 + \xi)^{k+1-r-p}} \right] \\
 &\quad * \left\{ P_{i-k-1+p,j-1,p}^{(1,1)}(t) + P_{i-k-1+p,j-1,p}^{(1,2)}(t) \right\} \Bigg) + \mu_2 e^{-(\lambda+\mu_1+\xi+k\theta)t} \sum_{p=0}^{k-1} \left(\lambda^{k-p} \right. \\
 &\quad \left. (p+1)\theta \left[\frac{1}{(\mu_1 + \mu_2 + \xi)^{k+1-p}} - e^{-(\mu_1+\mu_2+\xi)t} \sum_{r=0}^{k-p} \frac{t^r}{r!} \frac{1}{(\mu_1 + \mu_2 + \xi)^{k+1-r-p}} \right] \right. \\
 &\quad * \left\{ P_{i-k-p,j-1,p+1}^{(1,1)}(t) + P_{i-k-p,j-1,p+1}^{(1,2)}(t) \right\} \Bigg) + \mu_2 (k+1)\theta e^{-(\lambda+\mu_1+\xi+k\theta)t} \\
 &\quad \left[\frac{1}{\mu_1 + \mu_2 + \xi} - \frac{e^{-(\mu_1+\mu_2+\xi)t}}{\mu_1 + \mu_2 + \xi} \right] * \left\{ P_{i,j-1,k+1}^{(1,1)}(t) + P_{i,j-1,k+1}^{(1,2)}(t) \right\} + (k+1)\theta a_1 \\
 &\quad e^{-(\lambda+\mu_1+\xi+k\theta)t} * P_{i,j,k+1}^{(0)}(t) \quad i \geq j+k+1, j > k \geq 1 \quad (3.32)
 \end{aligned}$$

$$\begin{aligned}
 P_{i,j,k}^{(1,2)}(t) &= \lambda a_2 e^{-(\lambda+\mu_2+\xi+k\theta)t} * P_{i-1,j,k}^{(0)}(t) + \mu_1 e^{-(\lambda+\mu_2+\xi+k\theta)t} \sum_{p=0}^k \left(\lambda^{k+1-p} \right. \\
 &\quad \left[\frac{1}{(\mu_1 + \mu_2 + \xi)^{k+1-p}} - e^{-(\mu_1+\mu_2+\xi)t} \sum_{r=0}^{k-p} \frac{t^r}{r!} \frac{1}{(\mu_1 + \mu_2 + \xi)^{k+1-r-p}} \right] \\
 &\quad * \left\{ P_{i-k-1+p,j-1,p}^{(1,1)}(t) + P_{i-k-1+p,j-1,p}^{(1,2)}(t) \right\} \Bigg) + \mu_1 e^{-(\lambda+\mu_2+\xi+k\theta)t} \sum_{p=0}^{k-1} \left(\lambda^{k-p} \right. \\
 &\quad \left. (p+1)\theta \left[\frac{1}{(\mu_1 + \mu_2 + \xi)^{k+1-p}} - e^{-(\mu_1+\mu_2+\xi)t} \sum_{r=0}^{k-p} \frac{t^r}{r!} \frac{1}{(\mu_1 + \mu_2 + \xi)^{k+1-r-p}} \right] \right. \\
 &\quad * \left\{ P_{i-k-p,j-1,p+1}^{(1,1)}(t) + P_{i-k-p,j-1,p+1}^{(1,2)}(t) \right\} \Bigg) + \mu_1 (k+1)\theta e^{-(\lambda+\mu_2+\xi+k\theta)t} \\
 &\quad \left[\frac{1}{\mu_1 + \mu_2 + \xi} - \frac{e^{-(\mu_1+\mu_2+\xi)t}}{\mu_1 + \mu_2 + \xi} \right] * \left\{ P_{i,j-1,k+1}^{(1,1)}(t) + P_{i,j-1,k+1}^{(1,2)}(t) \right\} + (k+1)\theta a_2 \\
 &\quad e^{-(\lambda+\mu_2+\xi+k\theta)t} * P_{i,j,k+1}^{(0)}(t) \quad i \geq j+k+1, j > k \geq 1 \quad (3.33)
 \end{aligned}$$

$$\begin{aligned}
 P_{i,j,k}^{(2)}(t) &= \lambda e^{-(\lambda+\xi+\mu_1+\mu_2)t} * \left\{ P_{i-1,j,k}^{(1,1)}(t) + P_{i-1,j,k}^{(1,2)}(t) \right\} + e^{-(\lambda+\xi+\mu_1+\mu_2)t} \sum_{p=0}^{k-1} \left(\lambda^{k+1-p} \right. \\
 &\quad \frac{t^{k-p}}{(k-p)!} * \left\{ P_{i-k-1+p,j,p}^{(1,1)}(t) + P_{i-k-1+p,j,p}^{(1,2)}(t) \right\} \Bigg) + e^{-(\lambda+\xi+\mu_1+\mu_2)t} \sum_{p=0}^{k-1} \left(\lambda^{k-p} (p+1)\theta \right. \\
 &\quad \frac{t^{k-p}}{(k-p)!} * \left\{ P_{i-k+p,j,p+1}^{(1,1)}(t) + P_{i-k+p,j,p+1}^{(1,2)}(t) \right\} \Bigg) + (k+1)\theta e^{-(\lambda+\xi+\mu_1+\mu_2)t} \\
 &\quad * \left\{ P_{i,j,k+1}^{(1,1)}(t) + P_{i,j,k+1}^{(1,2)}(t) \right\} \quad i \geq j+k+2, j \geq 0, k \geq 1 \quad (3.34)
 \end{aligned}$$

$$Q_{i,0}(t) = \xi e^{-\tau t} * \left[P_{i,0,0}^{(0)}(t) + \sum_{k=0}^{i-1} \left(P_{i,0,k}^{(1,1)}(t) + P_{i,0,k}^{(1,2)}(t) \right) + \sum_{k=0}^{i-2} P_{i,0,k}^{(2)}(t) \right] \quad i \geq 1 \quad (3.35)$$

$$Q_{i,j}(t) = \xi e^{-\tau t} * \left[\sum_{k=0}^{i-j} P_{i,j,k}^{(0)}(t) + \sum_{k=0}^{i-j-1} \left(P_{i,j,k}^{(1,1)}(t) + P_{i,j,k}^{(1,2)}(t) \right) + \sum_{k=0}^{i-j-2} P_{i,j,k}^{(2)}(t) \right] \quad i \geq j \geq 1 \quad (3.36)$$

4 Verification of Results

- Taking $\xi = 0, \tau = 0$ and

$$P_{i,j,k}^{(0)}(t) = P_{i,j}^{(0)}(t) \text{ where } k = \text{number of arrivals - number of departures.}$$

$$P_{i,j,k}^{(1,l)}(t) = P_{i,j}^{(1,l)}(t) \text{ where } k = \text{number of arrivals - number of departures - 1.}$$

$$P_{i,j,k}^{(2)}(t) = P_{i,j}^{(2)}(t) \text{ where } k = \text{number of arrivals - number of departures - 2.}$$

in equations (2.1 – 2.7) we get,

$$\frac{d}{dt}P_{i,j}^{(0)}(t) = -(\lambda + (i-j)\theta)P_{i,j}^{(0)}(t) + \mu_1(1 - \delta_{0,j})P_{i,j-1}^{(1,1)}(t) + \mu_2(1 - \delta_{0,j})P_{i,j-1}^{(1,2)}(t) \quad i \geq j \geq 0 \quad (4.1)$$

$$\frac{d}{dt}P_{1,0}^{(1,1)}(t) = -(\lambda + \mu_1)P_{1,0}^{(1,1)}(t) + \lambda a_1 P_{0,0}^{(0)}(t) \quad (4.2)$$

$$\begin{aligned} \frac{d}{dt}P_{i,j}^{(1,1)}(t) = & -(\lambda + \mu_1 + (i-j-1)\theta)P_{i,j}^{(1,1)}(t) + \lambda a_1(1 - \delta_{0,j})P_{i-1,j}^{(0)}(t) \\ & + (i-j)\theta a_1 P_{i,j}^{(0)}(t) + \mu_2 P_{i,j-1}^{(2)}(t) \quad i > 1, i > j \geq 0 \end{aligned} \quad (4.3)$$

$$\frac{d}{dt}P_{1,0}^{(1,2)}(t) = -(\lambda + \mu_2)P_{1,0}^{(1,2)}(t) + \lambda a_2 P_{0,0}^{(0)}(t) \quad (4.4)$$

$$\begin{aligned} \frac{d}{dt}P_{i,j}^{(1,2)}(t) = & -(\lambda + \mu_2 + (i-j-1)\theta)P_{i,j}^{(1,2)}(t) + \lambda a_2(1 - \delta_{0,j})P_{i-1,j}^{(0)}(t) \\ & + (i-j)\theta a_2 P_{i,j}^{(0)}(t) + \mu_1 P_{i,j-1}^{(2)}(t) \quad i > 1, i > j \geq 0 \end{aligned} \quad (4.5)$$

$$\begin{aligned} \frac{d}{dt}P_{i,j}^{(2)}(t) = & -(\lambda + \mu_1 + \mu_2)P_{i,j}^{(2)}(t) + \lambda\{P_{i-1,j}^{(1,1)}(t) + P_{i-1,j}^{(1,2)}(t)\} + (i-j-1)\theta \\ & \{P_{i,j}^{(1,1)}(t) + P_{i,j}^{(1,2)}(t)\} + \lambda(1 - \delta_{i-2,j})P_{i-1,j}^{(2)}(t) \quad i > 1, i > j \geq 0 \end{aligned} \quad (4.6)$$

Furthermore, the model given in equations (4.1) – (4.6) can be converted into one-dimensional model by defining the probability $U_n^{(m)}(t)$ as:

$$U_n^{(m)}(t) = \text{Probability that there are } n \text{ customers in the orbit when } m(m = 0, 1, 2) \text{ servers are busy at time } t.$$

when both servers are free, it is represented as Probability:

$$U_n^{(0)}(t) = \sum_{j=0}^{\infty} P_{j+n,j}^{(0)}(t)$$

where n = number of arrivals - number of departures.

when only one server is busy($m = 1$), it is defined as Probability:

$$U_n^{(m,l)}(t) = \sum_{j=0}^{\infty} P_{j+n+m,j}^{(m,l)}(t) \quad l = 1, 2$$

where n = number of arrivals - number of departures - m .

when both servers are busy($m = 2$), it is defined as Probability:

$$U_n^{(m)}(t) = \sum_{j=0}^{\infty} P_{j+n+m,j}^{(m)}(t)$$

where n = number of arrivals - number of departures - m .

By using above definitions and putting $\mu_1 = \mu_2 = 1$ in (4.1) – (4.6) the equations in statistical equilibrium are:

$$(\lambda + n\theta)U_n^{(0)} = U_n^{(1,1)} + U_n^{(1,2)} \quad n \geq 0 \quad (4.7)$$

$$(\lambda + n\theta + 1)U_n^{(1,1)} = \lambda a_1 U_n^{(0)} + (n+1)\theta a_1 U_{n+1}^{(0)} + U_n^{(2)} \quad n \geq 1 \quad (4.8)$$

$$(\lambda + n\theta + 1)U_n^{(1,2)} = \lambda a_2 U_n^{(0)} + (n+1)\theta a_2 U_{n+1}^{(0)} + U_n^{(2)} \quad n \geq 1 \quad (4.9)$$

$$(\lambda + 2)U_n^{(2)} = \lambda\{U_n^{(1,1)} + U_n^{(1,2)}\} + \lambda(1 - \delta_{n,2})U_{n-1}^{(2)} + (n+1)\theta\{U_{n+1}^{(1,1)} + U_{n+1}^{(1,2)}\} \quad n \geq 2 \quad (4.10)$$

Using $U_n^{(1,1)} + U_n^{(1,2)} = U_n^{(1)}$ and letting $a_1 = a_2 = \frac{1}{2}$ and adding equations (4.8) and (4.9) we obtained the following set of equations:

$$(\lambda + n\theta)U_n^{(0)} = U_n^{(1)} \quad n \geq 0 \quad (4.11)$$

$$(\lambda + n\theta + 1)U_n^{(1)} = \lambda U_n^{(0)} + (n+1)\theta U_{n+1}^{(0)} + 2U_n^{(2)} \quad n \geq 1 \quad (4.12)$$

$$(\lambda + 2)U_n^{(2)} = \lambda U_n^{(1)} + \lambda(1 - \delta_{n,2})U_{n-1}^{(2)} + (n+1)\theta U_{n+1}^{(1)} \quad n \geq 2 \quad (4.13)$$

where

$$\delta_{n,2} = \begin{cases} 1 & \text{when } n = 2 \\ 0 & \text{otherwise} \end{cases}$$

which coincides with the results (2.1) and (2.3) of (Falin and Templeton, 1997)

5 Numerical Solution and Graphical Representation

The Numerical results are generated using MATLAB programming and following (Bunday, 1986) for the case $\rho = \left(\frac{\lambda}{\mu_1 + \mu_2}\right) = 0.7$, $\eta = \left(\frac{\theta}{\mu_1 + \mu_2}\right) = 0.6$, $\tau' = \left(\frac{\tau}{\mu_1 + \mu_2}\right) = 0.4$, $\xi' = \left(\frac{\xi}{\mu_1 + \mu_2}\right) = 0.3$, $r_1 = \left(\frac{\mu_1}{\mu_1 + \mu_2}\right) = 0.4$, $r_2 = (1 - r_1) = 0.6$, $a_1 = a_2 = 0.5$. In following tables, we observe some probabilities at various time instants whose sum approaches to 1.

Table 1: At time $t = 1$

t	$P_{0,0,0}^{(0)}$	$P_{1,0,0}^{(0)}$	$P_{1,1,0}^{(0)}$	$P_{2,0,0}^{(0)}$	$P_{2,1,0}^{(0)}$	$P_{1,0,0}^{(1,1)}$	$P_{2,1,0}^{(1,1)}$	$P_{1,0,0}^{(1,2)}$	$P_{2,0,0}^{(2)}$	$P_{3,0,1}^{(2)}$	$Q_{0,0}$
1	0.3954	0.0054	0.0559	0.0008	0.0003	0.1089	0.0175	0.0995	0.0566	0.0116	0.1542

$Q_{1,0}$	$Q_{1,1}$	$Q_{2,0}$	Sum
0.0419	0.0068	0.0076	0.9624

Table 2: At time $t = 25$

t	$P_{5,0,0}^{(0)}$	$P_{5,1,0}^{(0)}$	$P_{5,2,0}^{(0)}$	$P_{5,3,0}^{(0)}$	$P_{5,4,0}^{(0)}$	$P_{5,5,0}^{(0)}$	$Q_{5,0}$	$Q_{5,1}$	$Q_{5,2}$	$Q_{5,3}$	$Q_{5,4}$
25	0.0156	0.0597	0.1143	0.1398	0.1179	0.0708	0.0124	0.0467	0.0875	0.1047	0.0866

$Q_{5,5}$	$Q_{4,2}$	$Q_{4,3}$	$Q_{4,4}$	$Q_{2,1}$	$Q_{2,2}$	$Q_{3,0}$	$Q_{3,1}$	$Q_{3,2}$	$Q_{3,3}$	$Q_{4,0}$
0.0517	0.006	0.0038	0.0011	0.0029	0.0012	0.0022	0.0044	0.0036	0.0012	0.0019

$Q_{4,1}$	Sum
0.005	0.941

 Table 3: At time $t = 35$

t	$P_{5,0,0}^{(0)}$	$P_{5,1,0}^{(0)}$	$P_{5,2,0}^{(0)}$	$P_{5,3,0}^{(0)}$	$P_{5,4,0}^{(0)}$	$P_{5,5,0}^{(0)}$	$Q_{5,0}$	$Q_{5,1}$	$Q_{5,2}$	$Q_{5,3}$	$Q_{5,4}$
35	0.0172	0.0662	0.1264	0.1527	0.1258	0.0729	0.0131	0.0501	0.0951	0.1144	0.0939

$Q_{5,5}$	Sum
0.0544	0.9822

 Table 4: At time $t = 40$

t	$P_{5,0,0}^{(0)}$	$P_{5,1,0}^{(0)}$	$P_{5,2,0}^{(0)}$	$P_{5,3,0}^{(0)}$	$P_{5,4,0}^{(0)}$	$P_{5,5,0}^{(0)}$	$Q_{5,0}$	$Q_{5,1}$	$Q_{5,2}$	$Q_{5,3}$	$Q_{5,4}$
40	0.0175	0.0672	0.1281	0.1545	0.1268	0.0731	0.0132	0.0506	0.0962	0.1158	0.0949

$Q_{5,5}$	Sum
0.0548	0.9927

The probabilities against time are represented graphically in the following figures.

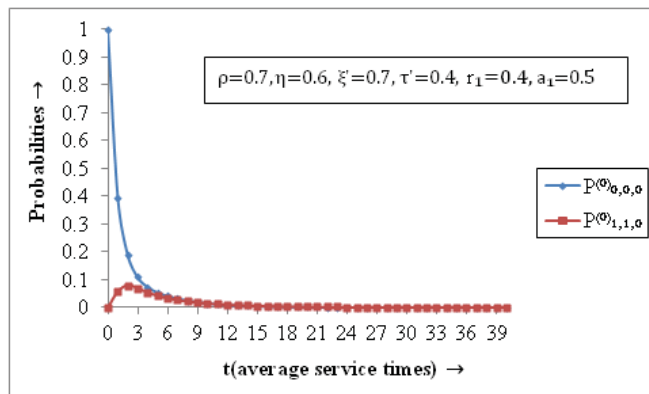


Figure 2

In figure 2, the probabilities $P_{0,0,0}^{(0)}$ and $P_{1,1,0}^{(0)}$ are plotted against average service time t for the given case. Here we observe that the probability $P_{0,0,0}^{(0)}$ decreases rapidly from its initial value

1 (at time $t = 0$) whereas the probability $P_{1,1,0}^{(0)}$ increases from its initial value 0 (at $t = 0$) in the beginning and then decreases.

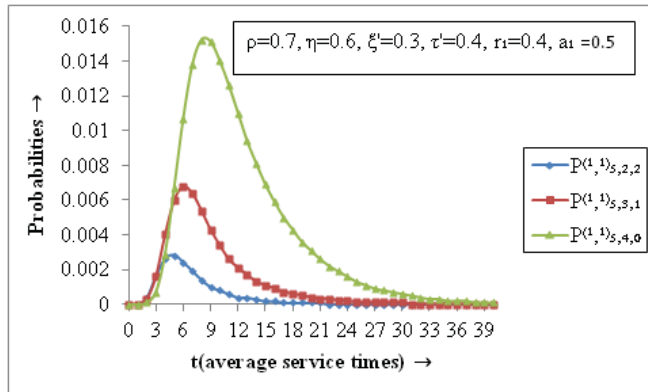


Figure 3

Figure 3 shows the comparison among the probabilities $P_{5,2,2}^{(1,1)}$, $P_{5,3,1}^{(1,1)}$ and $P_{5,4,0}^{(1,1)}$. Here we observe that all the probabilities increase rapidly in the beginning and then start decreasing after reaching their peak. Moreover, the probabilities are higher for large number of departures.

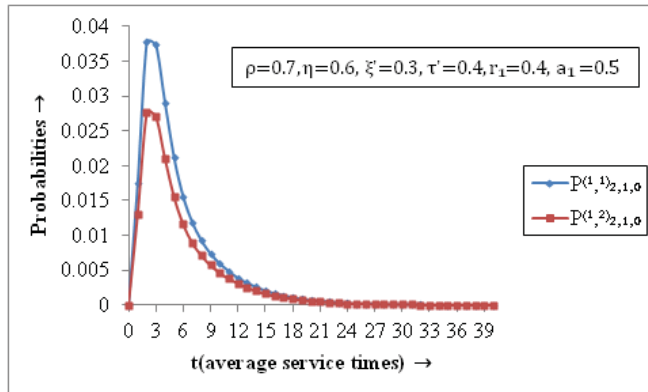


Figure 4

In figure 4, the probabilities $P_{2,1,0}^{(1,1)}$ and $P_{2,1,0}^{(1,2)}$ are plotted against time t for the given case. It is observed that the probabilities increase initially and then decrease. In addition, we can see that the probabilities of first server busy are higher than of second server busy since r_2 is greater than r_1 .

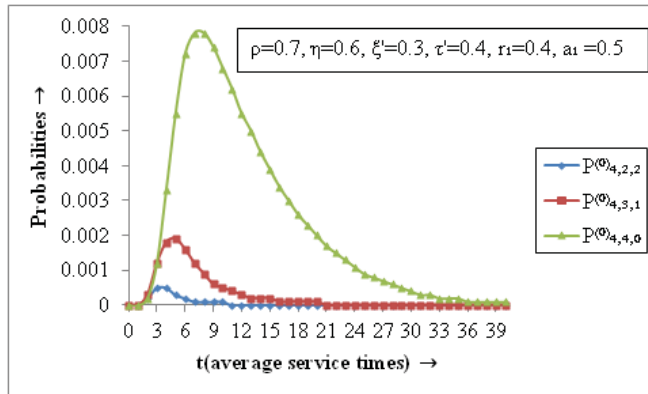


Figure 5

In figure 5, the probabilities $P_{4,2,2}^{(0)}$, $P_{4,3,1}^{(0)}$ and $P_{4,4,0}^{(0)}$ are compared against time t . The graph clearly indicates that all probabilities increase in their starting moments and then decrease gradually. In this graph we observe that the probability when both servers are free attains higher values for greater number of departures.

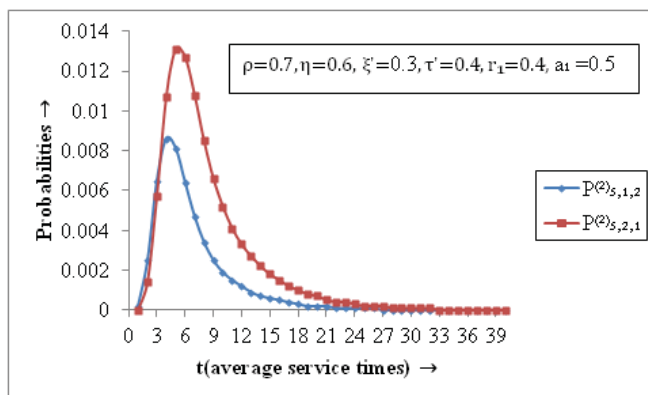


Figure 6

In figure 6, the probabilities $P_{5,1,2}^{(2)}$ and $P_{5,2,1}^{(2)}$ are plotted against time t . Here it is observed that the probability when both servers are busy is higher whenever the number of departures is larger.

5.1 Effect of Catastrophe rate per unit service time on the probability of server being under repair

To study the effect of change of ξ' (catastrophe rate per unit service time) on the various probabilities, data is summarized in the table given below. It can be seen that in general whenever the catastrophe rate per unit service time increases, the probability of server being under repair

also increases which is as desired.

	$\xi' = 0.3$			$\xi' = 0.6$			$\xi' = 0.9$		
t	$Q_{2,1}$	$Q_{3,1}$	$Q_{4,1}$	$Q_{2,1}$	$Q_{3,1}$	$Q_{4,1}$	$Q_{2,1}$	$Q_{3,1}$	$Q_{4,1}$
1	0.0028	0.0006	0.0001	0.0046	0.0009	0.0001	0.0055	0.0011	0.0002
3	0.0285	0.0126	0.0044	0.0509	0.0322	0.0052	0.0348	0.0145	0.0047
6	0.0393	0.0251	0.013	0.0509	0.0322	0.016	0.0514	0.0313	0.0148
9	0.0318	0.0256	0.0163	0.0495	0.0396	0.0244	0.0552	0.0423	0.0248
12	0.0226	0.0216	0.016	0.0424	0.0403	0.029	0.0525	0.0477	0.0327
15	0.015	0.0164	0.0137	0.0338	0.0369	0.0302	0.0464	0.0484	0.0377
18	0.0095	0.0117	0.0108	0.0256	0.0315	0.0287	0.039	0.0457	0.0397
21	0.0058	0.0079	0.008	0.0187	0.0255	0.0256	0.0316	0.041	0.0392
24	0.0035	0.0051	0.0056	0.0187	0.0255	0.0256	0.0249	0.0354	0.0369
27	0.002	0.0032	0.0038	0.0093	0.015	0.0177	0.0192	0.0297	0.0334
30	0.0012	0.002	0.0025	0.0064	0.0111	0.014	0.0145	0.0243	0.0293
33	0.0007	0.0012	0.0016	0.0043	0.008	0.0108	0.0109	0.0194	0.025
36	0.0004	0.0007	0.001	0.0029	0.0057	0.0082	0.008	0.0153	0.0209
39	0.0002	0.0004	0.0006	0.0019	0.004	0.0061	0.0058	0.0118	0.0172
t	$Q_{3,0}$	$Q_{3,1}$	$Q_{3,2}$	$Q_{3,0}$	$Q_{3,1}$	$Q_{3,2}$	$Q_{3,0}$	$Q_{3,1}$	$Q_{3,2}$
1	0.0012	0.0006	0.0001	0.0011	0.0006	0.0001	0.001	0.0005	0.0001
3	0.0104	0.0137	0.0077	0.0095	0.0126	0.0071	0.0078	0.0104	0.0059
6	0.0158	0.0284	0.025	0.014	0.0251	0.0215	0.0099	0.0182	0.0154
9	0.0167	0.0313	0.0288	0.0135	0.0256	0.0229	0.0072	0.0146	0.0132
12	0.0162	0.0307	0.0276	0.0111	0.0216	0.019	0.0041	0.0088	0.0081
15	0.0149	0.0284	0.0248	0.0083	0.0164	0.0142	0.002	0.0045	0.0041
18	0.0132	0.0251	0.0215	0.0059	0.0117	0.0099	0.0009	0.0021	0.0019
21	0.0114	0.0216	0.0182	0.0039	0.0079	0.0066	0.0004	0.0009	0.0008
24	0.0095	0.0182	0.015	0.0026	0.0051	0.0042	0.0001	0.0004	0.0003
27	0.0079	0.015	0.0122	0.0016	0.0032	0.0026	0.0001	0.0001	0.0001
30	0.0064	0.0121	0.0098	0.001	0.002	0.0016	0	0.0001	0
33	0.0051	0.0097	0.0077	0.0006	0.0012	0.001	0	0	0
36	0.0041	0.0077	0.006	0.0004	0.0007	0.0006	0	0	0
39	0.0032	0.0061	0.0047	0.0002	0.0004	0.0003	0	0	0

This can also be shown graphically by taking the probability of server being under repair when the number of arrivals are three and the departed units are zero for different values of ξ' .

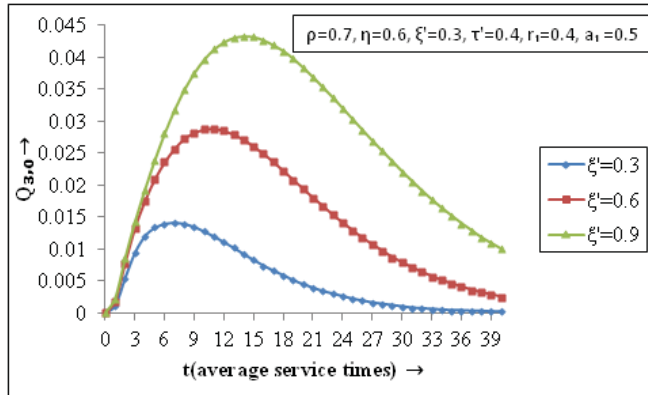


Figure 7

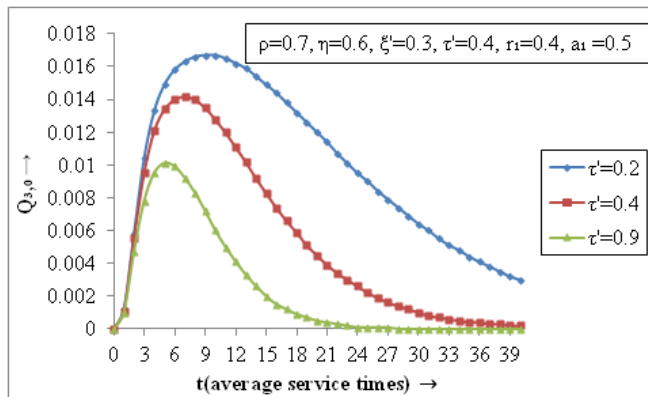


Figure 8

Similarly in figure 8, the effect of τ' (change of repair rate per unit service time) on the probability $Q_{3,0}$ is studied. It is observed from the graph that the probability of server being under repair decreases with the increasing values of repair rate per unit service time.

6 Conclusion

In this paper a three-state retrial queueing system with catastrophe having two non-identical parallel servers is discussed. We come across many situations where a sudden occurrence of any calamity may cause customer loss and affect the system profit as well. Here to model many real life situations the factors retrial, catastrophe, heterogeneity of servers are incorporated. The results are well-quantified from a three-state model. Using the Laplace transformation and recursive technique, the difference-differential equations are solved. Thereby the time dependent probabilities for exact number of arrivals in the system, exact number of departures after taking service from the system, exact number of units in the orbit with both, one or none

servers are busy, are obtained. Numerical solutions are generated and represented graphically to study the effect of various parameters.

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