

Partial Null-controllability of Evolution Equations by One-dimensional Controls

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ABSTRACT

The problem of partial null controllability for linear autonomous evolution equations, which are controlled by a one-dimensional control, is under consideration. The partial null-controllability conditions for coupled abstract evolution systems have been obtained using the moment problem approach.

Keywords: Partial null-controllability of evolution equations, coupled systems, eigenvalue problems, moment method, strongly minimal sequences.

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1 Introduction

Let X, Y, U be separable Hilbert spaces. Consider the evolution equation

$$\dot{x}(t) = Ax(t) + bu(t), \quad (1.1)$$

$$x(0) = x_0, \quad (1.2)$$

$$y(t) = Px(t), t \geq 0, \quad (1.3)$$

where $x(t) \in X$ is a current state, $x_0 \in X$ is an initial state; the operator A generates a strongly continuous C_0 -semigroup $S(t)$ of bounded operators (Hille and Phillips, 1977), $b \in X$, $P : X \rightarrow Y$ is a linear bounded operator. The input function $u(t)$ is assumed to be locally L^2 -integrated. Following (Ammar-Khodja, Chouly and Duprez, 2017) we assume, that the observation operator $P : X \rightarrow Y$ is an orthogonal projection operator,

Denote by $x(t, x_0, u(\cdot))$ the mild solution of equation (1.1) with initial condition (1.2).

Definition 1.1. Equation (1.1)–(1.3) is said to be completely null-controllable on $[0, t_1]$, if for each $x_0 \in X$ there is a control $u(t), u(\cdot) \in L_2([0, t_1], U)$, such that the corresponding solution $x(t, x_0, u(\cdot))$ satisfies the condition

$$x(t_1, x_0, u(\cdot)) = 0.$$

Definition 1.2. Equation (1.1)–(1.3) is said to be partially null-controllable on $[0, t_1]$, if for each $x_0 \in X$ there is a solution $x(t, x_0, u(\cdot))$ satisfies the condition

$$Px(t_1, x_0, u(\cdot)) = 0. \quad (1.4)$$

Further for the sake of brevity we use $x(t)$ instead of $x(t, x_0, u(\cdot))$.

The partial controllability has been well investigated for finite-dimensional systems (see, for example, (Gabasov and Kirillova, 1971) and references therein). However, as far as we know there are very few investigations on the partial controllability of infinite-dimensional equations (see (Bashirov, Mahmudov, Semi and Etikan, 2007), (Ammar-Khodja et al., 2017) and reference therein).

In recent years, there has been an increasing interest in the study of the controllability of coupled PDEs. Partial controllability of such systems is one of important applications to control of these systems. In (Ammar-Khodja et al., 2017) partial controllability conditions have been obtained for coupled parabolic systems (see also references therein). Controllability conditions of (Avdonin, Rivero and Teresa, 2007) (see also references therein) may be considered as partially null controllability conditions for coupled hyperbolic (namely waved) equations, where only initial condition conditions $x_0 \in Y$ are considered.

The main goal of this paper is to obtain results related to partial controllability issues for linear evolution equations using the moment method, which is a very powerful tool for investigation of (complete) controllability for linear evolution systems (see (Avdonin and Ivanov, 1995), (Ammar-Khodja, 2018), (Avdonin et al., 2007) and references therein). The paper aims to provide insights into controlling such systems and may discuss aspects related to hyperbolic and parabolic systems. The moment method is a mathematical technique employed to analyze and establish controllability in these systems.

We assume that A has the properties:

(I) the operator A has a purely point spectrum σ_A which is either finite or has no finite limit points, and each $\lambda \in \sigma_A$ has a finite multiplicity;

(II) the family of generalized eigenvectors of the operator A forms a Riesz basis of X .

It is well-known, that $x(t)$ is defined by the formula ¹:

$$x(t) = S(t)x_0 + \int_0^t S(t-\tau)bu(\tau)d\tau. \quad (1.5)$$

2 Coupled evolution systems

Let $X = Y \times Z$, where Y and Z are separable Hilbert spaces². Consider the linear evolution system

$$\dot{y}(t) = A_Y y(t) + A_{YZ} z(t) + b_Y u(t), \quad (2.1)$$

$$\dot{z}(t) = A_Z z(t) + b_Z u(t). \quad (2.2)$$

$$P = \begin{pmatrix} I_Y & 0 \\ 0 & 0 \end{pmatrix}, \quad (2.3)$$

¹The integral is understood in the Bochner sense.

²We will not distinguish between spaces X and $X \times \{0\}$, Y and $\{0\} \times Y$.

where the operators A_Y and A_Z generate strongly continuous semigroups $S_Y(t)$ and $S_Z(t)$ in X and Y correspondingly, and $S_{YZ}(t) : Z \rightarrow Y$ is a bounded strongly continuous operator.

This system is a partial case of system (1.1)–(1.3), where $x = \begin{pmatrix} y \\ z \end{pmatrix}$, $y \in Y, z \in Z$, $A =$

$$\begin{pmatrix} A_Y & A_{YZ} \\ 0 & A_Z \end{pmatrix}, A^* = \begin{pmatrix} A_Y^* & 0 \\ A_{YZ}^* & A_Z^* \end{pmatrix}, b = \begin{pmatrix} b_Y \\ b_Z \end{pmatrix}.$$

We assume, that eigenvalues of the operators A_Y and A_Z are simple, and the operator

$$A = \begin{pmatrix} A_Y & A_{YZ} \\ 0 & A_Z \end{pmatrix} \text{ satisfies the conditions (I)–(II).}$$

The linear bounded operator A_{YZ} will be called a couple operator.

We have

$$S(t) = \begin{pmatrix} S_Y(t) & S_{YZ}(t) \\ 0 & S_Z(t) \end{pmatrix},$$

where $S_Y(t)$ and $S_Z(t)$ are strongly continuous semigroups, generated by the operators A_Y and A_Z correspondingly, and $S_{YZ}(t) : Z \rightarrow Y$ is a bounded strongly continuous operator.

Let the eigenvalues $\lambda_i \in \sigma_A, i \in \mathbb{N}$ be enumerated in the order of non-decreasing of their absolute values.

Denote by σ_Y the spectrum of the operator A_Y , and by σ_Z the spectrum of the operator A_Z .

Denote also: $I_Y := \{i \in \mathbb{N} : \lambda_i \in \sigma_Y\}$, $I_Z := \{i \in \mathbb{N} : \lambda_i \notin \sigma_Y\}$. Obviously, $\sigma_Y \cup \sigma_Z = \sigma_A$, $I_Y \cup I_Z = \mathbb{N}$.

Denote $\sigma_{YZ} = \sigma_Y \cap \sigma_Z$.

3 The case $\sigma_{YZ} = \emptyset$

3.1 Eigenvectors and eigenvalues

In the beginning consider the case $\sigma_{YZ} = \emptyset$. Obviously, $I_Y \cap I_Z = \emptyset$ also.

3.1.1 Eigenvectors of operator A

Let $\varphi_i = \begin{pmatrix} \varphi_{Yi} \\ \varphi_{Zi} \end{pmatrix}$, $i \in \mathbb{N}$ be an eigenvector of the operator A , corresponding to its eigenvalue $\lambda_i, i \in \mathbb{N}$. We have

$$A\varphi_i = \begin{pmatrix} A_Y\varphi_{Yi} + A_{YZ}\varphi_{Zi} \\ A_Z\varphi_{Zi} \end{pmatrix} = \lambda_i \begin{pmatrix} \varphi_{Yi} \\ \varphi_{Zi} \end{pmatrix}, i \in \mathbb{N}.$$

Hence,

$$\begin{aligned} A_Y\varphi_{Yi} + A_{YZ}\varphi_{Zi} &= \lambda_i\varphi_{Yi}, i \in \mathbb{N}, \\ A_Z\varphi_{Zi} &= \lambda_i\varphi_{Zi}, i \in \mathbb{N}. \end{aligned}$$

1) Let $j \in I_Y \iff \lambda_j \in \sigma_Y$, and let φ_{Yj} be a corresponding eigenvector of the operator A_Y .

In this case the vector $\varphi_j = \begin{pmatrix} \varphi_{Yj} \\ 0 \end{pmatrix}$ is an eigenvector of the operator A .

2) Let $k \in I_Z \iff \lambda_k \in \sigma_Z$, and let $\varphi_{Zk} \neq 0$ be a corresponding eigenvector of the operator A_Z . In this case the vector $\varphi_k = \begin{pmatrix} \varphi_{Yk} \\ \varphi_{Zk} \end{pmatrix}$, where the vector φ_{Yk} is a solution of the operator equation

$$A_Y \varphi_{Yk} + A_{YZ} \varphi_{Zk} = \lambda_k \varphi_{Yk}, \varphi_{Zk} \neq 0, k \in I_Z, \quad (3.1)$$

is an eigenvector of the operator A .

Remark 3.1. Since λ_k is a regular value of the operator A_Y , the operator equation (3.1) is always solvable. Its solution is:

$$\varphi_{Yk} = (A_Y - \lambda_k I_Y)^{-1} A_{YZ} \varphi_{Zk}, k \in I_Z. \quad (3.2)$$

3.1.2 Eigenvectors of adjoint operator A^*

Let $\psi_i = \begin{pmatrix} \psi_{Yi} \\ \psi_{Zi} \end{pmatrix}, i \in \mathbb{N}$ be the eigenvector of the operator A^* , corresponding to its eigenvalue $\bar{\lambda}_i, i \in \mathbb{N}$. We have

$$A^* \psi_i = \begin{pmatrix} A_Y^* \psi_{Yi} \\ A_{YZ}^* \psi_{Yi} + A_Z^* \psi_{Zi} \end{pmatrix} = \bar{\lambda}_i \begin{pmatrix} \psi_{Yi} \\ \psi_{Zi} \end{pmatrix}, i \in \mathbb{N}.$$

Hence,

$$\begin{aligned} A_Y^* \psi_{Yi} &= \bar{\lambda}_i \psi_{Yi}, i \in \mathbb{N}, \\ A_{YZ}^* \psi_{Yi} + A_Z^* \psi_{Zi} &= \bar{\lambda}_i \psi_{Zi}, i \in \mathbb{N}. \end{aligned}$$

1) Let $k \in I_Z \iff \lambda_k \in \sigma_Z$, and let ψ_{Zk} be a corresponding eigenvector of the operator A_Z^* .

In this case the vector $\psi_k = \begin{pmatrix} 0 \\ \psi_{Zk} \end{pmatrix}$ is an eigenvector of the operator A^* .

2) Let $j \in I_Y \iff \lambda_j \in \sigma_Y$, and let ψ_{Yj} be a corresponding eigenvector of the operator A_Y^* .

In this case the vector $\psi_j = \begin{pmatrix} \psi_{Yj} \\ \psi_{Zj} \end{pmatrix}$, where the vector ψ_{Zj} is a solution of the equation

$$A_{YZ}^* \psi_{Yj} + A_Z^* \psi_{Zj} = \bar{\lambda}_j \psi_{Zj}, j \in I_Y, \quad (3.3)$$

is an eigenvector of the operator A^* .

Remark 3.2. Since $\bar{\lambda}_j$ is a regular value of the operator A_Z^* , the operator equation (3.3) is always solvable. Its solution is:

$$\psi_{Zj} = (A_Z^* - \bar{\lambda}_j I_Z)^{-1} A_{YZ}^* \psi_{Yj}, 0, k \in I_Z. \quad (3.4)$$

Denote: $b_i = (b, \psi_i), i \in \mathbb{N}$.

We have

$$b_i = \begin{pmatrix} b_{Yi} \\ b_{Zi} \end{pmatrix}, i \in \mathbb{N}.$$

where

$$b_{Yi} = (b_Y, \psi_{Yi}), b_{Zi} = (b_Z, \psi_{Zi}), i \in \mathbb{N}. \quad (3.5)$$

If $k \in I_Z$, then $\psi_{Yk} = 0$, so $b_k = \begin{pmatrix} 0 \\ b_{Zk} \end{pmatrix}$.

The eigenvectors $\varphi_i, \psi_i, i \in \mathbb{N}$ of operators A and A^* are chosen such that following orthogonality conditions hold:

$$(\varphi_m, \psi_n) = \delta_{mn}, \quad m, n \in \mathbb{N}.$$

Hence

$$(\varphi_{Yj}, \psi_{Yl}) = \delta_{jl}, \quad j, l \in I_Y, \quad (3.6)$$

$$(\varphi_{Zk}, \psi_{Zm}) = \delta_{km}, \quad k, m \in I_Z, \quad (3.7)$$

$$(\varphi_{Yk}, \psi_{Yj}) = -(\varphi_{Zk}, \psi_{Zj}), \quad k \in I_Z, j \in I_Y. \quad (3.8)$$

Denote:

$$\Phi_Y := \left\{ \varphi_j = \begin{pmatrix} \varphi_{Yj} \\ 0 \end{pmatrix}, j \in I_Y \right\}, \Phi_Z := \left\{ \varphi_k = \begin{pmatrix} \varphi_{Yk} \\ \varphi_{Zk} \end{pmatrix}, k \in I_Z \right\},$$

$$\Psi_Y := \left\{ \psi_j = \begin{pmatrix} \psi_{Yj} \\ \psi_{Zj} \end{pmatrix}, j \in I_Y \right\}, \Psi_Z := \left\{ \psi_k = \begin{pmatrix} 0 \\ \psi_{Zk} \end{pmatrix}, k \in I_Z \right\}$$

Lemma 3.1. *The sequences $\{\varphi_{Yj}, j \in I_Y\}$ and $\{\psi_{Yj}, j \in I_Y\}$ are Riesz basics of the space Y . The sequences $\{\varphi_{Zk}, k \in I_Z\}$ and $\{\psi_{Zk}, k \in I_Z\}$ are Riesz basics of the space Z .*

Proof. This follows from results of (Stoeva, 2018) Theorem 2.1 (see also (Bari, 1951)). ■

4 Partial controllability conditions

Below all definitions and notations of strongly minimal sequences belong to (Taldykin, 1977).

Let

$$\{x_1, x_2, \dots, x_n, \dots\} \quad (4.1)$$

be a minimal sequence of elements of a separable Hilbert space H .

The infinite-dimensional matrix

$$\{\varphi_{ij} = (x_i, x_j), i, j \in \mathbb{N}\} \quad (4.2)$$

is called a Gram matrix of family (4.1).

Let

$$G_n = \{\varphi_{ij} = (x_i, x_j), i, j = 1, 2, \dots, n\} \quad (4.3)$$

be $n \times n$ cutout of (4.2), and let $0 \leq \lambda_1^n \leq \lambda_2^n \leq \dots \leq \lambda_n^n$ be eigenvalues of (4.3)³. It is well-known, that $\lambda_1^n \geq 0$ is not increasing, λ_n^n is not decreasing, so there exists $\lim_{n \rightarrow \infty} \lambda_1^n \geq 0$.

Definition 4.1. If $\lim_{n \rightarrow \infty} \lambda_1^n > 0$, then sequence (4.1) is called strongly minimal sequence⁴.

Denote:

$$b_j(t) = b_j - e^{-\lambda_j(t_1-t)} (S_Z(t_1-t) b_Z, \psi_{Zj}), j \in I_Y, t \in [0, t_1].$$

³Matrix (4.3) is a positive Hermitian matrix for any $n \in \mathbb{N}$.

⁴The definition of strongly minimal sequences is equivalent to the definition of Riesz-Fisher sequences (Casazza, Christensen, Li and Lindner, 2002), (Young, 1980). In our opinion, the strong minimality concept better explains the relationship of such sequences with minimal systems.

Theorem 4.1. *Let the series*

$$\sum_{j \in I_Y} e^{-2 \operatorname{Re} \lambda_j t_1} \left| (S_Z(t_1) x_Z^0, \psi_{Zj}) \right|^2 \quad (4.4)$$

converges for any $x_Z^0 \in Z$.

Equation (2.1) is partially null controllable on $[0, t_1]$ if and only if the family of exponents

$$\left\{ e^{-\lambda_j t} b_j(t), j \in I_Y, t \in [0, t_1] \right\} \quad (4.5)$$

is strongly minimal.

Proof. Since the family of eigenvectors $\{\varphi_i, i \in \mathbb{N}\}$ of the operator A is a Riesz basic of X and the family of eigenvectors $\{\psi_i, i \in \mathbb{N}\}$ of the operator A^* is a Riesz basic of X as well, for any $x^0 \in X$ we have

$$x^0 = \sum_{i \in \mathbb{N}} (x^0, \psi_i) \varphi_i = \sum_{j \in I_Y} (x^0, \psi_j) \begin{pmatrix} \varphi_{Yj} \\ 0 \end{pmatrix} + \sum_{k \in I_Z} (x^0, \psi_k) \begin{pmatrix} \varphi_{Yk} \\ \varphi_{Zk} \end{pmatrix}, \quad (4.6)$$

$$x_Y^0 = \sum_{j \in I_Y} (x^0, \psi_j) \varphi_{Yj} + \sum_{k \in I_Z} (x^0, \psi_k) \varphi_{Yk}, \quad (4.7)$$

$$\begin{aligned} (x_Y^0, \psi_{Yj}) &= (x^0, \psi_j) + \left(\sum_{k \in I_Z} (x^0, \psi_k) \varphi_{Yk}, \psi_{Yj} \right) = \\ &= (x_Y^0, \psi_{Yj}) + (x_Z^0, \psi_{Zj}) + \left(\sum_{k \in I_Z} (x_Z^0, \psi_{Zk}) (\varphi_{Yk}, \psi_{Yj}) \right). \end{aligned} \quad (4.8)$$

After substituting $S(t_1) x^0$ instead of x^0 in (4.6), we obtain

$$\begin{aligned} x^0(t_1) &:= S(t_1) x^0 = \sum_{i \in \mathbb{N}} (S(t_1) x^0, \psi_i) \varphi_i = \\ &= \sum_{j \in I_Y} (x^0, \psi_j) e^{\lambda_j t_1} \begin{pmatrix} \varphi_{Yj} \\ 0 \end{pmatrix} + \sum_{k \in I_Z} (x^0, \psi_k) e^{\lambda_k t_1} \begin{pmatrix} \varphi_{Yk} \\ \varphi_{Zk} \end{pmatrix}. \end{aligned} \quad (4.9)$$

Using b instead of x^0 and $t_1 - \tau$ instead of t_1 in (4.9) we obtain after integration, that

$$\begin{aligned} &\int_0^{t_1} S(t_1 - \tau) b u(\tau) d\tau = \\ &= \int_0^{t_1} \left[\sum_{j \in I_Y} b_j e^{\lambda_j(t_1 - \tau)} \begin{pmatrix} \varphi_{Yj} \\ 0 \end{pmatrix} + \sum_{k \in I_Z} b_k e^{\lambda_k(t_1 - \tau)} \begin{pmatrix} \varphi_{Yk} \\ \varphi_{Zk} \end{pmatrix} \right] u(\tau) d\tau. \end{aligned} \quad (4.10)$$

Substituting (4.9) and (4.10) into (1.5), we obtain

$$x(t_1) = \sum_{j \in I_Y} \left((x^0, \psi_j) e^{\lambda_j t_1} + \left(\int_0^{t_1} b_j e^{\lambda_j(t_1 - \tau)} u(\tau) d\tau \right) \right) \begin{pmatrix} \varphi_{Yj} \\ 0 \end{pmatrix} + \quad (4.11)$$

$$\begin{aligned}
 & + \sum_{k \in I_Z} \left((x^0, \psi_k) e^{\lambda_k t_1} + \left(\int_0^{t_1} (b, \psi_k) e^{\lambda_k(t_1-\tau)} u(\tau) d\tau \right) \right) \begin{pmatrix} \varphi_{Yk} \\ \varphi_{Zk} \end{pmatrix}, \\
 x_Y(t_1) &= \sum_{j \in I_Y} (x^0, \psi_j) e^{\lambda_j t_1} \varphi_{Yj} + \sum_{k \in I_Z} (x^0, \psi_k) e^{\lambda_k t_1} \varphi_{Yk} + \\
 & + \int_0^{t_1} \sum_{j \in I_Y} \left(b_j e^{\lambda_j(t_1-\tau)} \varphi_{Yj} + \sum_{k \in I_Z} b_k e^{\lambda_k(t_1-\tau)} \varphi_{Yk} \right) u(\tau) d\tau.
 \end{aligned} \tag{4.12}$$

Taking into account, that

$$(x^0, \psi_k) = (x_Z^0, \psi_{Zk}) \quad \text{and} \quad (\varphi_{Yk}, \psi_{Yj}) + (\varphi_{Zk}, \psi_{Zj}) = 0, \forall j \in I_Y, k \in I_Z,$$

we obtain

$$\begin{aligned}
 & (x_Y(t_1), \psi_{Yj}) = \\
 & = e^{\lambda_j t_1} (x^0, \psi_j) + \left(\sum_{k \in I_Z} e^{\lambda_k t_1} (x_Z^0, \psi_{Zk}) \varphi_{Yk}, \psi_{Yj} \right) + \\
 & + \left(\int_0^{t_1} e^{\lambda_j(t_1-\tau)} b_j + \sum_{k \in I_Z} \left(e^{\lambda_k(t_1-\tau)} (b_Z, \psi_{Zk}) \varphi_{Yk}, \psi_{Yj} \right) \right) u(\tau) d\tau = \\
 & = e^{\lambda_j t_1} (x^0, \psi_j) - \sum_{k \in I_Z} \left(e^{\lambda_k t_1} (x_Z^0, \psi_{Zk}) \varphi_{Zk}, \psi_{Zj} \right) + \\
 & + \int_0^{t_1} \left(e^{\lambda_j(t_1-\tau)} b_j - \sum_{k \in I_Z} \left(e^{\lambda_k(t_1-\tau)} (b_Z, \psi_{Zk}) \varphi_{Yk}, \psi_{Yj} \right) \right) u(\tau) d\tau, \quad j \in I_Y.
 \end{aligned}$$

Since the sequence $\{\varphi_{Zk}, k \in I_Z\}$ is a Riesz basic of the space Z (see Lemma 3.1), we have

$$S_Z(t)z = \sum_{k \in I_Z} e^{\lambda_k t} (z, \psi_{Zk}) \varphi_{Zk}, \forall t > 0, \forall z \in Z. \tag{4.13}$$

Hence

$$\begin{aligned}
 & (x_Y(t_1), \psi_{Yj}) = e^{\lambda_j t_1} \left[(x^0, \psi_j) - e^{-\lambda_j t_1} (S_Z(t_1) x_Z^0, \psi_{Zj}) \right] + \\
 & + \left(\int_0^{t_1} e^{\lambda_j(t_1-\tau)} \left[e^{-\lambda_j \tau} b_j - e^{-\lambda_j t_1} (S_Z(t_1-\tau) b_Z, \psi_{Zj}) \right] \right) u(\tau) d\tau, \quad j \in I_Y.
 \end{aligned} \tag{4.14}$$

Thus, the partial null-controllability we are proving is equivalent to the existence of a solution $u(\cdot) \in L_2[0, t_1]$ to the moment problem

$$\begin{aligned}
 & - (x^0, \psi_j) + e^{-\lambda_j t_1} (S_Z(t_1) x_Z^0, \psi_{Zj}) = \\
 & = \int_0^{t_1} e^{-\lambda_j \tau} \left[b_j - e^{-\lambda_j(t_1-\tau)} (S_Z(t_1-\tau) b_Z, \psi_{Zj}) \right] u(\tau) d\tau,
 \end{aligned} \tag{4.15}$$

$$\forall j \in I_Y, \quad \forall x^0 \in X.$$

By convergence of series (4.4) we obtain

$$\sum_{j \in I_Y} \left| (x^0, \psi_j) - e^{-\lambda_j t_1} (S_Z(t_1) x_Z^0, \psi_{Zj}) \right|^2 < +\infty, \forall x^0 \in X.$$

Hence for any $\forall x^0 \in X$ moment problem (4.15) admits a solution $u(\cdot) \in L_2[0, t_1]$ if and only if the family (4.5) is strongly minimal.

This proves the theorem. ■

Remark 4.1. The convergence of series (4.4) essentially depends on the structure of operator A_{YZ} .

Example. Let $X = L_2[0, \pi] \times \mathbb{R}$, $Y = L_2[0, \pi]$, $Z = \mathbb{R}$, $A_Y = \Delta$ with boundary conditions $y(0) = y(\pi) = 0$, $A_Z = w \in \mathbb{R}$, $0 \leq w < 1$, $A_{YZ} = a \in Y$;

$I_Y = \mathbb{N} \setminus \{1\} = 2, 3, \dots$, $I_Z = \{1\}$; $\lambda_1 = w$, $\varphi_{Z1} = \psi_{Z1} = 1$,

$\lambda_l = -(l-1)^2$, $\varphi_{Yl} = \psi_{Yl} = \sqrt{\frac{2}{\pi}} \sin(l-1)x$, $l \in \mathbb{N} \setminus \{1\}$;

$A_{YZ}^* \psi_{Yl} + w \psi_{Zl} = \lambda_l \psi_{Zl} \iff$

$$\iff \psi_{Zl} = \frac{A_{YZ}^* \psi_{Yl}}{\lambda_l - w} = -\frac{(a, \psi_{Yl})}{(l-1)^2 + w} = -\frac{\sqrt{\frac{2}{\pi}} \int_0^\pi a(x) \sin((l-1)x) dx}{(l-1)^2 + w}, l \in \mathbb{N} \setminus \{1\},$$

Obviously, $S_Z(t_1) x_Z^0 = e^{wt_1} x_Z^0$. Using $j = l-1$, $l = 2, 3, \dots$, we obtain

$$\psi_{Zj+1} = -\sqrt{\frac{2}{\pi}} \frac{\int_0^\pi a(x) \sin jx dx}{j^2 + w}, j \in \mathbb{N}.$$

Hence, for any $l \in I_Y$ we have

$$\begin{aligned} e^{-\lambda_l t_1} (S_Z(t_1) x_Z^0, \psi_{Zl}) &= e^{(l-1)^2 t_1} e^{wt_1} x_0 \psi_{Zl} = \\ &= -\sqrt{\frac{2}{\pi}} x_0 e^{wt_1} e^{j^2 t_1} \frac{\int_0^\pi a(x) \sin jx dx}{j^2 + w}, j \in \mathbb{N}. \end{aligned}$$

Obviously, $\sum_{j=1}^\infty \frac{1}{j^2} < +\infty$. Hence, by Riesz-Fisher Theorem there exists $a(x) \in L_2[0, \pi]$, such

that $\int_0^\pi a(x) \sin jx dx = \frac{1}{j}$, $j \in \mathbb{N}$.

Also, $\sum_{j=1}^\infty e^{-2j^2 t_1} < +\infty$. Hence, by Riesz-Fisher Theorem there exists $a(x) \in L_2[0, \pi]$, such

that $\int_0^\pi a(x) \sin jx dx = e^{-j^2 t_1}$, $j \in \mathbb{N}$.

If $\int_0^\pi a(x) \sin jx dx = \frac{1}{j}$, then

$$\sum_{j \in I_Y} \left| e^{-\lambda_j t_1} (S_Z(t_1) x_Z^0, \psi_{Zj}) \right|^2 = \frac{2}{\pi} e^{2wt_1} |x_Z^0|^2 \sum_{j=1}^\infty e^{2j^2 t_1} \frac{1}{j^2 (j^2 + w)^2} = +\infty.$$

If $\int_0^\pi a(x) \sin jx dx = e^{-2j^2 t_1}$, then

$$\begin{aligned} \sum_{j \in I_Y} \left| e^{-\lambda_j t_1} (S_Z(t_1) x_Z^0, \psi_{Zj}) \right|^2 &= \frac{2}{\pi} \sum_{j=1}^\infty e^{2j^2 t_1} \sqrt{\frac{2}{\pi}} e^{2wt_1} |x_Z^0|^2 \frac{e^{-2j^2 t_1}}{(j^2 + w)^2} = \\ &= \frac{2}{\pi} e^{2wt_1} |x_Z^0|^2 \sum_{j=1}^\infty \frac{1}{(j^2 + w)^2} < +\infty. \end{aligned}$$

4.0.1 The case $b \in Y$

In this case problem moment (4.15) is simplified.

Let $b_z = 0$, i.e. $b = b_Y \in Y$.

Controllability of such partial parabolic control systems have important applications to many fields such as chemistry, physics or biology (see also Example 3 below).

If $b \in Y$, then (2.1)–(2.3) is not completely null-controllable on $[0, t_1]$, $\forall t_1 > 0$, because solutions of equation (2.2) are independent on control. However it may be partially null controllable.

In this chase problem moment (4.15) is simplified:

$$\begin{aligned} & - (x^0, \psi_j) + e^{-\lambda_j t_1} (S_Z(t_1) x_Z^0, \psi_{Zj}) = \\ & = \int_0^{t_1} b_j e^{-\lambda_j \tau} u(t) dt, j \in I_Y. \end{aligned} \quad (4.16)$$

Corollary 4.2. *If the series (4.4) converges for any $x_Z^0 \in Z$, then system (2.1)–(2.3) is partially null controllable on $[0, t_1]$ if and only if the family of exponent*

$$\left\{ e^{-\lambda_j t} b_j, j \in I_Y, t \in [0, t_1] \right\} \quad (4.17)$$

is strongly minimal.

Corollary 4.3. *If the series (4.4) converges for any $x_Z^0 \in Z$, then system (2.1)–(2.3) is partially null controllable on $[0, t_1]$ if and only if equation*

$$\dot{y}(t) = A_Y y(t) + b_Y u(t)$$

is null (completely) controllable on $[0, t_1]$.

Proof. Since the series (4.4) converges for any $x_Z^0 \in Z$, and the family $\psi_j, j \in Y$ forms a Riesz basic of Y , then for $x_Y^0 \in Y$ and $x_Z^0 \in Z$ there exists $y^0 \in Y$, such that

$$- (x^0, \psi_j) + e^{-\lambda_j t_1} (S_Z(t_1) x_Z^0, \psi_{Zj}) = - (y^0, \psi_j), j \in Y.$$

The corollary follows from the above formula and (4.16). ■

4.0.2 One sufficient condition for strong minimality

The direct method for checking the strong minimality of any sequence (4.1) is the computing of $\lim_{n \rightarrow \infty} \lambda_1^n > 0$ by numeric methods. It may be rather difficult. However if $b_Y \in Y$, then one can find more simpler strong minimality conditions for sequence (4.5) in some partial cases (see, for example, results of (Ammar-Khodja, 2018), (Young, 1980), (Young, 1998), (Fattorini and Russel, 1974) and references therein).

Consider a strongly decreasing sequence of negative numbers $\{\lambda_j, j \in I_Y\}$, with finite upper density (MandelBrojt, 1969).

Let the family

$$\{\exp(\lambda_j t), j \in I_Y, t \in [0, t_1]\} \quad (4.18)$$

be minimal, and let

$$\{q_j(t), j \in I_Y, t \in [0, t_1]\} \quad (4.19)$$

be biorthogonal family for (4.18).

Theorem 4.4. *If $b_j \neq 0, \forall j \in I_Y, \lim_{j \rightarrow \infty} \frac{2 \ln |b_j|}{\lambda_j}$ is finite, $t_1 > \lim_{j \rightarrow \infty} \frac{2 \ln |b_j|}{\lambda_j}$, and for any $\varepsilon > 0$ there exists a positive constant K_ε such that*

$$\|q_j(\cdot)\| < K_\varepsilon e^{-\lambda_j \varepsilon}, j \in I_Y, \quad (4.20)$$

then the family

$$\{b_j \exp(-\lambda_j t), j \in I_Y, t \in [0, t_1]\} \quad (4.21)$$

is strongly minimal.

Proof. Obviously, the family

$$\left\{ \frac{q_j(t_1 - t)}{b_j} e^{\lambda_j t_1}, j \in I_Y, t \in [0, t_1] \right\}$$

is biorthogonal to the family (4.21).

Let I_Y^f be any finite subsequence of I_Y . Following from Cauchy-Schwartz inequality, we have

$$\begin{aligned} \left\| \sum_{j \in I_Y^f} \beta_j \frac{q_j(t_1 - t)}{b_j} e^{\lambda_j t_1} \right\|^2 &\leq \left(\sum_{j \in I_Y^f} |\beta_j| \frac{1}{|b_j|} \|q_j(\cdot)\| e^{\lambda_j t_1} \right)^2 \leq \\ &\leq \left(\sum_{j \in I_Y^f} |\beta_j|^2 \right) \left(\sum_{j \in I_Y^f} \frac{1}{|b_j|^2} \|q_j(\cdot)\|^2 e^{2\lambda_j t_1} \right) \leq \\ &\leq \left(\sum_{j \in I_Y^f} |\beta_j|^2 \right) \left(\sum_{j \in I_Y^f} \frac{1}{|b_j|^2} K_\varepsilon^2 e^{-2\lambda_j \varepsilon} e^{2\lambda_j t_1} \right) = \\ &= K_\varepsilon^2 \left(\sum_{j \in I_Y^f} |\beta_j|^2 \right) \left(\sum_{j \in I_Y^f} \frac{1}{|b_j|^2} e^{2\lambda_j(t_1 - \varepsilon)} \right) < K_\varepsilon^2 \left(\sum_{j \in I_Y^f} |\beta_j|^2 \right) \left(\sum_{j \in I_Y^f} \frac{1}{|b_j|^2} e^{\lambda_j t_1} \right) \end{aligned}$$

for $\varepsilon < \frac{t_1}{2}$.

The series $\sum_{j \in I_Y} \frac{1}{|b_j|^2} e^{\lambda_j t_1}$ is the Dirichlet series with abscissa of the convergence (Mandelbrojt, 1969)

$$\sigma_c = \lim_{j \rightarrow \infty} \frac{2 \ln |b_j|}{\lambda_j}.$$

Hence

$$\left\| \sum_{j \in I_Y^f} \beta_j \frac{q_j(t_1 - t)}{b_j} e^{\lambda_j t_1} \right\|^2 \leq M^2 \sum_{j \in I_Y^f} |\beta_j|^2, \forall \beta_j \in \mathbb{R}, j \in I_Y^f \quad (4.22)$$

for any $t_1 > \lim_{j \rightarrow \infty} \frac{2 \ln |b_j|}{\lambda_j}$, where $M^2 = K_\varepsilon^2 \sum_{j \in I_Y} \frac{1}{|b_j|^2} e^{\lambda_j t_1}$. It follows from (4.22), that the family

$\left\{ \frac{q_j(t_1 - t)}{b_j} e^{\lambda_j t_1}, j \in I_Y, t \in [0, t_1] \right\}$ is a Bessel sequence. Hence (Casazza et.al., 2002), (Young, 1980), family (4.21) is a Riesz-Fisher sequence, i.e. this family is strongly minimal.

This proves the theorem. ■

The next theorem follows from previous Theorem 4.4 and results of (Mandelbrojt, 1969), (Leontiev, 1976).

Theorem 4.5. . Let $I_Y = \{j_1, j_2, \dots, j_n, \dots\}$, $j_1 < j_2 < \dots < j_n < \dots$, $\lambda_{j_n} < 0$, $|\lambda_{j_{n+1}}| > |\lambda_{j_n}|$, $n \in \mathbb{N}$. If:

1. there exists $\rho > 0$ such that $|\lambda_{j_1}| > \rho$, $|\lambda_{j_m} - \lambda_{j_n}| > |m - n| \rho$,

2. $\sum_{n \in I_Y} \frac{1}{|\lambda_{j_n}|} < +\infty$,

then the family (4.5) is strongly minimal on $[0, t_1]$ for any

$$t_1 > \lim_{j \rightarrow \infty, j \in I_Y} \frac{2 \ln |b_j|}{\lambda_j}.$$

Proof. The proof is derived from the fact that conditions specified in the theorem are adequate to establish the validity of (4.20). ■

Corollary 4.6. If conditions of Theorem 4.5 hold and series (4.4) converges for any $x_Z^0 \in Z$, then system (2.1)–(2.3) is partially null controllable on $[0, t_1]$ for any

$$t_1 > \lim_{j \rightarrow \infty, j \in I_Y} \frac{2 \ln |b_j|}{\lambda_j}.$$

4.1 Example 1

Consider the system of parabolic equations in the form (1.1):

$$\begin{cases} y'_t = \Delta y + y + z + xu(t), & (t, x) \in [0, +\infty) \times [0, \pi], \\ z'_t = z''_{xx}, \end{cases} \quad (4.23)$$

$$y(0, x) = y_0(x), z(0, x) = z_0(x), x \in L_2[0, \pi],$$

$$y(t, 0) = y(t, \pi) = 0,$$

$$z(t, 0) = z(t, \pi) = 0,$$

where $y_0(x), z_0(x) \in L_2[0, \pi]$, $b(x) = \begin{pmatrix} x \\ 0 \end{pmatrix}$, $u(t) \in L_2[0, t_1]$.

Let $P = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$. In this case

$$Y = \left\{ \begin{pmatrix} y(x) \\ 0 \end{pmatrix}, y(\cdot) \in L_2[0, \pi] \right\}, Z = \left\{ \begin{pmatrix} 0 \\ z(x) \end{pmatrix}, z(\cdot) \in L_2[0, \pi] \right\},$$

$$A = \begin{pmatrix} \Delta + 1 & 1 \\ 0 & \Delta \end{pmatrix}, b = \begin{pmatrix} x \\ 0 \end{pmatrix},$$

so

$$A_Y = \Delta + 1, A_Z = \Delta$$

with domains (Balakrishnan, 1976, page 214, Example 4.6.6, page 241, Example 4.10.1)

$$D(A_Y) = D(A_Z) = \{f, f' \text{ a.c.} : f', f'' \in L_2[0, \pi] \text{ and } f(0) = f(\pi) = 0\}.$$

After computations and enumeration of the eigenvalues $\lambda_i \in \sigma_A, i \in \mathbb{N}$ in the order of non-decreasing of their absolute values, we obtain:

1) $I_Y = \{1, 3, 5, \dots\}$, $I_Z = \{2, 4, 6, \dots\}$

2) the eigenvalues λ_j and eigenvectors $\varphi_j, j \in \mathbb{N}$ of the operator A are:

$$\lambda_{2m-1} = 1 - m^2, \varphi_{2m-1}(x) = \sqrt{\frac{2}{\pi}} \begin{pmatrix} \sin mx \\ 0 \end{pmatrix}, j = 2m - 1 \in I_Y, m \in \mathbb{N}, \quad (4.24)$$

$$\lambda_{2m} = -m^2, \varphi_{2m}(x) = \sqrt{\frac{2}{\pi}} \begin{pmatrix} -\sin mx \\ \sin mx \end{pmatrix}, j = 2m \in I_Z, m \in \mathbb{N}, \quad (4.25)$$

3) the eigenvalues λ_i and eigenvectors ψ_j of the operator A^* are:

$$\lambda_{2m-1} = 1 - m^2, \psi_{2m-1}(x) = \sqrt{\frac{2}{\pi}} \begin{pmatrix} \sin mx \\ \sin mx \end{pmatrix}, j = 2m - 1 \in I_Y, m \in \mathbb{N}, \quad (4.26)$$

$$\lambda_{2m} = -m^2, \psi_{2m}(x) = \sqrt{\frac{2}{\pi}} \begin{pmatrix} 0 \\ \sin mx \end{pmatrix}, j = 2m \in I_Z, m \in \mathbb{N}. \quad (4.27)$$

Substituting (4.26) and (4.27) to the formula $b_i = (b, \psi_i)$, $i \in \mathbb{N}$, we obtain:

$$b_j = \begin{cases} \int_0^\pi x \sin mx dx = \frac{(-1)^{m-1}}{m} \pi, & j = 2m - 1 \in I_Y, m \in \mathbb{N}, \\ 0, & j = 2m \in I_Z, m \in \mathbb{N}. \end{cases} \quad (4.28)$$

Using the characterization of Riesz bases via biorthogonal sequences developed in (Stoeva, 2018) (see Theorem 2.1), one can prove, that

Lemma 4.7. *The sequence*

$$\left\{ \sqrt{\frac{2}{\pi}} \begin{pmatrix} \sin mx \\ 0 \end{pmatrix}, \sqrt{\frac{2}{\pi}} \begin{pmatrix} -\sin mx \\ \sin mx \end{pmatrix}, m \in \mathbb{N} \right\} \quad (4.29)$$

and sequence

$$\left\{ \sqrt{\frac{2}{\pi}} \begin{pmatrix} \sin mx \\ \sin mx \end{pmatrix}, \sqrt{\frac{2}{\pi}} \begin{pmatrix} 0 \\ \sin mx \end{pmatrix}, m \in \mathbb{N} \right\} \quad (4.30)$$

are a Riesz basis of $L_2([0, \pi], \mathbb{R}^2)$.

Proof. We will prove, that 1) sequence (4.29) is complete, and 2) sequences (4.29) and (4.30) are Bessel sequences.

1) If for $f(x) = \begin{pmatrix} f_Y(x) \\ f_Z(x) \end{pmatrix}$, where $f_Y(x), f_Z(x) \in L_2[0, \pi]$, equalities $(\varphi_{2m-1}, f) = 0$, $(\varphi_{2m}, f) = 0$ hold for any $m \in \mathbb{N}$, i.e.

$$\begin{aligned} (\sin mx, f_Y) &= 0, \\ -(\sin mx, f_Y) + (\sin mx, f_Z) &= 0, \\ \forall m &\in \mathbb{N}, \end{aligned}$$

then from the completeness of the sequence $\{\sin mx, m \in \mathbb{N}\}$ in $L_2[0, \pi]$, it follows, $f_Y = f_Z = 0$.

Hence, sequence (4.29) is complete.

2) For any $f(x) = \begin{pmatrix} f_Y(x) \\ f_Z(x) \end{pmatrix}$, where $f_Y(x), f_Z(x) \in L_2[0, \pi]$, we have

$$\begin{aligned} & \sum_{m=1}^{\infty} |(\varphi_{2m-1}, f)|^2 + \sum_{m=1}^{\infty} |(\varphi_{2m}, f)|^2 \leq \\ & \leq 3 \sum_{m=1}^{\infty} \left| \left(\sqrt{\frac{2}{\pi}} \sin mx, f_Y \right) \right|^2 + 2 \sum_{m=1}^{\infty} \left| \left(\sqrt{\frac{2}{\pi}} \sin mx, f_Z \right) \right|^2 < +\infty. \end{aligned}$$

Hence, sequence (4.29) is a Bessel sequence.

By the same way, for any $f(x) = \begin{pmatrix} f_Y(x) \\ f_Z(x) \end{pmatrix}$, where $f_Y(x), f_Z(x) \in L_2[0, \pi]$, we have

$$\begin{aligned} & \sum_{m=1}^{\infty} |(\psi_{2m-1}, f)|^2 + \sum_{m=1}^{\infty} |(\psi_{2m}, f)|^2 \leq \\ & \leq 2 \sum_{m=1}^{\infty} \left| \left(\sqrt{\frac{2}{\pi}} \sin mx, f_Y \right) \right|^2 + 3 \sum_{m=1}^{\infty} \left| \left(\sqrt{\frac{2}{\pi}} \sin mx, f_Z \right) \right|^2 < +\infty. \end{aligned}$$

Hence the sequence (4.30) is a Bessel sequence also.

This proves the lemma. ■

Lemma 4.7 shows that equation (4.23) satisfies all the conditions of equation (2.1).

$$\begin{aligned} & \text{For } j = 2m - 1, m \in \mathbb{N}, \text{ we have } e^{-2\lambda_j t_1} (S_Z(t_1) x_Z^0, \psi_{Zj}) = \\ & = e^{-\lambda_{2m-1} t_1} \left(\sum_{n=1}^{\infty} (x_Z^0, \psi_{Z(2n)}) e^{\lambda_{2n} t_1} \varphi_{Z(2n)}, \psi_{Z(2m-1)} \right) = \\ & = e^{(m^2-1)t_1} \left(\int_0^\pi \left(\sum_{n=1}^{\infty} \sqrt{\frac{2}{\pi}} \left(\int_0^\pi z^0(\xi) \sin n\xi d\xi \right) e^{-n^2 t_1} \sqrt{\frac{2}{\pi}} \sin nx \right) \sqrt{\frac{2}{\pi}} \sin mx \right) dx = \\ & = e^{(m^2-1)t_1} e^{-m^2 t_1} \frac{2}{\pi} \left(\int_0^\pi z^0(x) \sin mx dx \right) dx = \sqrt{\frac{2}{\pi}} e^{-t_1} \left(\int_0^\pi z^0(x) \sin mx dx \right), \end{aligned}$$

so

$$\begin{aligned} & \sum_{\substack{j=1, \\ j=1,3,5,\dots}}^{\infty} \left| e^{-\lambda_j t_1} (S_Z(t_1) x_Z^0, \psi_{Zj}) \right|^2 = \\ & = \frac{2}{\pi} e^{-2t_1} \sum_{m=1}^{\infty} \left| \left(\int_0^\pi z^0(x) \sin mx dx \right) dx \right|^2 < +\infty. \end{aligned}$$

Also we have

$$e^{-\lambda_j t_1} b_j = e^{(m^2-1)t_1} \frac{(-1)^{m-1}}{m} \pi = \pi e^{-1} \frac{(-1)^{m-1}}{m} e^{m^2 t_1}, j = 2m - 1 \in I_Y, m \in \mathbb{N}.$$

When the family

$$\left\{ \frac{(-1)^{m-1}}{m} e^{m^2 t_1}, m = 1, 2, \dots \right\}$$

is strongly minimal?

The eigenvalues $\lambda_j = -j^2$ satisfy the conditions of Theorem 4.5 and $\sigma_c = \lim_{m \rightarrow \infty} \ln \left(\frac{1}{m} \right) \frac{1}{-m^2} =$

$$\lim_{m \rightarrow \infty} \frac{1}{(\ln m)m^2} = 0.$$

In accordance with Theorem 4.5 and Corollary 4.6 system (4.23) is partially null-controllable on $[0, t_1]$ for any $t_1 > 0$.

However there is no $t_1 > 0$, such that system (4.23) is completely null-controllable on $[0, t_1]$, because the second equation of this system doesn't depend on a control.

5 The case $\sigma_{YZ} \neq \emptyset$

5.1 Eigenvectors of the operators A and A^*

If $j \in I_Y \setminus I_{YZ}$ and $k \in I_Z \setminus I_{YZ}$, we are in conditions of the previous section, so eigenvectors of the operator A and A^* satisfy orthogonality conditions (3.8).

If $m \in I_{YZ}$, then the eigenvalue λ_m of the operator

$$A = \begin{pmatrix} A_Y & A_{YZ} \\ 0 & A_Z \end{pmatrix}$$

may have two linear independent eigenvectors.

The first one is $\phi_{1m} = \begin{pmatrix} \phi_{1Ym} \\ 0 \end{pmatrix}$, where $\phi_{1Ym} \neq 0$ is an eigenvector of the operator A_Y .

The second one is: $\phi_{2m} = \begin{pmatrix} \phi_{2Ym} \\ \phi_{2Zm} \end{pmatrix}$, where ϕ_{2Zm} is an eigenvector of the operator A_Z , provided that there exists a solution of the operator equation

$$A_Y \phi_{2Ym} + A_{YZ} \phi_{2Zm} = \lambda_m \phi_{2Ym} \quad (5.1)$$

for $m \in I_{YZ}$. In this case the vector ϕ_{2Ym} is a solution of above equation.

Since λ_m is an eigenvalue of the operator A_Y , the operator equation (5.1) may be not solvable.

In this case the operator A has an unique eigenvector $\phi_{1m} = \begin{pmatrix} \phi_{1Ym} \\ 0 \end{pmatrix}$

The same considerations can be performed for the operator

$$A^* = \begin{pmatrix} A_Y^* & 0 \\ A_{YZ}^* & A_Z^* \end{pmatrix},$$

i.e. for $m \in I_{YZ}$ the eigenvalue $\bar{\lambda}_m$ may have two linear independent eigenvectors.

The first one is $\psi_{2m} = \begin{pmatrix} 0 \\ \psi_{2Zm} \end{pmatrix}$, where $\psi_{2Zm} \neq 0$ is an eigenvector of the operator A_Z^* .

The second one is: $\psi_{1m} = \begin{pmatrix} \psi_{1Ym} \\ \psi_{1Zm} \end{pmatrix}$, where ψ_{1Ym} is an eigenvector of the operator A_Y^* , provided that there exists a solution the operator equation

$$A_Y^* \psi_{1Ym} + A_Z^* \psi_{1Zm} = \bar{\lambda}_m \psi_{1Zm}. \quad (5.2)$$

Since $\bar{\lambda}_m$ is an eigenvalue of the operator A_Z^* , the operator equation (5.2) may be not solvable.

In this case the operator A has an unique eigenvector $\psi_{2m} = \begin{pmatrix} 0 \\ \psi_{2Zm} \end{pmatrix}$

The eigenvectors ϕ_{1m}, ψ_{2m} and the generalized eigenvectors ϕ_{2m}, ψ_{1m} of the operators A and A^* correspondingly are chosen such that the following orthogonality conditions hold:

$$\begin{pmatrix} (\phi_{1l}, \psi_{1j}) = \delta_{lj} & (\phi_{1l}, \psi_{2j}) = 0, \\ (\phi_{2l}, \psi_{1j}) = 0 & (\phi_{2l}, \psi_{2j}) = \delta_{lj}, \end{pmatrix}, l, j \in I_{YZ}. \quad (5.3)$$

Hence, $(\phi_{2l}, \psi_{1j}) = (\phi_{2Yl}, \psi_{1Yj}) + (\phi_{2Zl}, \psi_{1Zj}) = 0, l, j \in I_{YZ} \iff$

$$(\phi_{2Yl}, \psi_{1Yj}) = -(\phi_{2Zl}, \psi_{1Zj}), l, j \in I_{eq:YZ}. \quad (5.4)$$

If equations (5.1) does not have a solution, we consider the following alternative equation

$$(A_Y - \lambda_m I) \phi_{2Ym} = c_m \phi_{1Ym} - A_{YZ} \phi_{2Zm}. \quad (5.5)$$

Theorem 5.1. *The operator equation*

$$A\phi_{2m} - \lambda_m \phi_{2m} = c_m \phi_{1m}, \quad (5.6)$$

has a solution $\phi_{2m} = \begin{pmatrix} \phi_{2Ym} \\ \phi_{2Zm} \end{pmatrix}$ if and only if

$$c_m = (A_{YZ} \phi_{2Zm}, \psi_{1Ym}). \quad (5.7)$$

Proof. Equation (5.6) is equivalent to equation (5.5).

Equation (5.5) has a solution if and only if (Krein, 1971)

$$(c\phi_{1Ym} - A_{YZ}\phi_{2Zm}, \psi_{1Ym}) = 0.$$

We have

$$\begin{aligned} & (c_m \phi_{1Ym} - A_{YZ} \phi_{2Zm}, \psi_{1Ym}) = \\ & = c_m (\phi_{1Ym}, \psi_{1Ym}) - (A_{YZ} \phi_{2Zm}, \psi_{1Ym}) = \\ & = c_m - (A_{YZ} \phi_{2Zm}, \psi_{1Ym}), \end{aligned}$$

i.e. the last condition is equivalent to the condition (5.7).

This proves the theorem. ■

If $c_m = 0$, then ϕ_{2m} is a solution of the equation

$$A\phi_{2m} - \lambda_m \phi_{2m} = 0,$$

so ϕ_{2m} is the second eigenvector of the operator A of the same simple eigenvalue λ_m having the multiplicity 2.

If $c_m \neq 0$, then the ϕ_{2m} is a generalized eigenvector of the operator A , corresponding to the eigenvalue λ_m , with a multiplicity of 2.

If equation (5.2) does not have a solution, we consider the following alternative equation

$$(A_Z^* - \bar{\lambda}_m I) \psi_{1Zm} = c_m \psi_{2Zm} - A_{YZ}^* \psi_{1Ym}, \quad (5.8)$$

Theorem 5.2. *The operator equation*

$$A^* \psi_{1m} - \lambda_m \psi_{1m} = c_m \psi_{2m}, \quad (5.9)$$

has a solution $\psi_{1m} = \begin{pmatrix} \psi_{1Ym} \\ \psi_{1Zm} \end{pmatrix}$, where $\phi_{2Zm} \neq 0$ is an eigenvector of the operator A_Z , corresponding to its eigenvalue λ_m , if and only if

$$c_m = (\phi_{2Zm}, A_{YZ}^* \psi_{1Ym}). \quad (5.10)$$

Proof. The operator equation (5.9) is equivalent to operator equation (5.8). Equation (5.8) has a solution if and only if (Krein, 1971)

$$(\phi_{2Zm}, c_m \psi_{2Zm} - A_{YZ}^* \psi_{1Ym}) = 0.$$

We have $(\phi_{2Zm}, c_m \psi_{2Zm} - A_{YZ}^* \psi_{1Ym}) = c_m (\phi_{2Zm}, \psi_{2Zm}) - (\phi_{2Zm}, A_{YZ}^* \psi_{1Ym}) = c_m - (\phi_{2Zm}, A_{YZ}^* \psi_{1Ym})$, i.e. the last condition is equivalent to the condition (5.10).

This proves the theorem. ■

Conditions (5.7) and (5.10) are equivalent.

If $c_m = 0$, then ψ_{1m} is a solution of the equation

$$A^* \psi_{1m} - \lambda_m \psi_{1m} = 0,$$

so the vector $\psi_{1m} = \begin{pmatrix} \psi_{1Ym} \\ \psi_{1Zm} \end{pmatrix}$ is a second eigenvector of the operator A^* , corresponding the same simple eigenvalue $\bar{\lambda}_m$ of the multiplicity 2.

If $c_m \neq 0$, then ψ_{1m} is a generalized eigenvector of the operator A^* , corresponding the eigenvalue $\bar{\lambda}_m$ of the multiplicity 2.

Lemma 5.3. *The sequence $\{c_m, m \in I_{YZ}\}$ is bounded.*

Proof. We have

$$|c_m| = |(A_{YZ} \phi_{2Zm}, \psi_{1Ym})| \leq \|A_{YZ}\| \|\phi_{2Zm}\| \|\psi_{1Ym}\|, m \in I_{YZ}.$$

The sequences $\{\phi_{2Zm}, m \in I_{YZ}\}$ and $\{\psi_{1Ym}, m \in I_{YZ}\}$ are bounded, because in accordance with Lemma 3.1 the first one is a Riesz basis of Z , and the second one is a Riesz basis of Y .

This proves the Lemma. ■

Lemma 3.1 holds true for the case $\sigma_{YZ} \neq \emptyset$ as well.

6 Partial controllability conditions

Denote:

$$b_{1j} = \begin{cases} (b, \psi_j), & j \in I_Y \setminus I_{YZ}, \\ (b, \psi_{1j}), & j \in I_{YZ}, \end{cases}, b_{2j} = (b, \psi_{2j}), j \in I_{YZ}.$$

$$b_j(\tau) = \begin{cases} b_{1j} - e^{-\lambda_j(t_1-\tau)} (S_Z(t_1-\tau) b_Z, \psi_{2j}), & j \in I_Y \setminus I_{YZ}, \\ b_{1j} + b_{2j} c_j(t_1-\tau) - e^{-\lambda_j(t_1-\tau)} (S_Z(t_1-\tau) b_Z, \psi_{1Zj}), & j \in I_{YZ}. \end{cases}$$

Theorem 6.1. *Let the series*

$$\sum_{j \in I_Y} e^{-2\lambda_j t_1} |(S_Z(t_1) x_Z^0, \psi_{Zj})|^2 \quad (6.1)$$

converges for any $x_Z^0 \in Z$.

Equation (2.1)–(2.3) is partially null controllable on $[0, t_1]$ if and only if the family of exponent

$$\left\{ e^{-\lambda_j t} b_j(t), j \in I_Y, t \in [0, t_1] \right\} \quad (6.2)$$

is strongly minimal.

Proof. Since the family of eigenvectors $\{\varphi_i, i \in \mathbb{N}\}$ of the operator A is a Riesz basic of X and the family of eigenvectors $\{\psi_i, i \in \mathbb{N}\}$ of the operator A^* is a Riesz basic of X as well, for any

$x^0 = \begin{pmatrix} x_Y^0 \\ x_Z^0 \end{pmatrix} \in X = Y \times Z$ we have

$$\begin{aligned} x^0(t_1) &:= S(t_1) x^0 = \sum_{l \in I_Y \setminus I_{YZ}} e^{\lambda_l t_1} (x^0, \psi_l) \begin{pmatrix} \varphi_{Yl} \\ 0 \end{pmatrix} + \\ &+ \sum_{l \in I_{YZ}} e^{\lambda_l t_1} \left[(x^0, \psi_{1l}) \begin{pmatrix} \phi_{1Yl} \\ 0 \end{pmatrix} + (x^0, \psi_{2l}) \left(c_j t_1 \begin{pmatrix} \phi_{1Yl} \\ 0 \end{pmatrix} + \begin{pmatrix} \phi_{2Yl} \\ \phi_{2Zl} \end{pmatrix} \right) \right] + \\ &+ \sum_{k \in I_Z \setminus I_{YZ}} (x^0, \psi_k) e^{\lambda_k t_1} \begin{pmatrix} \varphi_{Yk} \\ \varphi_{Zk} \end{pmatrix}, \end{aligned}$$

$$\begin{aligned} x_Y^0(t_1) &:= PS(t_1)x^0 = \sum_{l \in I_Y \setminus I_{YZ}} e^{\lambda_l t_1} (x^0, \psi_l) \varphi_{Yl} + \\ &+ \sum_{l \in I_{YZ}} e^{\lambda_l t_1} [(x^0, \psi_{1l}) \phi_{1Yl} + (x^0, \psi_{2l}) (c_j t_1 \phi_{1Yl} + \phi_{2Yl})] + \\ &+ \sum_{k \in I_Z \setminus I_{YZ}} e^{\lambda_k t_1} (x^0, \psi_k) \varphi_{Yk}. \end{aligned}$$

Using (3.6)–(3.8), (5.3)–(5.4), we obtain the following:

If $j \in I_Y \setminus I_{YZ}$, then

$$\begin{aligned} (x_Y^0(t_1), \psi_{Yj}) &= \sum_{l \in I_Y \setminus I_{YZ}} (x^0, \psi_l) e^{\lambda_l t_1} (\varphi_{Yl}, \psi_{Yj}) + \\ &+ \sum_{l \in I_{YZ}} e^{\lambda_l t_1} (x^0, \psi_{2l}) [c_m t_1 (\phi_{1Yl}, \psi_{Yj}) + (\phi_{2Yl}, \psi_{Yj})] + \\ &+ \sum_{l \in I_{YZ}} e^{\lambda_l t_1} (x^0, \psi_{1l}) (\phi_{1Yl}, \psi_{Yj}) + \\ &+ \sum_{k \in I_Z \setminus I_{YZ}} (x^0, \psi_k) e^{\lambda_k t_1} (\varphi_{Yk}, \psi_{Yj}) = \\ &= e^{\lambda_j t_1} (x^0, \psi_{Yj}) + \\ &+ \sum_{l \in I_{YZ}} (x_Z^0, \psi_{2Zl}) e^{\lambda_l t_1} (\phi_{2Yl}, \psi_{Yj}) + \\ &+ \sum_{k \in I_Z \setminus I_{YZ}} (x^0, \psi_{Zk}) e^{\lambda_k t_1} (\varphi_{Yk}, \psi_{Yj}) = \\ &= e^{\lambda_j t_1} (x^0, \psi_j) - \\ &- \sum_{k \in I_{YZ}} (x_Z^0, \psi_{Zk}) e^{\lambda_k t_1} (\varphi_{2Zk}, \psi_{Zj}) - \\ &- \sum_{k \in I_Y \setminus I_{YZ}} (x_Z^0, \psi_{Zk}) e^{\lambda_k t_1} (\varphi_{Zk}, \psi_{Zj}), \end{aligned}$$

and since the sequence $\{\varphi_{2Zk}, k \in I_Z\}$ is a Riesz basic of Z , we obtain, that

$$\sum_{k \in I_{YZ}} (x_Z^0, \psi_{Zk}) e^{\lambda_k t_1} (\varphi_{2Zk}, \psi_{Zj}) + \sum_{k \in I_Y \setminus I_{YZ}} (x_Z^0, \psi_{Zk}) e^{\lambda_k t_1} (\varphi_{Zk}, \psi_{Zj}) = S_Z(t_1) x_Z^0, \text{ so}$$

$$(x_Y(t_1), \psi_{Yj}) = e^{\lambda_j t_1} (x^0, \psi_j) - (S_Z(t_1) x_Z^0, \psi_{Zj}).$$

If $j \in I_{YZ}$, then

$$\begin{aligned} (PS(t_1) x^0, \psi_{1Yj}) &= \\ &= \sum_{l \in I_{YZ}} e^{\lambda_l t_1} (x^0, \psi_{1l}) (\phi_{1Yl}, \psi_{1Yj}) + \\ &+ \sum_{l \in I_{YZ}} e^{\lambda_l t_1} (x^0, \psi_{2l}) (c_j t_1 (\phi_{1Yl}, \psi_{1Yj}) + (\phi_{2Yl}, \psi_{1Yj})) + \\ &+ \sum_{k \in I_Z \setminus I_{YZ}} e^{\lambda_k t_1} (x^0, \psi_k) (\varphi_{Yk}, \psi_{1Yj}) = \\ &= e^{\lambda_j t_1} [(x^0, \psi_{1j}) + c_j t_1 (x^0, \psi_{2j})] + \sum_{l \in I_{YZ}} e^{\lambda_l t_1} (x^0, \psi_{2l}) (\phi_{2Yl}, \psi_{1Yj}) \\ &+ \sum_{k \in I_Z \setminus I_{YZ}} e^{\lambda_k t_1} (x^0, \psi_k) (\varphi_{Yk}, \psi_{1Yj}) = e^{\lambda_j t_1} [(x^0, \psi_{1j}) + c_j t_1 (x^0, \psi_{2j})] - \\ &- \sum_{l \in I_{YZ}} e^{\lambda_l t_1} (x_Z^0, \psi_{2Zl}) (\phi_{2Zl}, \psi_{1Zj}) - \sum_{k \in I_Z \setminus I_{YZ}} e^{\lambda_k t_1} (x_Z^0, \psi_{Zk}) (\varphi_{Zk}, \psi_{1Zj}), \end{aligned}$$

and again, since the sequence $\{\varphi_{2Zk}, k \in I_Z\}$ is a Riesz basic of Z , we obtain, that

$$(x_Y(t_1), \psi_{Yj}) = e^{\lambda_j t_1} [(x^0, \psi_{1j}) + (x^0, \psi_{2j}) c_j t_1] - (S_Z(t_1) x_Z^0, \psi_{1Zj}).$$

Using $b, t_1 - \tau$ instead of x^0, t_1 in above computations, we obtain :

if $j \in I_Y \setminus I_{YZ}$, then

$$(PS(t_1 - \tau) b, \psi_j) = e^{\lambda_j(t_1 - \tau)} b_{1j} - (S_Z(t_1) b_Z, \psi_{Zj}),$$

if $j \in I_{YZ}$, then

$$\begin{aligned} (PS(t_1 - \tau) b, \psi_{1j}) &= \\ &= [b_{1j} + b_{2j} c_j (t_1 - \tau)] e^{\lambda_j(t_1 - \tau)} - (S_Z(t_1 - \tau) b_Z, \psi_{1Zj}). \end{aligned}$$

After integration in $[0, t_1]$ we obtain:

if $j \in I_Y \setminus I_{YZ}$, then

$$\begin{aligned} (x_Y(t_1), \psi_j) &= e^{\lambda_j t_1} (x^0, \psi_j) - (S_Z(t_1) x_Z^0, \psi_{Zj}) + \\ &+ \int_0^{t_1} [b_{1j} e^{\lambda_j(t_1 - \tau)} - (S_Z(t_1 - \tau) b_Z, \psi_{Zj})] u(\tau) d\tau; \end{aligned}$$

if $j \in I_{YZ}$, then

$$\begin{aligned} (x_Y(t_1), \psi_j) &= [(x^0, \psi_{1j}) + (x^0, \psi_{2j}) c_j t_1] e^{\lambda_j t_1} - (S_Z(t_1) x_Z^0, \psi_{2Zj}) + \\ &+ \int_0^{t_1} ([b_{1j} + b_{2j} c_j (t_1 - \tau)] e^{\lambda_j(t_1 - \tau)} - (S_Z(t_1 - \tau) b_Z, \psi_{2Zj})) u(\tau) d\tau. \end{aligned}$$

Hence, equation (2.1)–(2.3) is partial controllable on $[0, t_1]$ if and only if the moment problem

$$\begin{aligned} &- (x^0, \psi_j) + e^{-\lambda_j t_1} (S_Z(t_1) x_Z^0, \psi_{Zj}) = \\ &= \int_{t_0}^{t_1} [b_{1j} e^{-\lambda_j \tau} - e^{-\lambda_j t_1} (S_Z(t_1 - \tau) b_Z, \psi_{Zj})] u(\tau) d\tau, j \in I_Y \setminus I_{YZ}, \\ &- (x^0, \psi_{1j}) - (x^0, \psi_{2j}) c_j t_1 + e^{-\lambda_j t_1} (S_Z(t_1) x_Z^0, \psi_{2Zj}) = \end{aligned} \tag{6.3}$$

$$\begin{aligned}
 &= \int_0^{t_1} [b_{1j} + b_{2j}c_j(t_1 - \tau)] e^{-\lambda_j \tau} u(\tau) d\tau - \\
 &- \int_0^{t_1} e^{-\lambda_j t_1} (S_Z(t_1 - \tau) b_Z, \psi_{1Zj}) u(\tau) d\tau, j \in I_{YZ}.
 \end{aligned} \tag{6.4}$$

has a solution $u(\cdot) \in L_2[0, t_1]$.

Since the series (6.1) converges for any $x_z^0 \in Z$, the sequence

$$\{(x^0, \psi_j) - e^{-\lambda_j t_1} (S_Z(t_1) x_Z^0, \psi_{Zj}), j \in I_Y \setminus I_{YZ}\} \in l^2,$$

and, in accordance with Lemma 5.3, the sequence $\{c_j, j \in I_{YZ}\}$ is bounded, so

$$\{(x^0, \psi_{1j}) + (x^0, \psi_{2j}) c_j t_1 - e^{-\lambda_j t_1} (S_Z(t_1) x_Z^0, \psi_{2Zm}), j \in I_{YZ}\} \in l^2 \text{ as well.}$$

Hence (Boaz, 1941), (Young, 1998) above problem moment has a solution if and only if family of exponential (6.2) is strongly minimal.

This proves the theorem. ■

If $b \in Y$, i.e. $b = \begin{pmatrix} b_Y \\ 0 \end{pmatrix}$, then moment problem (6.3)–(6.4) may be simplified:

$$(x^0, \psi_j) - e^{-\lambda_j t_1} (S_Z(t_1) x_Z^0, \psi_{Zj}) = \tag{6.5}$$

$$= \int_0^{t_1} b_{1j} e^{-\lambda_j \tau} u(\tau) d\tau, j \in I_Y \setminus I_{YZ},$$

$$(x^0, \psi_{1j}) + (x^0, \psi_{2j}) c_j t_1 - e^{-\lambda_j t_1} (S_Z(t_1) x_Z^0, \psi_{1Zj}) = \tag{6.6}$$

$$= \int_0^{t_1} b_{1j} e^{-\lambda_j \tau} u(\tau) d\tau, j \in I_{YZ},$$

or

$$d_j = \int_0^{t_1} b_{1j} e^{-\lambda_j \tau} u(\tau) d\tau, \tag{6.7}$$

where

$$d_j = \begin{cases} (x^0, \psi_{1j}) - e^{-\lambda_j t_1} (S_Z(t_1) x_Z^0, \psi_{Zj}), j \in I_Y \setminus I_{YZ} \\ (x^0, \psi_{1j}) + (x^0, \psi_{2j}) c_j t_1 - e^{-\lambda_j t_1} (S_Z(t_1) x_Z^0, \psi_{1Zj}), j \in I_{YZ}. \end{cases}$$

Since series (6.1) converges for any $x_Z^0 \in Z$, we have $\{d_j \in I_Y\} \in l^2$.

Hence the following corollaries are valid.

Corollary 6.2. *If series (6.1) converges for any $x_Z^0 \in Z$, then system (2.1)–(2.3) is partially null controllable on $[0, t_1]$ if and only if the family of exponent*

$$\{e^{-\lambda_j t} b_{1j}, j \in I_Y, t \in [0, t_1]\} \tag{6.8}$$

is strongly minimal.

Corollary 6.3. *If the series (6.1) converges for any $x_Z^0 \in Z$, then system (2.1)–(2.3) is partially null controllable on $[0, t_1]$ if and only if equation*

$$\dot{y}(t) = A_Y y(t) + b_Y u(t)$$

is null (completely) controllable on $[0, t_1]$.

Theorems 4.4–4.5 and Corollaries 4.2–4.6 hold true for $\sigma_{YZ} \neq \emptyset$ as well.

6.1 Example 2

The author of (Ammar-Khodja, 2018, page 4, system (12)) has proved the partial controllability conditions for the two-dimensional system

$$\begin{cases} y'_t = \Delta y + \alpha z + 1_\omega u, \\ z'_t = \Delta z \end{cases} \quad (t, x) \in Q_T = [0, T] \times Q, \quad (6.9)$$

$$\begin{aligned} y(0, x) &= y_0(x), z(0, x) = z_0(x), x \in Q, \\ y(t, x) &= z(t, x) = 0, t \in [0, T], x \in \partial Q, \end{aligned}$$

where Q is a bounded domain in $\mathbb{R}$ with a C^2 -boundary ∂Q , ω is a non-empty open subset of Q and $T > 0$,

$$y_0(\cdot), z_0(\cdot) \in L_2(Q), u(\cdot, \cdot) \in L_2(Q_T), \alpha(\cdot, \cdot) \in L_\infty(Q_T, \mathcal{L}(\mathbb{R})).$$

As it is noted in (Ammar-Khodja, 2018), controllability of such partial parabolic control systems are investigated in many fields such as chemistry, physics or biology.

However, even if $u(t, x) \in \mathbb{R}$ for all $t \in [0, T]$ and $x \in Q$, the control $u(\cdot, \cdot) \in L_2(Q_T)$ cannot be considered as a single control force⁵. The given paper focuses on achieving controllability using a single control force, so we will define $u(t, x)$ in the form $\cdot \in L_2(Q)$ is predetermined and $u(t), u(\cdot) \in L_2([0, T], \mathbb{R})$ represents a single control force.

The system being studied in this section is

$$\begin{cases} y'_t = \Delta y + \alpha z + xu, \\ z'_t = \Delta z, \end{cases} \quad (6.10)$$

$$(t, x) \in Q_T = [0, T] \times [0, \pi], \alpha \in \mathbb{R} \setminus \{0\}, \hat{b}(\cdot) \in L_2([0, \pi], \mathbb{R}), u(\cdot) \in L_2[0, T],$$

$$y(0, x) = y_0(x), z(0, x) = z_0(x), x \in [0, \pi], \quad (6.11)$$

$$y(t, 0) = z(t, 0) = 0, t \in [0, T], \quad (6.12)$$

$$y_0(\cdot), z_0(\cdot) \in L_2([0, \pi]).$$

One can write system (6.10)–(6.12) in form (2.1)–(2.3), where

$$A = \begin{pmatrix} \Delta & \alpha \\ 0 & \Delta \end{pmatrix}, b = \begin{pmatrix} x \\ 0 \end{pmatrix}, P = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}.$$

Here

$$Y = \left\{ \begin{pmatrix} y(x) \\ 0 \end{pmatrix}, y(\cdot) \in L_2[0, \pi] \right\}, Z = \left\{ \begin{pmatrix} 0 \\ z(x) \end{pmatrix}, z(\cdot) \in L_2[0, \pi] \right\},$$

$$A_Y = A_Z = \Delta$$

with domains (Balakrishnan, 1976, page 214, Example 4.6.6, page 241, Example 4.10.1)

$$D(A_Y) = D(A_Z) = \{f, f' \text{ a.c.} : f', f'' \in L_2[0, \pi] \text{ and } f(0) = f(\pi) = 0\},$$

$$b_Y = \hat{b}, b_Z = 0.$$

Let λ_i and $\varphi_i, i \in \mathbb{N}$ be eigenvalues and eigenvectors of the operator Δ with boundary conditions (6.11). Obviously, eigenvalues $\lambda_n = -n^2, n = 1, 2, \dots$, of Δ are eigenvalues of A , and to every eigenvalues λ_n correspond the eigenvectors $\phi_{1n} = \sqrt{\frac{2}{\pi}} \begin{pmatrix} \sin nx \\ 0 \end{pmatrix}, n = 1, 2, \dots$ of the

⁵it is infinite-dimensional control

operator A , and the eigenvectors $\psi_{2n} = \sqrt{\frac{2}{\pi}} \begin{pmatrix} 0 \\ \sin nx \end{pmatrix}$ of the operator A^* . In this example both operator equations (5.5) and (5.8) are reduced to the equation

$$\begin{aligned} y'' + \alpha \sqrt{\frac{2}{\pi}} \sin nx &= -n^2 y + c_n \sqrt{\frac{2}{\pi}} \sin nx, \\ y(0) &= y(\pi) = 0. \end{aligned}$$

where according to (5.7) we obtain

$$c_n = \alpha \int_0^\pi \frac{2}{\pi} \sin^2 nx dx = \alpha \neq 0, n \in \mathbb{N}.$$

Hence equations $y'' + \alpha \sqrt{\frac{2}{\pi}} \sin nx = -n^2 y + c_n \sqrt{\frac{2}{\pi}} \sin nx$, $n \in \mathbb{N}$, are equivalent to equations

$$y'' = -n^2 y, n \in \mathbb{N}, y(0) = y(\pi) = 0, \quad (6.13)$$

so the vector

$$\phi_{2n} = \begin{pmatrix} \phi_{2Yn}(x) \\ \sqrt{\frac{2}{\pi}} \sin nx \end{pmatrix}, x \in [0, \pi]$$

is a generalized eigenvector of A , and the vector

$$\psi_{1n} = \begin{pmatrix} \sqrt{\frac{2}{\pi}} \sin nx \\ \psi_{1Zn}(x) \end{pmatrix}, x \in [0, \pi]$$

is a generalized eigenvector of the operator A^* , where $\phi_{2Yn}(x)$ and $\psi_{1Zn}(x)$ are solutions of equation (6.13). For any $n \in \mathbb{N}$ equation (6.13) has the infinite number of solutions. Let's choose the solutions $\phi_{2Yn} = -\sqrt{\frac{2}{\pi}} \sin nx$ and $\psi_{1Zn}(x) = \sqrt{\frac{2}{\pi}} \sin nx$, so the vector

$$\phi_{2n} = \begin{pmatrix} -\sqrt{\frac{2}{\pi}} \sin nx \\ \sqrt{\frac{2}{\pi}} \sin nx \end{pmatrix}, x \in [0, \pi]$$

is a generalized eigenvector of the operator A , and the vector

$$\psi_{1n} = \begin{pmatrix} \sqrt{\frac{2}{\pi}} \sin nx \\ \sqrt{\frac{2}{\pi}} \sin nx \end{pmatrix}, x \in [0, \pi]$$

is a generalized eigenvector of the operator A^* , corresponding to the eigenvalue $\lambda_n = -n^2$, $n \in \mathbb{N}$.

One can prove (see (Ammar-Khodja et al., 2017) and Example 1), that the sequences

$$\{\phi_{1n}, \phi_{2n}, n \in \mathbb{N}\}, \quad (6.14)$$

and

$$\{\psi_{1n}, \psi_{2n}, n \in \mathbb{N}\}, \quad (6.15)$$

are Riesz basic in $X = L_2([0, \pi], \mathbb{R}^2)$. Hence system (6.10)–(6.12) satisfies to conditions of system (2.1)–(2.3)⁶, so all theorems and corollaries proven in previous sections, are applicable to Example 2.

⁶See the bottom of Pages 2 and 3.

Denote $b_n = \sqrt{\frac{2}{\pi}} \int_0^\pi x \sin nx dx = \frac{(-1)^{n-1}}{n} \pi, n \in \mathbb{N}$.

From (6.5)–(6.6) it follows, that⁷

$$(x^0, \psi_{1n}) + (x^0, \psi_{2n}) \alpha t_1 - e^{n^2 t_1} (S_Z(t_1) x_Z^0, \psi_{1Zn}) = \int_0^{t_1} b_n e^{n^2 \tau} u(\tau) d\tau, n \in \mathbb{N}, \quad (6.16)$$

where $\psi_{1Zn} = \sqrt{\frac{2}{\pi}} \sin nx, n \in \mathbb{N}$.

Now we will check the convergence of series (6.1).

We have

$S_Z(t_1) x_Z^0 = \sum_{m=1}^\infty x_{Zm}^0 e^{-m^2 t_1} \sqrt{\frac{2}{\pi}} \sin mx dx, 0 \leq x \leq \pi$, where

$$x_{Zm}^0 = \int_0^\pi x_Z^0(\theta) \sqrt{\frac{2}{\pi}} \sin m\theta d\theta,$$

$$(S_Z(t_1) x_Z^0, \psi_{1Zn}) = \sum_{m=1}^\infty e^{-m^2 t_1} x_{Zm}^0 \int_0^\pi \left(\sqrt{\frac{2}{\pi}} \sin mx \sqrt{\frac{2}{\pi}} \sin nx \right) dx = x_{Zn}^0 e^{-n^2 t_1},$$

$$\sum_{n=1}^\infty e^{2n^2 t_1} |(S_Z(t_1) x_Z^0, \psi_{1Zn})|^2 = \sum_{n=1}^\infty e^{2n^2 t_1} e^{-2n^2 t_1} |x_{Zn}^0|^2 = \|x_Z^0\|^2 < +\infty, \forall t_1 > 0.$$

Hence from Corollary 6.2 it follows, that system (6.10)–(6.12) is partially null controllable on $[0, t_1]$ if and only if the family

$$\left\{ e^{n^2 t} b_n, n \in \mathbb{N}, t \in [0, t_1] \right\} \quad (6.17)$$

is strongly minimal.

Again, the eigenvalues $\lambda_n = -n^2, n \in \mathbb{N}$ satisfy the sufficient conditions for strong minimality of family (6.17), obtained in Theorem 4.5 with $\rho = 1$, and $\sigma_c = \lim_{n \rightarrow \infty} \frac{2 \ln |n|}{-n^2} = 0$, so the family $\{e^{n^2 t} b_n = e^{n^2 t} \frac{(-1)^{n-1}}{n} \pi, n \in \mathbb{N}, t \in [0, t_1]\}$ is strongly minimal (see Example 1).

In accordance with Theorem 4.5 and Corollary 4.6 system (6.10)–(6.12) is partially null controllable on $[0, t_1]$ for any $t_1 > 0$.

7 Conclusion

The partial exact null-controllability conditions for abstract coupled control equations (1.1)–(1.3) governed by a single distributed control $u(t)$ have been obtained using the moment problem approach. These conditions strengthen the partial controllability conditions for one of coupled parabolic systems obtained in Ammar Khodja et al. (2016). This is because, as demonstrated in Example 2, even when $u(t, x) \in \mathbb{R}$ for all $t \in [0, T]$ and $x \in Q$, the control $u(\cdot, \cdot) \in L(Q)$ cannot be treated as a single control force in the conventional sense⁸.

The controllability results to date have focused on control problems for coupled PDEs, mostly of the same structure. The majority of results have been obtained for coupled parabolic equations, and a few have been done for the controllability of coupled wave equations. The abstract approach developed here allows us to investigate various controllability problems, including partial controllability, for coupled systems containing equations of different structures.

All examples are simple; they have been chosen only for illustrative purposes.

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⁷Here $I_Y = I_Z$, so $I_Y \setminus I_{YZ}$ is an empty set.

⁸it is infinite-dimensional control

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