

Shifted Chebyshev Polynomials for a Generalized Abel-Volterra-Fredholm integral equations

Mohamed Nasseh NADIR¹ and Adel JAWAHDOU²

¹Department of Mathematics, Faculty of Sciences of Bizerte. University of Carthage.
 Jarzouna. Tunisia
 nadir.mohamednasseh@yahoo.com

²Department of Mathematics, Institute of Engineering of Bizerte. University of
 Carthage. Jarzouna. Tunisia
 adeljaw2002@yahoo.com

ABSTRACT

The aim of our work is to solve numerically a Generalized Volterra Abel-Fredholm integral equations using a new spectral method based on a shifted first kind Chebyshev Polynomials, this method leads us to convert the equation to a system of a linear algebraic equations in the unknown expansion coefficients. The numerical solutions obtained in some examples by our technical show its high accuracy and efficiency compared with others methods.

Keywords: Shifted Chebyshev polynomials, Generalized Volterra Abel-Fredholm integral equations, collocation method, Projection method.

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1 Introduction

The phenomena of the Volterra-Abel-Fredholm integral equations is applicable in many areas of physics, biological and fluid dynamics, considerable effort of researchers has been presented to solving this kind of equations. The most of the initial and boundary value problems associated with the ordinary and partial differential equations can be modeled into the Volterra Abel-Fredholm integral equation of the form

$$\varphi(x) - \int_a^x \frac{1}{(x-t)^\alpha} \varphi(t) dt - \int_a^b k(x,t) \varphi(t) dt = f(x), \quad 0 \leq \alpha < 1, \quad (1.1)$$

where $f(x)$ is a given function defined at every point x in the interval $[a, b]$, $k(x, t)$ is a given function defined for every pair of point x and t in $[a, b]$ and $\varphi(x)$ is the unknown function in $[a, b]$ to be determined. Denoting the equation (1.1) by the operator equation

$$\varphi(x) - A_1 \varphi(x) - A_2 \varphi(x) = f(x), \quad (1.2)$$

where the operators $A_1 \varphi(x)$ and $A_2 \varphi(x)$ are given by

$$A_1 \varphi(x) = \int_a^x \frac{1}{(x-t)^\alpha} \varphi(t) dt, \quad A_2 \varphi(x) = \int_a^b k(x,t) \varphi(t) dt, \quad (1.3)$$

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or still

$$A_1 \varphi(x) = \int_a^x \frac{1}{(x-t)^\alpha} (\varphi(t) - \varphi(x)) dt + \varphi(x) \int_a^x \frac{1}{(x-t)^\alpha} dt \quad (1.4)$$

Many authors are tried to solve Abel-Volterra integral equation by different methods, in (Abdelkhani, 1990) the author replaces the weakly kernel by a polynomial of degree N in $(x-t)$. On the other hand, the authors in (M. Nadir, 2016), (M. Nadir, 2018), (Nadir, 2020) and (A. Lakhali and Nadir, 2022) replace the density function by Legendre, Lagrange, modified Lagrange and Chebyshev polynomials. Two numerical methods based on Touchard and Laguerre polynomials were presented to solve Abel-Volterra integral equation in (J. Abdullah, 2021). Modified Navot-Simpson's Quadrature and Pseudospectral method based on Muntz-Legendre wavelets (M. Araghi, 2008), (I. Dassios, 2022). Euler's operational matrix of integration and Euler polynomials are used in (R. Arora, 2022), in (J. Mouley, 2021) the authors use Differential Transform Method, Daubechies wavelet basis and the linear approximation respectively.

2 Chebyshev polynomials

i- The First-kind polynomial T_n

The Chebyshev polynomial $T_n(x)$ of the first kind is a polynomial in x of degree n ; defined by the relation

$$T_n(x) = \cos n\theta \quad \text{when } x = \cos \theta, \quad (2.1)$$

where $x \in [-1, 1]$, this involves that the corresponding variable $\theta \in [0, \pi]$. It is easy to see that $T_0(x) = 1$, $T_1(x) = x$ and by the recurrence formula satisfied by Chebyshev polynomials

$$\cos n\theta + \cos(n-2)\theta = 2 \cos \theta \cos(n-1)\theta,$$

we obtain the fundamental relation

$$T_n(x) = 2xT_{n-1}(x) - T_{n-2}(x), \quad n = 2, 3, \dots \quad (2.2)$$

Noting that the functions $\{T_n(x), n = 0, 1, 2, \dots\}$ form an orthogonal system on the interval $[-1, 1]$ with respect to the weight $w(x) = \frac{1}{\sqrt{1-x^2}}$ and so the polynomial system $S_n(x)$ given by

$$\left\{ S_0(x) = \sqrt{\frac{1}{\pi}} T_0(x), \dots, S_n(x) = \sqrt{\frac{2}{\pi}} T_n(x), \dots, n \geq 1 \right\}, \quad (2.3)$$

form an orthonormal system on the interval $[-1, 1]$ with respect to the weight $w(x) = \frac{1}{\sqrt{1-x^2}}$.

In other words

$$\langle S_k(x), S_l(x) \rangle = \int_{-1}^1 \frac{S_k(x) S_l(x)}{\sqrt{1-x^2}} dx = \begin{cases} 0 & \text{if } k \neq l \\ 1 & \text{if } k = l \end{cases} \quad (2.4)$$

ii- Shifted Chebyshev Polynomials

We construct the shifted Chebyshev polynomials through the change of variable $x = \frac{2}{b-a}t - \frac{b+a}{b-a}$. Therefore, the shifted Chebyshev polynomials $T^s(t)$, $t \in [a, b]$; $a, b \in \mathbb{R}$ with $a < b$.

This implies that $T^s(x) = T\left(\frac{2}{b-a}x - \frac{b+a}{b-a}\right)$ and so $T_0^s(x) = 1$, $T_1^s(x) = \frac{2}{b-a}x - \frac{b+a}{b-a}$. Hence we obtain the fundamental relation for the shifted polynomial

$$T_n^s(x) = \left(\frac{4}{b-a}x - 2\frac{(b+a)}{b-a}\right)T_{n-1}^s(x) - T_{n-2}^s(x), \quad n = 2, 3, \dots \quad (2.5)$$

Noting that the functions $\{T_n^s(x), n = 0, 1, 2, \dots\}$ form an orthogonal system on the interval $[a, b]$ with respect to the weight

$$w(x) = \frac{1}{\sqrt{1 - \left(\frac{2}{b-a}x - \frac{b+a}{b-a}\right)^2}} \quad (2.6)$$

and so the polynomial system $S_n^s(x)$ given by

$$\left\{S_n^s(x) = \sqrt{\frac{2}{b-a}}S_n\left(\frac{2}{b-a}x - \frac{b+a}{b-a}\right), \dots, n \geq 0\right\}, \quad (2.7)$$

form an orthonormal system on the interval $[a, b]$ with respect to the weight function (2.6). In other words

$$\begin{aligned} \langle S_k^s(x), S_l^s(x) \rangle &= \int_a^b \frac{S_k^s(x)S_l^s(x)}{\sqrt{1 - \left(\frac{2}{b-a}x - \frac{b+a}{b-a}\right)^2}} dx \\ &= \int_a^b \frac{S_k\left(\frac{2}{b-a}x - \frac{b+a}{b-a}\right)S_l\left(\frac{2}{b-a}x - \frac{b+a}{b-a}\right)}{\sqrt{1 - \left(\frac{2}{b-a}x - \frac{b+a}{b-a}\right)^2}} \frac{2}{b-a} dx \end{aligned} \quad (2.8)$$

with the change of variables $t = \left(\frac{2}{b-a}x - \frac{b+a}{b-a}\right)$, we obtain (2.4).

3 Discretization of integral equation

Applying a collocation method to the equation (1.1) in order to discretize and convert the equation (1.1) to a system of linear equations on the interval $[a, b]$, and approximate the unknown function $\varphi(x)$ by a finite sum of the form

$$\varphi(x) = \sum_{k=0}^N \alpha_k S_k^s(x), \quad (3.1)$$

where $S_n^s(x)$ denotes the n th shifted Chebyshev polynomial of the first kind. After substitution of the expansion (3.1) into the equation (1.1) using the decomposition (1.4) this latter becomes an approximate equation as

$$\begin{aligned} \sum_{k=0}^N \alpha_k S_k^s(x) - \int_a^x \frac{1}{(x-t)^\alpha} \sum_{k=0}^N \alpha_k (S_k^s(t) - S_k^s(x)) dt + \\ \sum_{k=0}^N \alpha_k S_k^s(x) \int_a^x \frac{1}{(x-t)^\alpha} dt - \int_a^b k(x, t) \sum_{k=0}^N \alpha_k S_k^s(t) dt = f(x). \end{aligned} \quad (3.2)$$

Choosing the Fourier's coefficients α_k such that (3.2) is satisfied on the interval $[a, b]$. For this technical we take the equidistant collocation points as follows

$$t_j = a + jh, \quad h = \frac{b-a}{N}, \quad j = 0, 1, \dots, N, \quad (3.3)$$

and define the residual as

$$R_N(x) = \sum_{k=0}^N \alpha_k S_k^s(x) - A_1 \left(\sum_{k=0}^N \alpha_k S_k^s(t) \right) - A_2 \left(\sum_{k=0}^N \alpha_k S_k^s(t) \right) - f(x). \quad (3.4)$$

Then, by imposing conditions at collocation points

$$R_N(x_j) = 0, \quad j = 0, 1, \dots, N, \quad (3.5)$$

the integral equation (3.2) is converted to a system of linear equations.

Theorem

Let $A = A_1 + A_2 : X \rightarrow X$ be compact operator and suppose that the equation

$$(I - A)\varphi = f, \quad (3.6)$$

admits a unique solution. For the projections $P_n : X \rightarrow X_n$ such that

$$\|P_n A - A\| \rightarrow 0, \quad n \rightarrow \infty.$$

The approximate equation

$$\varphi_n - P_n A \varphi_n = P_n f, \quad (3.7)$$

has a unique solution for all $f \in X$ with sufficiently large n , besides

$$\|\varphi - \varphi_n\| \leq M \|\varphi - P_n \varphi\|, \quad (3.8)$$

with some positive constant M depending on A .

Proof

As it is known for all sufficiently large n the inverse operators $(I - P_n A)^{-1}$ exist and are uniformly bounded, see [5, 7]. To verify the error bound, we apply the projection operator P_n to the equation (3.6) and get

$$P_n \varphi - P_n A \varphi = P_n f, \quad (3.9)$$

or again

$$\varphi - P_n A \varphi = P_n f + \varphi - P_n \varphi. \quad (3.10)$$

Subtracting (3.10) from (3.7) we find

$$(I - P_n A)(\varphi - \varphi_n) = (I - P_n)\varphi.$$

Hence the estimate (3.8) follows.

4 Illustrating examples

Example 1

Consider the linear Volterra-Fredholm integral equation

$$\varphi(x) - \int_0^x \frac{1}{\sqrt{x-t}} \varphi(t) dt = f(x),$$

where the function

$$f(x) = \frac{1}{2}\pi x + \sqrt{x},$$

chosen so that the solution $\varphi(x)$ is given by

$$\varphi(x) = \sqrt{x}.$$

Applying the first Shifted Chebyshev polynomial $S_8^s(x)$ to approximate the solution $\varphi(x)$, that is to say $\varphi_N(x)$ solution of the algebraic system of linear equations and the error.

Points of x	Exact sol	Approx sol	Error	(Abdelkhani, 1990)	(I. Dassios, 2022)
0.0000e+00	0.0000e+00	1.5753e-05	1.5753e-05	3.0712e-02	1.7909e-03
2.5000e-01	5.0000e-01	5.0022e-01	2.2297e-04	3.0712e-02	1.7909e-03
5.0000e-01	7.0710e-01	7.0706e-01	4.1769e-05	3.0712e-02	1.7909e-03
7.5000e-01	8.6602e-01	8.6587e-01	1.5138e-04	3.0712e-02	1.7909e-03
1.0000e+00	1.0000e+00	1.0004e+00	4.4268e-04	3.0712e-02	1.7909e-03

Table 1. The exact, the approximate solution and the error of example 1 in some arbitrary points, using the first shifted Chebyshev polynomial $S_n^s(x)$, $n = 8$.

Example 2

Consider the linear Volterra-Fredholm integral equation

$$\varphi(x) - \frac{1}{4} \int_0^x \frac{1}{\sqrt{x-t}} \varphi(t) dt = f(x),$$

where the function

$$f(x) = \frac{1}{\sqrt{x+1}} + \frac{\pi}{8} - \frac{1}{4} \arcsin\left(\frac{1-x}{1+x}\right),$$

chosen so that the solution $\varphi(x)$ is given by

$$\varphi(x) = \frac{1}{\sqrt{x+1}}$$

Applying the first Shifted Chebyshev polynomial $S_8^s(x)$ to approximate the solution $\varphi(x)$, that is to say $\varphi_N(x)$ solution of the algebraic system of linear equations and the error.

Points of x	Exact sol	Approx sol	Error	(S. Mondal, 2018)
0.0000e+00	1.0000e+00	1.0000e+00	6.0369e-07	0.0000e+00
2.5000e-01	8.9442e-01	8.9442e-01	4.6757e-07	6.8300e-04
5.0000e-01	8.1649e-01	8.1649e-01	2.9359e-07	4.1700e-04
7.5000e-01	7.5592e-01	7.5592e-01	1.7936e-07	3.0700e-04
8.7500e-01	7.3029e-01	7.3029e-01	2.0123e-07	2.7100e-04

Table 2. The exact, the approximate solution and the error of example 2 in some arbitrary points, using the first shifted Chebyshev polynomial $S_n^s(x)$, $n = 8$.

Example 3

Consider the linear Volterra-Fredholm integral equation

$$\varphi(x) - \int_0^x \frac{1}{\sqrt{x-t}} \varphi(t) dt - \int_0^1 (xt) \varphi(t) dt = f(x),$$

where the function

$$f(x) = x^2 - \frac{16}{15}x^2\sqrt{x} - \frac{1}{4}x$$

chosen so that the solution $\varphi(x)$ is given by

$$\varphi(x) = x^2$$

Applying the first Shifted Chebyshev polynomial $S_8^s(x)$ to approximate the solution $\varphi(x)$, that is to say $\varphi_N(x)$ solution of the algebraic system of linear equations and the error.

Points of x	Exact sol	Approx sol	Error
0.0000e+00	0.0000e+00	-4.7205e-09	4.7205e-09
2.5000e-01	6.2500e-02	6.2498e-02	1.5318e-06
5.0000e-01	2.5000e-01	2.4999e-01	5.4495e-06
7.5000e-01	5.6250e-01	5.6248e-01	1.4381e-05
1.0000e+00	1.0000e+00	9.9996e-01	3.4311e-05

Table 3. The exact, the approximate solution and the error of example 3 in some arbitrary points, using the first shifted Chebyshev polynomial $S_n^s(x)$, $n = 8$.

Example 4

Consider the linear Volterra-Fredholm integral equation

$$\varphi(x) - \int_0^x \frac{1}{(x-t)^{\frac{2}{3}}} \varphi(t) dt - \int_0^1 (\sqrt{x-t}) \varphi(t) dt = f(x),$$

where the function

$$\begin{aligned} f(x) = & (x^2 - 1) - (x)^{\frac{1}{3}} \left(\frac{27}{14}x^2 - 3 \right) + \frac{2}{105} (x-1)^{\frac{3}{2}} (12x + 8x^2 - 20) \\ & - \frac{2}{105} (x)^{\frac{3}{2}} (8x^2 - 35) \end{aligned}$$

chosen so that the solution $\varphi(x)$ is given by

$$\varphi(x) = x^2 - 1$$

Applying the first Shifted Chebyshev polynomial $S_8^s(x)$ to approximate the solution $\varphi(x)$, that is to say $\varphi_N(x)$ solution of the algebraic system of linear equations and the error.

Points of x	Exact sol	Approx sol	Error
0.0000e+00	-1.0000e+00	-1.0000e+00	6.5221e-06
2.5000e-01	-9.3750e-01	-9.3750e-01	7.2568e-06
5.0000e-01	-7.5000e-01	-7.5000e-01	7.4959e-06
7.5000e-01	-4.3750e-01	-4.3750e-01	8.9801e-06
1.0000e+00	0.0000e+00	4.1050e-06	1.5703e-05

Table 4. The exact, the approximate solution and the error of example 4 in some arbitrary points, using the first shifted Chebyshev polynomial $S_n^s(x)$, $n = 8$.

Example 5

Consider the linear Volterra-Fredholm integral equation

$$\varphi(x) - \int_0^x \frac{1}{(x-t)^{\frac{1}{4}}} \varphi(t) dt - \int_0^1 (x+t)^2 \varphi(t) dt = f(x),$$

where the function

$$f(x) = x^3 - \left(\frac{512}{1155} x^3 (x)^{\frac{3}{4}} \right) - \left(\frac{1}{4} x^2 + \frac{2}{5} x + \frac{1}{6} \right)$$

chosen so that the solution $\varphi(x)$ is given by

$$\varphi(x) = x^3$$

Applying the first Shifted Chebyshev polynomial $S_8^s(x)$ to approximate the solution $\varphi(x)$, that is to say $\varphi_N(x)$ solution of the algebraic system of linear equations and the error.

Points of x	Exact sol	Approx sol	Error
0.0000e+00	0.0000e+00	-8.6721e-08	8.6721e-08
2.5000e-01	1.5625e-02	1.5624e-02	1.8910e-07
5.0000e-01	1.2500e-01	1.2499e-01	2.5697e-07
7.5000e-01	4.2187e-01	4.2187e-01	2.7262e-07
1.0000e+00	1.0000e+00	9.9999e-01	2.1642e-07

Table 5. The exact, the approximate solution and the error of example 5 in some arbitrary points, using the first shifted Chebyshev polynomial $S_n^s(x)$, $n = 8$.

5 Conclusion

Solving numerically a Generalized Volterra Abel-Fredholm integral equations is to approximate the unknown the function $\varphi(x)$ by a finite sum of shifted first kind Chebyshev Polynomials. Substituting this finite sum into the integral equation in order to obtain a system of linear equations

with $N + 1$ unknowns. Some numerical examples have been given to show the high accuracy and efficiency of this technical compared with others methods.

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