

Stability of Neural Networks with Delays and Hybrid Impulses

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ABSTRACT

We study stability and stabilization issue for a time delays continuously distributed on a specified duration of time system of integro-differential equations, under introduced certain conditions. Characteristics of global stability of these systems subject to with hybrid impulse conditions in the continuous-time case investigated.

Keywords: Impulsive equations, hybrid impulses, neural networks, exponential stability, stabilizing and destabilizing, global stability.

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1 Introduction

Impulsive and delay neural networks model has applications in various electronic or biological networks systems and these systems are encountering instantaneous and abrupt changes frequently within period of time. During past few decades, dynamic networks extensively studied in various field of science and engineering, because of its broad applications. Generally speaking impulses can be categorized into two groups depending on effects of the system. If the impulses potentially destroy the stability of dynamical systems; called destabilizing, and if the impulses stabilize dynamical systems; called impulsive controllers, stabilizing dynamical systems. Stabilizing and destabilizing impulses have been studied by many researchers (Li, Shen and Feng, 2008), (Li, Li and Liu, 2009), (Li, Caraballo, Rakkiyappan and Han, 2015), (Li, Yang and Huang, 2019), (Zou, Li, Wang, Lou and Lu, 2020) and references there in. The hybrid effect of impulses for the system stabilizing and destabilizing occurs simultaneously. In various dynamical systems existence of impulses and delays investigated by many researchers (Akca, Alassar, Covachev, Covacheva and Al-Zahrani, 2004), (Akça, Covachev and Covacheva, 2004), (Gopalsamy, 2004), (Li, O'Regan and Akca, 2015), (Mohamad, Gopalsamy and Akca, 2008). There are three classes of impulsive hybrid systems namely; impulsive differential systems, sample data control systems and impulsive switched systems.

The paper is organized as follows. In Section 2, we will provide some notations and definitions useful all along the paper. In Section 3, model description, using average impulsive strength to describe the hybrid impulses and some new sufficient conditions on uniform stability of system of integro-differential equations are derived. The standard notations are used such as $\mathbb{R}^+$ and $\mathbb{R}^n$ denote the set of nonnegative real numbers and the n-dimensional Euclidean space. $\mathbb{N}_+$ denotes the set of positive integers. For $x \in \mathbb{R}^n$, $|x|$ denotes the Euclidean norm of x .

Definition 1. (Average Impulsive Interval) (Lu, Ho and Cao, 2010),(Rakkiyappan, Gayathri, Velmurugan and Cao, 2019),(Zou et al., 2020)

T_a is called the average impulsive interval of the impulsive sequence $\zeta = \{t_1, t_2, \dots\}$ if

$$\frac{T-t}{T_a} - N_0 \leq N_\zeta(T, t) \leq \frac{T-t}{T_a} + N_0, \quad \text{for all } T \geq t \geq 0$$

where N_0 is a positive integer and T_a is a positive number. $N_\zeta(T, t)$ denotes the impulsive times of the impulsive sequence ζ in the time interval (t, T) and the average impulsive interval of the impulsive sequence $\zeta = \{t_1, t_2, \dots\}$ defined the number $i(0, t)$ of moments of impulsive effect between 0 and t satisfies

$$N_\zeta(T, t) = \lim_{t \rightarrow +\infty} \frac{i(0, t)}{t} = p < +\infty$$

Definition 2. (Average Impulsive Strength) (Zou et al., 2020)

$$\zeta(T, t) = \{t_l, t_{l+1}, \dots, t_{l+N_\zeta(T, t)-1}\} \text{ for all } T \geq t \geq 0 \text{ if} \quad (1.1)$$

where α is called the average impulsive strength of the sequence.

$$\frac{|\alpha_{N_\zeta(t, 0)}| + |\alpha_{N_\zeta(t, 0)+1}| + \dots + |\alpha_{N_\zeta(T, 0)-1}|}{\alpha} - N_1 \leq N_\zeta(T, t) \quad (1.2)$$

$$\leq \frac{|\alpha_{N_\zeta(t, 0)}| + |\alpha_{N_\zeta(t, 0)+1}| + \dots + |\alpha_{N_\zeta(T, 0)-1}|}{\alpha} + N_1 \quad (1.3)$$

where N_1 is a positive integer and α is a positive number. Using these inequality we can easily obtain

$$\alpha(1 - \frac{N_1}{N_\zeta(T, t)}) \leq \frac{|\alpha_{N_\zeta(t, 0)}| + |\alpha_{N_\zeta(t, 0)+1}| + \dots + |\alpha_{N_\zeta(T, 0)-1}|}{N_\zeta(T, t)} \leq \alpha(1 + \frac{N_1}{N_\zeta(T, t)}) \quad (1.4)$$

Henceforth we assumed that each activation function $f_i(\cdot)$, $i = 1, 2, \dots, m$ satisfies following conditions:

(H1) $f_i : \mathbb{R} \rightarrow \mathbb{R}$ is Lipschitzian with Lipschitz constant, $L_i > 0$,

$$|f_i(x) - f_i(y)| \leq L_i|x - y|, \text{ for all } x, y \in \mathbb{R}.$$

(H2) $|f_i(x)| \leq M_i, x \in \mathbb{R}$, for some constant $M_i > 0$.

2 Model Description

In the last two decades there has been great interest and introduced results in the neural networks, such as Hopfield, Cellular, Cohen-Grossberg, bidirectional associative and interesting

applications in many areas such as , optimization, signal and image processing, pattern recognitions, associative memory, neural networks, cryptography, solving nonlinear algebraic equations, and so on. Consider impulsive hybrid Hopfield-type model (Akca, Alassar, Covachev, Covacheva and Al-Zahrani, 2004; Kaslik and Sivasundaram, 2011; Gopalsamy, 2004).

$$\begin{cases} \frac{dx_i}{dt} = -a_i x_i(t) + \sum_{j=1}^m b_{ij} f_j(x_j(t)) + c_i, & t > 0, \quad t \neq t_k, \\ \Delta x_i(t_k) = I_i(x_i(t_k)), & i \in \{1, 2, \dots, m\}, \quad k = 1, 2, \dots, \end{cases} \quad (2.1)$$

where $\Delta x(t_k) = x(t_k + 0) - x(t_k - 0)$ are impulses at the moments t_k and $t_1 < t_2, \dots$ strictly increasing sequence. Assume that conditions **H1**, **H2** are satisfied. Suppose further that:

i) $a_i > 0, \quad b_{ij} \in \mathbb{R}, \quad c_i \in \mathbb{R}, \quad i, j \in \{1, \dots, m\}$,

ii) the following inequalities hold:

$$a_i - L_i \sum_{j=1}^m |b_{ji}| > 0, \quad i \in \{1, \dots, m\},$$

iii) the impulsive operators $I_i(x_i(t))$ satisfy

$$I_i(x_i(t_k)) = -\gamma_{ik}(x_i(t_k) - x_i^*), \quad 0 < \gamma_{ik} < 2, \quad i \in \{1, \dots, m\}, \quad k \in \mathbb{Z}^+.$$

Then the equilibrium $\mathbf{x}^* = (x_1^*, x_2^*, \dots, x_m^*)^T$ of (5) is unique. Most of the physical problems future state of the problems depends on not only the present state, but also on its past history. Therefore delay form of the physical problem ensures a realistic better modeling, such as pulse frequency modulation, optimization of drug distribution in the human body and control systems with changing reference signal and so on. Generalizing system (5) by inserting time delays in the neural networks, then we get following neural network (Akca, Alassar, Covachev, Covacheva and Al-Zahrani, 2004; Kaslik and Sivasundaram, 2011; Gopalsamy, 2004):

$$\frac{dx_i}{dt} = -a_i x_i(t) + \sum_{j=1}^m b_{ij} f_j(x_j(t - \tau_{ij})) + c_i, \quad t > 0, t \neq t_k \quad (2.2)$$

where $i \in \{1, 2, \dots, m\}$ and $\tau_{ij} \geq 0$ instantaneous transmission time delays neurons process input and output. The impulsive conditions are

$$\Delta x_i(t) = I_i(x_i(t)), \quad t = t_k, k = 1, 2, \dots \quad (2.3)$$

And the system is supplemented with initial functions of the form

$$x_i(s) = \psi_i(s), \quad s \in [-\tau, 0], i = \{1, 2, \dots, m\}, \tau = \max_{i,j \in \{1, 2, \dots, m\}} \{\tau_{ij}\} \quad (2.4)$$

where $\psi_i(s)$ is continuous for $s \in [-\tau, 0]$. It is assumed in the formulation of the system (5) that neurons process input and produce output instantaneously. Such instantaneous processing and delivery is not always straight on time and there are in general always significant time delays both in neural processing and axonal transmission. Therefor system having delays in neural network formulation denoted by $\tau_{ij} \geq 0, i, j = 1, 2, \dots$

Let x^* be the equilibrium point of (5). Because of convenience, we can shift the equilibrium x^* by letting $y = x - x^*$ and then system (5) transformed into

$$\begin{cases} \frac{dy_i}{dt} = -a_i y_i(t) + \sum_{j=1}^m b_{ij} f_j(y_j(t - \tau_{ij})), & t > 0, t \neq t_k, \\ y(t_k^+) = \alpha_k y(t_k^-), k \in \mathbb{R}^+, \\ y(t) = \phi(t) \in C([- \tau, 0], \mathbb{R}) \end{cases} \quad (2.5)$$

where $\{t_1, t_2, \dots\}$ sequence of strictly increasing impulsive moments,

$$y(t) = (y_1(t), y_2(t), \dots, y_n(t))^T \in \mathbb{R}^n,$$

state vector of the transformed system. $\alpha_k \in \mathbb{R}$ denotes the strength of impulses. The superscript " T " represents the transpose of the matrix or vectors. Obviously $y(t)$ is right continuous at the points $t = t_k$, i.e., $y(t_k) = y(t_k^+)$. Therefore solutions of the system (8) piecewise right-hand continuous functions which are discontinuous at the points $t = t_k$ for $k \in \mathbb{R}$, and $y(t) = \phi(t) \in C([- \tau, 0], \mathbb{R})$ the initial condition. Following (Akca, Alassar, Covachev, Covacheva and Al-Zahrani, 2004) define a Lyapunov function $V(\cdot)$ by

$$V(t) = V(y_1, y_2, \dots, y_m)(t) = \sum_{i=1}^m |y_i| = \sum_{i=1}^m |x_i - x_i^*|$$

and for $t \geq 0$ then we have

$$\begin{aligned} \frac{d^+ V(t)}{dt} &= \sum_{i=1}^m \frac{d^+}{dt} |y_i(t)| \\ &\leq \sum_{i=1}^m \left(-a_i |y_i| + \sum_{j=1}^m |b_{ij} L_j| |y_j| \right) \\ &= -\sum_{i=1}^m \left(a_i - L_i \sum_{j=1}^m |b_{ij}| \right) |y_j| \end{aligned}$$

Under the assumption (H1) and (H2) and $a_i - L_i \sum_{j=1}^m |b_{ij}| > 0, i \in \{1, 2, \dots, m\}$ there exists a real number $\alpha > 0$ such that

$$\frac{d^+ V(t)}{dt} \leq -\alpha V(t), \quad t > 0, \quad t \neq t_k \quad (2.6)$$

And because of the exponential stability theorem (Akca, Alassar, Covachev, Covacheva and Al-Zahrani, 2004; Gopalsamy, 2004; Mohamad et al., 2008; Mohamad and Gopalsamy, 2000) we have

$$\frac{d^+ V}{dt} \leq -\alpha V(t) \Rightarrow V(t) \leq e^{-\alpha t} V(0), \quad \forall t > 0.$$

In the model (9) we assumed that the time delays are discrete. More satisfactory hypothesis is that the time delays are continuously distributed over a certain duration of time. Now consider a system of integro-differential equations (Akça, Covachev and Covacheva, 2004; Gopalsamy, 2004; Li, O'Regan and Akca, 2015; Li, Caraballo, Rakkiyappan and Han, 2015; Li et al., 2019; Mohamad and Gopalsamy, 2000)

$$\begin{aligned} \frac{dx_i(t)}{dt} &= -a_i x_i(t) + \sum_{j=1}^m b_{ij} f_j \left(\int_{-\infty}^t K_{ij}(t-s) x_j(s) ds \right) + c_i, \\ i &= 1, 2, \dots, m, \quad t > 0, \end{aligned}$$

or

$$\frac{dx_i(t)}{dt} = -a_i x_i(t) + \sum_{j=1}^m b_{ij} f_j \left(\int_0^\infty K_{ij}(s) x_j(t-s) ds \right) + c_i, \quad (2.7)$$

$$i = 1, 2, \dots, m, \quad t > 0,$$

where for $i, j \in \{1, \dots, m\}$ the delay kernels $K_{ij}(s)$ are assumed to satisfy the following conditions:

H3. $K_{ij} : [0, \infty) \rightarrow [0, \infty)$ are bounded and continuous.

H4. $\int_0^\infty K_{ij}(s) ds = 1.$

H5. There exists a positive number μ such that

$$\int_0^\infty K_{ij}(s) e^{\mu s} ds < \infty.$$

For an integro-differential equation an impulsive condition including both the functional value and its integral also seems natural. We take the impulse conditions in the form

$$\Delta x_i(t_k) = I_i(x_i(t_k)) = B_{ik} x_i(t_k) + \int_{t_{k-1}}^{t_k} c_{ik}(s) x_i(s) ds + \alpha_{ik}, \quad k \in \mathbb{Z}^+, \quad (2.8)$$

where $t_k > t_0 = 0$ and $c_{ik} : [t_{k-1}, t_k] \rightarrow \mathbb{R}$ are measurable functions, essentially bounded on the respective interval, B_{ik} and α_{ik} are some real constants. For more details about impulse conditions of this and more general form see (Akça, Berezansky and Braverman, 1996) and reference therein.

The initial conditions associated with the system (2.7), (2.8) are given by

$$x_i(s) = \psi_i(s), \quad s \in (-\infty, 0], \quad i = 1, 2, \dots, m,$$

where $\psi_i(s)$ are bounded and continuous on $(-\infty, 0]$. Following the procedure from (Akca, Alassar, Covachev, Covacheva and Al-Zahrani, 2004) We have from (2.7) that

$$\frac{d^+}{dt} |x_i(t) - x_i^*| \leq -a_i |x_i(t) - x_i^*| + \sum_{j=1}^m |b_{ij}| L_j \int_0^\infty K_{ij}(s) |x_j(t-s) - x_j^*| ds, \quad (2.9)$$

where $i \in \{1, \dots, m\}$, $t > 0$, $t \neq t_k$.

Let us consider the functions $F_i : [0, \mu] \rightarrow (-\infty, \infty)$ defined by

$$F_i(\varepsilon_i) = a_i - \varepsilon_i - L_i \sum_{j=1}^m |b_{ji}| \int_0^\infty K_{ji}(s) e^{\varepsilon_i s} ds, \quad \varepsilon_i \in [0, \mu], \quad i = 1, 2, \dots, m.$$

Now, because of the assumptions **H3**, **H4** and **H5** each $F_i(\cdot)$ is well defined on $[0, \mu]$. We have

$$F_i(0) = a_i - L_i \sum_{j=1}^m |b_{ji}| > 0$$

by virtue of condition ii) of Theorem 3.1 (Akca, Alassar, Covachev, Covacheva and Al-Zahrani, 2004). Since $F_i(\cdot)$ is continuous and decreasing on $[0, \mu]$, it follows that there exists $\varepsilon_i^* \in (0, \mu]$ such that $F_i(\varepsilon_i) > 0$ for $\varepsilon_i \in (0, \varepsilon_i^*)$, $i \in \{1, \dots, m\}$. Choosing $\varepsilon^* = \min\{\varepsilon_1^*, \varepsilon_2^*, \dots, \varepsilon_m^*\}$, then we have

$$F_i(\varepsilon) = a_i - \varepsilon - L_i \sum_{j=1}^m |b_{ji}| \int_0^\infty K_{ji}(s) e^{\varepsilon s} ds > 0, \quad (2.10)$$

$$\varepsilon \in (0, \varepsilon^*), \quad i \in \{1, \dots, m\}.$$

Correspondingly, define

$$y_i(t) = |x_i(t) - x_i^*| e^{\varepsilon t},$$

where $i \in \{1, \dots, m\}$, $t \in \mathbb{R}$, and from (13) we derive

$$\frac{d^+ y}{dt} \leq -(a_i - \varepsilon) y_i(t) + \sum_{j=1}^m |b_{ij}| L_j \int_0^\infty K_{ij}(s) e^{\varepsilon s} y_j(t-s) ds, \quad t > 0, \quad t \neq t_k.$$

We define a Lyapunov functional $V(\cdot)$ by

$$V(t) = \sum_{i=1}^m \left\{ y_i(t) + \sum_{j=1}^m |b_{ij}| L_j \int_0^\infty K_{ij}(s) e^{\varepsilon s} \left(\int_{t-s}^t y_j(w) dw \right) ds \right\}, \quad t > 0.$$

It is found that $V(t) \geq 0$ for $t > 0$ and that

$$V(0) \leq \sum_{i=1}^m \left(y_i(0) + \sum_{j=1}^m |b_{ij}| L_j \int_0^\infty K_{ij}(s) e^{\varepsilon s} ds \sup_{w \in (-\infty, 0]} y_j(w) \right).$$

This implies that $0 < V(0) < \infty$ since $\int_0^\infty K_{ij}(s) e^{\varepsilon s} ds < \infty$ for $\varepsilon < \mu$.

We can now calculate

$$\begin{aligned} \frac{d^+ V(t)}{dt} &\leq \sum_{i=1}^m \left\{ -(a_i - \varepsilon) y_i(t) + \sum_{j=1}^m |b_{ij}| L_j \left(\int_0^\infty K_{ij}(s) e^{\varepsilon s} ds \right) y_j(t) \right\} \\ &= - \sum_{i=1}^m \left\{ a_i - \varepsilon - L_i \sum_{j=1}^m |b_{ji}| \int_0^\infty K_{ji}(s) e^{\varepsilon s} ds \right\} y_i(t), \quad t > 0, \quad t \neq t_k. \end{aligned}$$

As an application of (14) we have $\frac{d^+ V}{dt} \leq 0$ for $t > 0$, $t \neq t_k$. It follows that

$$V(t) \leq V(t_{k-1} + 0) \quad \text{for} \quad t \in (t_{k-1}, t_k], \quad k \in \mathbb{Z}^+. \quad (2.11)$$

We study the stability and stabilization problem of the delay neural networks with hybrid impulses. Using the Lyapunov stability theory along with comparisons principle and concept of average impulsive strength introduced exponential stability of delayed neural networks with hybrid impulses. The derived stabilization and criterion and convergence rate closely related with the references (Akca, Alassar, Covachev, Covacheva and Al-Zahrani, 2004; Akça, Covachev and Covacheva, 2004; Akça et al., 1996; Kaslik and Sivasundaram, 2011; Gopalsamy, 2004; Li, Caraballo, Rakkiyappan and Han, 2015; Li et al., 2019; Mohamad et al., 2008; Mohamad and Gopalsamy, 2000) average impulsive intervals of the neural networks

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