

Finite continuous Radon transforms with applications

Nitu Gupta¹ and V. R. Lakshmi Gorty²

¹Basic Sciences & Humanities

Mukesh Patel School of Technology Management & Engineering, SVKM's Narsee
Monjee Institute of Management Studies (NMIMS)
Deemed-to-University, Mumbai-400056, India.
nitugupta@nmims.edu

²Basic Sciences & Humanities

Mukesh Patel School of Technology Management & Engineering, SVKM's Narsee
Monjee Institute of Management Studies (NMIMS)
Deemed-to-University, Mumbai-400056, India.
VR.Lakshmigorty@nmims.edu

ABSTRACT

In this paper, finite continuous Radon transforms and inversion formula are studied. Sturm-Liouville theory with a self-adjoint operator and operational calculus for finite continuous Radon transforms are discussed. The developed finite continuous Radon transforms in this study is supported with examples from mathematical physics in the concluding section.

Keywords: Finite continuous Radon transform, Fourier-Radon expansion, adjoint operator, testing spaces, inversion.

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1 Introduction

Johann Radon (Radon, 1917) demonstrated how to construct a function of two variables from its integrals over all straight lines in the plane. The generalizations involving the reconstruction of a function from its integrals over smooth curves as well as the reconstruction of a function of n variables from its integrals over all hyperplanes were also studied in (Radon, 1917). The continuous Radon transform was defined by (Deans, 1983) as

$$\hat{f}(p, \phi) = R\{f(x, y)\} = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x, y) \delta(p - x \cos \theta - y \sin \theta) dx dy \quad (1.1)$$

where $\delta(\cdot)$ is Dirac delta function ($\delta(x) = 1$ if $x = 0$ and 0 elsewhere), $\theta \in [0, \pi]$ and $p \in [-\infty, \infty]$. In other words, $R\{f(x, y)\}$ is the integral of f over the line $L_{(p, \theta)}$ defined by $p = x \cos \theta - y \sin \theta$.

In (He and Liu, 2005), author studied $D(\Omega, \Phi)$ as a unbounded realization of the classical domain $(D - I)$ of type one. In this article, author defined the Radon transform on N and obtained

an inversion formula $R - 1 = (pi)(-2d)LR L$ in terms of a determinantal differential operator. In (Seeger, 1998), the author formulated conditions of finite type Radon transforms which leads modulo endpoints to regularity properties in two dimensions. Numerical integration of the finite Radon transforms on classes in multiple dimensions were studied in (Revers, 1995). In 2019, (Choi, Ginting, Jafari and Mnatsakanov, 2020) the relationship of the moments of the Radon transform and the moments of its modified Radon transform were derived. The author, used modified Radon transform data to provide a uniform approximation to the original density function. The reconstruction algorithm was then implemented and a simple density function was reconstructed from moments of its modified Radon transform.

The purpose of this paper, is to develop a classical work on finite continuous Radon transforms. Sturm-Liouville theory of finite continuous Radon transforms using (Al-Gwaiz, 2008) is established in this study. Fourier-Radon type of series expansions, analogous to (Malgonde and Lakshmigorty, 2008) is also demonstrated. The self-adjoint operator and operational calculus are derived using (Al-Gwaiz, 2008) and (Sneddon, 1972). The testing function spaces, inversion formula and uniqueness conditions have been established as in (Zemanian, 1968). The developed operational calculus of finite continuous Radon transforms in this text has been applied to partial differential equations involving boundary value problems.

2 Preliminary Results

Consider the Sturm-Liouville problem analogous to (Al-Gwaiz, 2008)

$$(\Delta_{x,y,\theta}) \phi(x, y) = 0, \quad (2.1)$$

where

$$\Delta_{x,y,\theta} = [(\sin^2 \theta) \Delta_x - (\cos^2 \theta) \Delta_y] \quad (2.2)$$

for the differential operator considered as

$$\Delta_x = D_x^2 \quad (2.3)$$

$$\Delta_y = D_y^2. \quad (2.4)$$

Here l, k are real constants and $-k \leq x \leq k$; $-l \leq y \leq l$, and $D_x = \frac{d}{dx}$, $D_y = \frac{d}{dy}$, with periodic boundary conditions (Al-Gwaiz, 2008, pp. 43-44):

$$\phi(-k, y) = \phi(k, y); \phi'(-k, y) = \phi'(k, y) \quad (2.5)$$

$$\phi(x, -l) = \phi(x, l); \phi'(x, -l) = \phi'(x, l). \quad (2.6)$$

The solution of the (2.1) can be obtained by assuming

$$\phi(x, y) = e^{-i\pi p} X_m(x) Y_n(y), \quad (2.7)$$

where p is an integer and $X_m(x)$, $Y_n(y)$ are functions of x , y respectively. Using (2.7), (2.1) can be written as (Al-Gwaiz, 2008, p. 118)

$$\frac{X_m''(x)}{\cos^2 \theta X_m(x)} = \frac{Y_n''(y)}{\sin^2 \theta Y_n(y)} = -\lambda^2 \quad (2.8)$$

where λ is an integer.

Further from (2.5) and (2.6), the solution of (2.8) is given as:

$$X_m(x) = c_1 \cos(\lambda_m x \cos \theta) \quad (2.9)$$

and

$$Y_n(y) = c_3 \cos(\lambda_n y \sin \theta) \quad (2.10)$$

where $\lambda_m = \frac{m\pi}{k \cos \theta}$, $-\infty < m < \infty$ and $\lambda_n = \frac{n\pi}{l \sin \theta}$, $-\infty < n < \infty$ are considered as eigenvalues of (2.8).

Using (2.9) and (2.10), (2.7) can be obtained as

$$\phi(x, y) = \phi_{m,n}(x, y) = c_1 c_3 e^{-\left(ip\pi - \frac{im\pi}{k}x - \frac{in\pi}{l}y\right)}. \quad (2.11)$$

Considering boundary conditions (2.5) and (2.6) as in (Bell, 2004), (2.8) can now be written as

$$(\Delta_x + \lambda_m^2 \cos^2 \theta) X_m(x) = 0. \quad (2.12)$$

$$(\Delta_y + \lambda_n^2 \sin^2 \theta) Y_n(y) = 0. \quad (2.13)$$

Here $X_m(x)$ and $Y_n(y)$ are the eigenfunctions of (2.8).

Hence (2.11) is eigenfunction of the problem (2.1)-(2.6) which corresponds to the nonzero eigenvalues λ_m and λ_n .

Then the orthogonality and orthonormality condition of (2.11) by (Al-Gwaiz, 2008, p. 94 and pp. 137-138) is given by

$$|\langle \phi_{m_1, n_1}(x, y), \phi_{m_2, n_2}(x, y) \rangle|^2 = \begin{cases} (c_1 c_3)^2 e^{-2i\pi p k l} & ; m_1, n_1 = m_2, n_2 \\ 0 & ; m_1, n_1 \neq m_2, n_2 \end{cases}. \quad (2.14)$$

3 Main results (The Fourier-Radon series and finite continuous Radon transformation)

Assume $f(x, y)$ is a square integrable function over rectangle (Al-Gwaiz, 2008, p. 122). Fourier-Radon series expansion follows from (2.14), (Al-Gwaiz, 2008, p. 124) and using (2.11):

$$f(x, y) = \phi_{m,n}(x, y) = c_{m,n} \left[1 + 2 \sum_{m,n=1}^{\infty} \cos \pi \left(p - \frac{m}{k}x - \frac{n}{l}y \right) \right]. \quad (3.1)$$

Multiplying (3.1) by $\left[1 + 2 \cos \pi \left(p - \frac{m}{k}x - \frac{n}{l}y \right) \right]$ and integrating w.r.t x, y over $[-k, k], [-l, l]$ yields:

$$\begin{aligned} & \int_{-l}^l \int_{-k}^k f(x, y) \left[1 + 2 \sum_{m,n=1}^{\infty} \cos \pi \left(p - \frac{m}{k}x - \frac{n}{l}y \right) \right] dx dy \\ &= \sum_{m,n=1}^{\infty} c_{m,n} \int_{-l}^l \int_{-k}^k \left[1 + 2 \sum_{m,n=1}^{\infty} \cos \pi \left(p - \frac{m}{k}x - \frac{n}{l}y \right) \right]^2 dx dy \\ &= \sum_{m,n=1}^{\infty} 8kl c_{m,n}. \end{aligned}$$

Then

$$c_{m,n} = \frac{1}{8kl} \int_{-l}^l \int_{-k}^k f(x, y) \left[1 + 2 \sum_{m,n=1}^{\infty} \cos \pi \left(p - \frac{m}{k}x - \frac{n}{l}y \right) \right] dx dy. \quad (3.2)$$

From (Kanwal, 1998, p. 39), $\left[1 + 2 \sum_{m,n=1}^{\infty} \cos \pi \left(p - \frac{m}{k}x - \frac{n}{l}y \right) \right]$ represents the regularized Dirac delta function and can be denoted as (Lee and Kim, 2012) by

$$\left[1 + 2 \sum_{m,n=1}^{\infty} \cos \pi \left(p - \frac{m}{k}x - \frac{n}{l}y \right) \right] = \delta \left(\pi \left(p - \frac{m}{k}x - \frac{n}{l}y \right) \right). \quad (3.3)$$

Hence (3.2) becomes

$$c_{m,n} = \frac{1}{8kl} \int_{-l}^l \int_{-k}^k f(x, y) \delta \left(\pi \left(p - \frac{m}{k}x - \frac{n}{l}y \right) \right) dx dy. \quad (3.4)$$

The convergence of the series (3.1) is straightforward by (Sneddon, 1972, p. 433) and (Ramm and Katsevich, 1996) and the following theorem 3.1 follows from (Sneddon, 1972, p. 425 and p. 432).

Theorem 3.1. *Let $f(x, y)$ be a function defined and absolute integrable on the rectangle $\{(x, y) : -k < x < k, -l < y < l\}$ then*

$$c_{m,n} = \frac{1}{8kl} \int_{-l}^l \int_{-k}^k f(x, y) \delta \left(\pi \left(p - \frac{m}{k}x - \frac{n}{l}y \right) \right) dx dy,$$

at each point of the open interval $[-k, k] \times [-l, l]$ at which $f(x, y)$ is continuous.

At any point of the interval at which $f(x, y)$ has a finite discontinuity the symbol $f(x, y)$ is taken to mean $\frac{1}{2} [f(x+, y+) + f(x-, y-)]$ and at $x = -k, y = -l$ or $x = k, y = l$ it is taken to mean $\frac{1}{2} [f(-k, -l) + f(k, l)]$.

Remark 3.1. Now let $V_{m,n} = \left(\frac{m}{k}x + \frac{n}{l}y \right) \in \mathbb{R}^2$ and $p \in \mathbb{R}$, substituting

$$H(V_{m,n}, p) = \left\{ (x, y) \in \mathbb{R}^2 \mid V_{m,n} \cdot (x, y) = \frac{m}{k}x + \frac{n}{l}y = p \right\},$$

where $H(V_{m,n}, p)$ is a hyperplane, more precisely: $H(V_{m,n}, p)$ is the right in the rectangle $[-k, k] \times [-l, l] \subset \mathbb{R}^2 \cdot \mathbb{P}^2$, $\{H(V_{m,n}, p) \mid (V, p) \in \mathbb{R}^2 \times \mathbb{R}\}$ is the differentiable variate.

Let $S(\mathbb{P}^2)$ be the Schwartz space, the expressions (3.1), (3.4) and theorem 3.1 suggest introducing the finite continuous Radon transform analogous to (Sneddon, 1972, p. 425) as follows:

$$Rf(H(V_{m,n}, p)) = \frac{1}{8kl} \int_{-l}^l \int_{-k}^k f(x, y) \delta(\pi(p - V_{m,n} \cdot (x, y))) dx dy \quad (3.5)$$

where $V_{m,n} \cdot (x, y) = \frac{m}{k}x + \frac{n}{l}y$. Thus implies $R : S(\mathbb{R}^2) \rightarrow S(\mathbb{P}^2) : f \rightarrow Rf$.

Inversion formula of (3.5) is given by

$$R^{-1}(Rf(H(V_{m,n}, p))) = f(x, y) = \sum_{m,n=1}^{\infty} Rf(H(V_{m,n}, p)) \delta(\pi(p - V_{m,n} \cdot (x, y))). \quad (3.6)$$

We point out the following operational rules:

(1) If $f(x, y) \in \mathbb{R}^2[-k, k] \times [-l, l]$, then (3.5) becomes:

$$R \left\{ \frac{\partial^2 f}{\partial x^2} \right\} (H(V_{m,n}, p)) = \frac{1}{8kl} \int_{-l}^l \int_{-k}^k \left\{ \frac{\partial^2 f}{\partial x^2} \right\} \delta(\pi(p - V_{m,n} \cdot (x, y))) dx dy. \quad (3.7)$$

From (Kanwal, 1998, p. 39), we deduce

$$\int f(x) \delta^n(x) dx = - \int \frac{\partial f(x)}{\partial x} \delta^{n-1}(x) dx. \quad (3.8)$$

Using (3.8); (3.7) becomes

$$R \left\{ \frac{\partial^2 f}{\partial x^2} \right\} (H(V_{m,n}, p)) = \frac{1}{8lk} \left[2I_1 - 2I_2 + \int_{-l}^l \int_{-k}^k f(x, y) \frac{\partial^2 \delta(\pi(p - V_{m,n} \cdot (x, y)))}{\partial x^2} dx dy \right] \quad (3.9)$$

where

$$I_1 = \int_{-l}^l f'(k, y) \delta(\pi(p - V_{m,n} \cdot (k, y))) dy \quad (3.10)$$

and

$$I_2 = \int_{-l}^l f'(-k, y) \delta(\pi(p - V_{m,n} \cdot (-k, y))) dy. \quad (3.11)$$

By (Kanwal, 1998, p. 25) and (Ramm and Katsevich, 1996, p. 260)

$$\int \delta(x - \xi) f(\xi) d\xi = f(x). \quad (3.12)$$

Substituting $\frac{n\pi}{l}y = t$, $dy = \frac{l}{n\pi}dt$ in (3.10), (3.11) and using (3.12) follows:

$$I_1 = \frac{l}{n\pi} f' \left(k, \frac{l}{n} (p - m) \right). \quad (3.13)$$

and

$$I_2 = \frac{l}{n\pi} f' \left(-k, \frac{l}{n} (p + m) \right). \quad (3.14)$$

Thus from (2.11) and (3.3), we obtain

$$\delta \left(\pi \left(p - \frac{m}{k}x - \frac{n}{l}y \right) \right) = e^{-(ip\pi - \frac{im\pi}{k}x - \frac{in\pi}{l}y)}.$$

Hence

$$\frac{\partial^2 \delta \left(\pi \left(p - \frac{m}{k}x - \frac{n}{l}y \right) \right)}{\partial x^2} = - \left(\frac{m\pi}{k} \right)^2 e^{-(ip\pi - \frac{im\pi}{k}x - \frac{in\pi}{l}y)}.$$

Thus

$$\frac{\partial^2 \delta \left(\pi \left(p - \frac{m}{k}x - \frac{n}{l}y \right) \right)}{\partial x^2} = - \left(\frac{m\pi}{k} \right)^2 \delta \left(\pi \left(p - \frac{m}{k}x - \frac{n}{l}y \right) \right). \quad (3.15)$$

Considering (3.12), (3.13) and (3.14), (3.9) can be written as

$$R \left\{ \frac{\partial^2 f}{\partial x^2} \right\} (H(V_{m,n}, p)) = \frac{1}{8lk} \left\{ \begin{aligned} & 2 \frac{l}{n\pi} f' \left(k, \frac{l}{n\pi} (p - m) \right) - 2 \frac{l}{n\pi} f' \left(-k, \frac{l(p+m)}{n\pi} \right) \\ & - \left(\frac{m\pi}{k} \right)^2 \int_{-l}^l \int_{-k}^k f(x, y) \delta(\pi(p - V_{m,n} \cdot (x, y))) dx dy \end{aligned} \right\}. \quad (3.16)$$

Using (3.5), (3.15) can be written as

$$R \left\{ \frac{\partial^2 f}{\partial x^2} \right\} (H(V_{m,n}, p)) = \frac{1}{8lk} \left\{ \begin{array}{l} 2 \frac{l}{n\pi} f' \left(k, \frac{l}{n\pi} (p-m) \right) \\ -2 \frac{l}{n\pi} f' \left(-k, \frac{l(p+m)}{n\pi} \right) \end{array} \right\} - \left(\frac{m\pi}{k} \right)^2 Rf(H(V_{m,n}, p)). \quad (3.17)$$

Similarly if $f(x, y) \in \mathbb{R}^2[-k, k] \times [-l, l]$, upon integrating by parts, we deduce the relation

$$R \left\{ \frac{\partial^2 f}{\partial y^2} \right\} (H(V_{m,n}, p)) = \frac{1}{8lk} \left\{ \begin{array}{l} 2 \frac{k}{m\pi} f' \left(\frac{k(p-n)}{m}, l \right) \\ -2 \frac{k}{m\pi} f' \left(\frac{k(p+n)}{m}, -l \right) \end{array} \right\} - \left(\frac{n\pi}{l} \right)^2 Rf(H(V_{m,n}, p)). \quad (3.18)$$

- (2) If $f(x, y)$ at boundary is zero as $f^r(-k, y) = f^r(k, y) = 0, -l < y < l$ and $f^r(x, -l) = f^r(x, l) = 0, -k < x < k$, then from (3.17) and (3.18) the derivative is given by

$$R \left\{ \frac{\partial^{2r} f}{\partial x^{2r}} \right\} (H(V_{m,n}, p)) = (-1)^r \left(\frac{m\pi}{k} \right)^{2r} Rf(H(V_{m,n}, p)) \quad (3.19)$$

and

$$R \left\{ \frac{\partial^{2r} f}{\partial y^{2r}} \right\} (H(V_{m,n}, p)) = (-1)^r \left(\frac{n\pi}{l} \right)^{2r} Rf(H(V_{m,n}, p)) \quad (3.20)$$

respectively, for r being a positive integer.

- (3) If $f(x, y)$ at boundary is zero as $f(-k, y) = f(k, y) = 0, -l < y < l$ and $f(x, -l) = f(x, l) = 0, -k < x < k$.

$$R \left\{ \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} \right\} (H(V_{m,n}, p)) = -\pi^2 \left[\left(\frac{m}{k} \right)^2 + \left(\frac{n}{l} \right)^2 \right] Rf(H(V_{m,n}, p)). \quad (3.21)$$

And also

$$R \left\{ \frac{\partial^{2r} f}{\partial x^{2r}} + \frac{\partial^{2r} f}{\partial y^{2r}} \right\} (H(V_{m,n}, p)) = (-1)^r \left[\left(\frac{m\pi}{k} \right)^{2r} + \left(\frac{n\pi}{l} \right)^{2r} \right] Rf(H(V_{m,n}, p)). \quad (3.22)$$

4 The testing function space A and A^* and their duals

In this, we employ the same notation and terminology as those used in (Zemanian, 1968). Thus I denote the interval $[-k, k] \times [-l, l]$. $L_2(I)$ and $L_2^*(I)$ represent the space of equivalence class of functions that are quadratically integrable on I .

A mixed inner product is define on $L_2(I) \times L_2^*(I)$ for $f \in L_2(I), g \in L_2^*(I)$ by (Al-Gwaiz, 2008) as follows,

$$\langle f(X), g(X) \rangle = \int_I f(X) \overline{g(X)} dX \quad (4.1)$$

where $(x, y) = X, \overline{g(X)}$ denotes the complex conjugate of $g(X)$.

Definition (4.1) is consistent with the inner product on $L_2(I)$ and $L_2^*(I)$. And note that $f(X)$ and $g(X)$ both are in $L_2(I)$ and $L_2^*(I)$. The symbol $D(I)$ will denote the space of infinitely differential function on $I = [-k, k] \times [-l, l]$, which has a compact support on I .

The topology of $D(I)$ and it's dual $D'(I)$ is considered from Schwartz's distribution. $E(I)$ will denote the space of distribution with compact support.

Remark 4.1. The adjoint operator of Δ_x is Δ_x^* (if it exists). If $\Delta_x = \Delta_x^*$, then Δ_x is said to be self-adjoint operator (Al-Gwaiz, 2008, p. 55).

Theorem 4.1. (2.9) gives the eigenfunctions and λ_m are eigenvalues (positive roots) of (2.12). Using (2.3), we deduce

$$\langle \Delta_x X_m, u \rangle = \langle X_m, \Delta_x^* u \rangle \quad (4.2)$$

where u is the dummy function having every characteristic of X_m .

Proof. L.h.s of (4.2) gives

$$\langle \Delta_x X_m, u \rangle = \int_{-k}^k u \frac{d^2 X_m}{dx^2} dx. \quad (4.3)$$

On integrating by parts, we get

$$\langle \Delta_x X_m, u \rangle = J(u, X_m) + \int_{-k}^k \Delta_x^* u X_m dx \quad (4.4)$$

where $J(u, X_m) = \left[u \frac{dX_m}{dx} \right]_{-k}^k - \left[\frac{du}{dx} X_m \right]_{-k}^k$ is called Bi-linear concomitant.

$$\begin{aligned} J(u, X_m) &= \left[u \frac{dX_m}{dx} \right]_{-k}^k - \left[\frac{du}{dx} X_m \right]_{-k}^k \\ &= u(k) \frac{dX_m(k)}{dx} - u(-k) \frac{dX_m(-k)}{dx} - \frac{du(k)}{dx} X_m(k) + \frac{du(-k)}{dx} X_m(-k). \end{aligned}$$

Applying boundary conditions (2.5), follows

$$J(u, X_m) = \frac{dX_m(k)}{dx} [u(k) - u(-k)] - X_m(k) [u'(k) - u'(-k)]. \quad (4.5)$$

Bi-linear concomitant vanishes in (4.5), only if

$$(a) \quad u(k) - u(-k) = 0$$

$$(b) \quad u'(k) - u'(-k) = 0.$$

Therefore (4.5) becomes

$$J(u, X_m) = 0. \quad (4.6)$$

And (4.4) becomes

$$\langle \Delta_x X_m, u \rangle = \int_{-k}^k \Delta_x^* X_m dx$$

which can be written as

$$\langle \Delta_x X_m, u \rangle = \langle X_m, \Delta_x^* u \rangle.$$

This concludes the proof. Thus (2.9) gives the eigenfunctions and λ_m are eigenvalues (which are the positive roots of (2.12)). $\square$

Similarly from (2.4), it follows:

$$\langle \Delta_y Y_n, v \rangle = \langle Y_n, \Delta_y^* v \rangle \quad (4.7)$$

for $\Delta_y^* = \Delta_y$, where v is the dummy function having each characteristic of Y_n , which means adjoint of Δ_y exists and is also a self-adjoint operator.

Hence (2.10) gives the eigenfunctions of Δ_y^* . The positive roots of (2.4) are given by the eigenvalues λ_n .

Since Δ_x and Δ_y are self-adjoint operators, so $\Delta_{x,y,\theta}$ is also a self-adjoint operator, whose eigenfunction from (3.1) is given by

$$\phi_{m,n}(X) = \sum_{m,n=1}^{\infty} c_{m,n} \delta \left(\pi \left(p - \frac{m}{k}x - \frac{n}{l}y \right) \right) \quad (4.8)$$

This is equivalent to say that $\{\phi_{m,n}(X)\}_{m,n=1}^{\infty}$ is orthogonal function of differential operator using (2.14) as

$$\Delta_{x,y,\theta} \phi_{m,n}(X) = 0. \quad (4.9)$$

A is defined as a testing function space of all infinitely differentiable complex-valued functions $\phi(X)$ on I (Zemanian, 1968, p. 252) such that

$$(i) \quad \xi_{\gamma} \phi(X) = \left[\int_I \left| \Delta_{x,y,\theta}^{\gamma} \phi(X) \right|^2 dX \right]^{\frac{1}{2}} \text{ exists for every } \gamma = 0, 1, 2, \dots$$

(ii) For each m, n and γ

$$\begin{aligned} \langle \Delta_{x,y,\theta}^{\gamma} \phi, \phi_{m,n} \rangle &= \int_I \Delta_{x,y,\theta}^{\gamma} \phi(X) \phi_{m,n} dX \\ &= \int_I \phi(X) \Delta_{x,y,\theta}^{\gamma} \phi_{m,n} dX \\ \langle \Delta_{x,y,\theta}^{\gamma} \phi, \phi_{m,n} \rangle &= \langle \phi, \Delta_{x,y,\theta}^{\gamma} \phi_{m,n} \rangle. \end{aligned} \quad (4.10)$$

A is the countable multinormed space having the topology generated by $\{\xi\}$. A is also complete. Consequently A is a Fréchet space.

In our context, we can establish a result analogous to (Zemanian, 1968).

Theorem 4.2. Every member $\phi \in A$ can be expanded into a series of the form

$$\phi = \sum_{m,n=1}^{\infty} (\phi, \phi_{m,n}) \phi_{m,n} \quad (4.11)$$

which converges in A .

Proof. Note that $\Delta_{x,y,\theta}^{\gamma} \phi \in L_2(I)$. Hence by (4.10) and (4.11), we have

$$\Delta_{x,y,\theta}^{\gamma} \phi = \sum_{m,n=1}^{\infty} \langle \Delta_{x,y,\theta}^{\gamma} \phi, \phi_{m,n} \rangle \phi_{m,n}$$

$$\begin{aligned}
 &= \sum_{m,n=1}^{\infty} \langle \phi, \Delta_{x,y,\theta}^{\gamma} \phi_{m,n} \rangle \phi_{m,n} \\
 &= \sum_{m,n=1}^{\infty} \langle \phi, \phi_{m,n} \rangle \Delta_{x,y,\theta}^{\gamma} \phi_{m,n}.
 \end{aligned}$$

where the series involved converge in $L_2(I)$.

Therefore

$$\xi_{\gamma} \left[\phi - \sum_{m,n=1}^{\infty} \langle \phi, \phi_{m,n} \rangle \phi_{m,n} \right] \rightarrow 0$$

as $m, n \rightarrow \infty$. This complete the proof of theorem 4.1.

A' is dual space of A and is also complete. □

We now list some of the properties of these spaces:

- (a) $D(I) \subset A \subset E(I)$. $E'(I)$ is a space of A' .
- (b) It can be seen that $\phi_{m,n}$, given by (4.8) belongs to A .
- (c) The operation $\phi \rightarrow \Delta_{x,y,\theta}^{\gamma} \phi$ is a continuous linear mapping of A into itself.

Consequently, the operation $f \rightarrow \Delta_{x,y,\theta} f$ defined on A' by

$$\langle \Delta_{x,y,\theta} f, \phi \rangle = \langle f, \Delta_{x,y,\theta} \phi \rangle \quad (4.12)$$

is also a continuous linear mapping of A' into itself.

It is important to note that operator $\Delta_{x,y,\theta}$ is self-adjoint on A . Therefore A is equivalent to A^* and A' is equivalent to A'^* by (Zemanian, 1968, p. 255).

Remark 4.2. Since the $\{\phi\}$ is an orthogonal system on I , verifying the orthogonality condition (2.14), we propose to consider finite continuous Radon transform

$$Rf(H(V_{m,n}, p)) = \frac{1}{8kl} \int_{-l}^l \int_{-k}^k f(x, y) \delta(\pi(p - V_{m,n} \cdot (x, y))) dx dy. \quad (4.13)$$

The inversion formula given by

$$f(x, y) = \sum_{m,n=1}^{\infty} Rf(H(V_{m,n}, p)) \delta(\pi(p - V_{m,n} \cdot (x, y))). \quad (4.14)$$

Remark 4.3. A may be identified with a subspace of A' that is $A \subset A'$.

Indeed, every member $f \in A$ generates a regular distribution in A' represented by

$$\langle f, \phi \rangle = \int_I f(X) \phi(X) dX, \quad \phi \in A,$$

since $|(f, \phi)| \leq \xi_0(\phi) \xi_0(\phi)$.

Furthermore, two members of A which give rise to the same member of A' must be identical.

In similar manner, A can be considered as a subspace of A' .

5 Inversion formula

The main result of this section can be stated as follows:

Theorem 5.1. Every member $f \in A'$, then

$$f = \sum_{m,n=1}^{\infty} \langle f, \phi_{m,n} \rangle \phi_{m,n} \quad (5.1)$$

where the series converges in A' .

Proof. By virtue of theorem 4.1, it is inferred that

$$\begin{aligned} \langle f, \phi \rangle &= \left\langle f, \sum_{m,n=1}^{\infty} (\phi, \phi_{m,n}) \phi_{m,n} \right\rangle \\ &= \sum_{m,n=1}^{\infty} \langle f, \phi_{m,n} \rangle \overline{\langle \phi, \phi_{m,n} \rangle} \\ &= \sum_{m,n=1}^{\infty} \langle f, \phi_{m,n} \rangle \langle \phi, \phi_{m,n} \rangle \end{aligned}$$

$\forall \phi \in A$. This implies (5.1) converges in A' .

In the view of theorem 5.1, the distributional finite continuous Radon transform of $f \in A'$ is defined by

$$R'f(H(V_{m,n}, p)) = \frac{1}{8kl} \int_{-l}^l \int_{-k}^k f(x, y) \delta(\pi(p - V_{m,n} \cdot (x, y))) dx dy. \quad (5.2)$$

Its corresponding inversion formula is supplied by theorem 5.1 and can be expressed as

$$R'^{-1}(R'f(H(V_{m,n}, p))) = f(x, y) = \sum_{m,n=1}^{\infty} Rf(H(V_{m,n}, p)) \delta(\pi(p - V_{m,n} \cdot (x, y))). \quad (5.3)$$

We invoke (4.11) to get

$$\begin{aligned} &\frac{1}{8lk} \int_{-l}^l \int_{-k}^k \left\{ \Delta_{x,y,\theta}^{\gamma} f(x, y) \right\} \delta(\pi(p - V_{m,n} \cdot (x, y))) dx dy \\ &= (-\lambda^2)^{\gamma} [\sin^2 \theta \cos^2 \gamma \theta - \cos^2 \theta \sin^2 \gamma \theta] \frac{1}{8lk} \int_{-l}^l \int_{-k}^k f(x, y) \delta(\pi(p - V_{m,n} \cdot (x, y))) dx dy \end{aligned}$$

for all $f \in A'$ and $\gamma = 0, 1, 2, \dots$.

$$\begin{aligned} &\frac{1}{8lk} \int_{-l}^l \int_{-k}^k \left\{ [(\sin^2 \theta) \Delta_x^{\gamma} - (\cos^2 \theta) \Delta_y^{\gamma}] f(x, y) \right\} \delta(\pi(p - V_{m,n} \cdot (x, y))) dx dy \\ &= (-\lambda^2)^{\gamma} [\sin^2 \theta \cos^2 \gamma \theta - \cos^2 \theta \sin^2 \gamma \theta] R'f(H(V_{m,n}, p)). \end{aligned} \quad (5.4)$$

□

Theorem 5.2 (The Uniqueness Theorem). *Let $f, g \in A'$ and let finite continuous Radon transform of f and g be $Rf(H(V_{m,n}, p))$ and $Rg(H(V_{m,n}, p))$ respectively, as defined by (4.13). If $Rf(H(V_{m,n}, p)) = Rg(H(V_{m,n}, p))$, then $f = g$ in the sense of equality in $D'(I)$. By (4.14)*

$$f - g = \sum_{m,n=1}^{\infty} [Rf(H(V_{m,n}, p)) - Rg(H(V_{m,n}, p))] \delta(\pi(p - V_{m,n} \cdot (x, y)))$$

Thus $f - g = 0$ as $Rf(H(V_{m,n}, p)) = Rg(H(V_{m,n}, p))$.

Hence $f = g$ in the sense of equality in $D'(I)$.

6 Applications using finite continuous Radon transforms to solve boundary value problems

The solutions of Poisson's equation and wave equation has been illustrated in this section, using finite continuous Radon transforms with boundary value problems.

Example 6.1. Find the solution of Poisson's equation, using finite continuous Radon transforms using boundary conditions

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + f(x, y) = 0, \quad (6.1)$$

$$-a < x < a, \quad -b < y < b.$$

Solution 6.1. Applying finite continuous Radon transforms on both sides of (6.1), we get

$$R \left[\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right] (H(V_{m,n}, p)) + Rf(H(V_{m,n}, p)) = 0. \quad (6.2)$$

Then from (3.19); (6.2) is given by

$$-\pi^2 \left[\left(\frac{m}{k} \right)^2 + \left(\frac{n}{l} \right)^2 \right] Ru(H(V_{m,n}, p)) + Rf(H(V_{m,n}, p)) = 0 \quad (6.3)$$

Applying inversion formula (4.14) on both the sides of (6.3), it is obtained as

$$u(x, y, t) = R^{-1} \left[\frac{1}{\pi^2 \left[\left(\frac{m}{k} \right)^2 + \left(\frac{n}{l} \right)^2 \right]} Rf(H(V_{m,n}, p)) \right]. \quad (6.4)$$

We may now invoke the inversion formula (4.14), which provides the required results:

$$u(x, y, t) = \frac{1}{\pi^2} \sum_{m,n=1}^{\infty} \left[\frac{1}{\left[\left(\frac{m}{k} \right)^2 + \left(\frac{n}{l} \right)^2 \right]} Rf(H(V_{m,n}, p)) \right] \delta(\pi(p - V_{m,n} \cdot (x, y))). \quad (6.5)$$

Example 6.2. Consider two-dimensional wave equation

$$\frac{\partial^2 u}{\partial t^2} = c^2 \left[\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right], \quad (6.6)$$

with initial conditions

$$u(x, y, 0) = f(x, y) \text{ and } u_t(x, y, 0) = 0. \quad (6.7)$$

Solution 6.2. Taking finite continuous Radon transform on both sides of (6.6) and using (6.7)

$$R \left[\frac{\partial^2 u}{\partial t^2} \right] (H(V_{m,n}, p)) = c^2 R \left[\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right] (H(V_{m,n}, p)).$$

From (3.19), we get

$$R \left[\frac{\partial^2 u}{\partial t^2} \right] (H(V_{m,n}, p)) = -c^2 \pi^2 \left[\left(\frac{m}{k} \right)^2 + \left(\frac{n}{l} \right)^2 \right] Ru(H(V_{m,n}, p)). \quad (6.8)$$

$$R \left[\frac{\partial^2 u}{\partial t^2} \right] (H(V_{m,n}, p)) = -\psi^2 Ru(H(V_{m,n}, p)) \quad (6.9)$$

where $\psi^2 = -c^2 \pi^2 \left[\left(\frac{m}{k} \right)^2 + \left(\frac{n}{l} \right)^2 \right]$.

Similarly (6.6) gives

$$Ru(x, y, 0)(H(V_{m,n}, p)) = Rf(H(V_{m,n}, p)); \quad Ru_t(x, y, 0)(H(V_{m,n}, p)) = 0. \quad (6.10)$$

Solving (6.8) with initial conditions (6.9) and applying inversion formula (4.14) gives

$$u(x, y, t) = \sum_{m,n=1}^{\infty} Rf(H(V_{m,n}, p)) \cos(\psi t) \delta(\pi(p - V_{m,n} \cdot (x, y))). \quad (6.11)$$

Conclusion

The finite continuous Radon transforms and its inversion formula is studied in this paper. Using Sturm-Liouville theory the operational calculus of finite continuous Radon transforms is analyzed. The solutions of Poisson's equation and two-dimensional wave equation has been illustrated in the concluding section using boundary conditions.

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