

Several soliton, solitary waves and their interactions for variable time dependent coefficient Burger's equation

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ABSTRACT

We implement the explicit soliton solutions of the Burger's model of time dependent coefficient. Created by similarity transformation besides the generalized tanh function technique, we report several kinds of analytical solutions of the desired model. The essential and important feature of this study is the formulation of different sequences of explicit solutions of the Burger's equation generated by the known solution of the model using the Backlund transformations. Particularly, we investigate the emergent soliton and solitary constructions by a distinctive choice of the self-similar variable. Our consequences display the forms of both solitons and solitary waves can be usefully controlled by the detailed formula of the self-similar variable. The interaction of two and three soliton are discussed. We discuss how to construct another solution to some famous mathematical physical models such as KdVBurger's, linear heat, coupled Burger's and symmetric coupled Burger's models from the known solution of Burger's model.

Keywords: Exact solutions, Backlund transformations, Burger's equation. time dependent coefficient, self-similar transformation..

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1. INTRODUCTION

Nonlinear dynamical models in mathematical physics are widely applied to the description of complex phenomena in various sciences, examples include, plasma physics, fluid mechanics, space science, astronomy, optics and quantum mechanics. In order to understand the physical mechanism of the phenomena in those fields, one has to study the solutions and their properties of the associated equations. The explicit solution of these differential models plays a vital significant and fundamental role in description of natural phenomena. knowing the explicit and analytical solutions of these differential models helps us in facilitating the confirmation of numerical treatments and also contribute the stability studies of the results. Exploring explicit and numerical solutions to these nonlinear models has long been paid much attention by many researchers in mathematical and physical fields. In particular, great growth has been done in finding explicit solutions to these differential models in mathematical physics. several powerful techniques have been introduced, examples include the

inverse scattering Ablowitz, et al 1973, Hirota technique Tain and Zhang 2010, Backlund transformation (BT) Miura 1979, Darboux transformation Matveev and Salle 1991, hyperbolic tangent function procedure Malfliet 1992, sine-cosine procedure Yang. 1994, the nonlinear transformation technique Kudryashov 2020, procedure by Liu et al 2001, truncated Painleve' technique Wiess et al 1983, homogenous balance method Wang et al 1996 and other methods Nouh and Abdel-Salam 2018, Wazwaz 2012, Wazwaz 2018, Abdel-Salam, and Nouh 2020, Abdel-Salam et al 2022, Azzam et al 2021 and Abdel-Salam 2010. BT are essential and important tool for finding of several states of nonlinear integrable models. It is a scheme of equations concerning the solution of the desired model to solution of the similar model or to a solution of another model. To date, much research has been dedicated to searching for BT, for example Painleve' property, Ablowitz-Kaup-Newell-Segur (AKNS) system, nonclassical symmetries, homogenous balance technique.

We assume the desired generalized Burger's model with time-dependent coefficient, that represented the dynamical of the model,

$$U_t - \beta(t)U_{xx} + \alpha(t)U U_x = 0, \quad (1)$$

with $\alpha(t) = a\beta(t)$, where $U = U(x, t)$ is a real function, t is time, and x is space coordinate.

$\alpha(t)$, $\beta(t)$ are time dependent variable coefficients. When $\alpha(t) = \beta(t) = 1$, Eq. (1) is simplified as the standard Burger's equation. Suppose that

$$U(x, t) = u(x, T), \quad (2)$$

with $T = T(t)$ real functions of the time t . Considering that $\beta(t) = \frac{\partial T}{\partial t}$ substitute this transformation

(2) into Eq. (1), thus Eq. (1) take the form

$$u_T + \alpha u u_x - u_{xx} = 0, \quad (3)$$

The Burger's equation (3), where α is an arbitrary constant, is an important nonlinear wave equation and a variety of nonlinear cases can be elucidated by this equation. The Burger's model is an significant diffusion model in physics, that designate the far fields propagation of waves in the corresponding dissipative system. It is a nonlinear differential model which arises in models of traffic and fluid flow. This model owns a essential quadratic nonlinearity and is regarded as an suitable beginning ensemble for discussing turbulence. It contributes not only to the exploration and validation of numerical treatments but it is also used as a basic way for studying physical phenomena such as traffic flow, shock waves, acoustic transmission, waves under the influence of diffusion, supersonic flow around airfoils, the dynamics in a liquid and wave dissemination in a thermo flexible intermediate

This research leaf is prepared in the form: in section 2, application of generalized tanh function technique to the Burger's model and get several kinds of explicit solutions. In section 3, by means of the Painleve' property, we can write the BT of the Burger's model and use the obtained solutions given in section 2 to generate a new several sequences of analytical solutions of the Burger's model. The interaction of two and three soliton are discussed analytically and graphically in section 4. In

section5, We discuss how to construct another solution to some famous mathematical physical models such as KdVBurger's, linear heat, coupled Burger's and symmetric coupled Burger's models from the known solution of Burger's model. Lastly, we conclude the manuscript and give some features and explanations.

2. ANALYTICAL SOLUTIONS FOR THE BURGER'S MODEL

The tanh function technique is stronger solution technique for the calculation of analytical traveling wave solution [5]. Different modification formulae's of the tanh function technique have been established. Firstly a power series of the tanh function was applied to get explicit solutions of abundant nonlinear differential models. Overcoming complication, Malfliet was employed the tanh function procedure by using the function tanh as original variable, where all differentials of the these function are characterized by the function tanh itself. A straight forward application which can be performed so that the procedure is adequate to a great number of models. In the next, we contribute a transitory explanation of the generalized tanh function technique and use it to obtain various kinds of explicit solution of the Burger's model.

We now demonstrate the generalized tanh function expansion technique. Suppose the nonlinear differential model

$$H(u, u_x, u_T, u_{Tx}, u_{xx}, u_{TT}, \dots) = 0. \quad (4)$$

Generalized tanh function expansion technique which solving equation (2) describes in the coming steps:

Step 1: we pursue its travelling wave solution of model (2) as the formulae

$$u(T, x) = u(\xi), \quad \xi = \varphi + k(x - \omega T), \quad (5)$$

With φ , k and ω are phase shift, wave number and wave velocity respectively. Substituting (5) into (4) gives ordinary differential equation (ODE)

$$\tilde{H}(u, u', u'', \dots) = 0, \quad u' = \frac{du}{d\xi}, \dots \text{etc.} \quad (6)$$

Step 2: assuming that $u(\xi)$ can be formulated by a finite power chain of $F(\xi)$

$$u(\xi) = a_0 + \sum_{j=1}^s a_j F^j(\xi), \quad a_s \neq 0, \quad (7)$$

since s is the maximum power of the chain, it can be calculated by equating maximum orders of the linear term with the nonlinear term in equation (6) and a_j are some constants to be computed. The main idea of the generalized tanh function expansion technique is to take the Riccati equation which tanh function satisfies and to use its solutions to get new explicit solution of the Burger's model. The desired Riccati equation

$$F' = A + B F + C F^2, \quad ' \equiv \frac{d}{d\xi}, \quad (8)$$

with A, B, C are constants.

Step 3: Substituting generalized tanh function expansion method (7) into ODE (6), therefore the LHS of equation (6) can be transformed into an equality in the power of F . putting every coefficient of F^i to zero gives algebraic equations in $a_0, a_1, \dots, a_s, \omega$ and k .

Step 4: by solving this algebraic system given in step 3, then $a_0, a_1, \dots, a_s, \omega$ and k can be obtained by A, B, C . Substitute the computed results into the generalized tanh function expansion (7), then a general formulae of travelling wave solution of equation (4) can be calculated.

Select properly A, B, C in ODE (6) so the equivalent solution $F(\xi)$ is one of hyperbolic or triangular functions of equation (8) given bellow.

Case 1: $F(\xi) = \tan \xi$ when $B = 0$ and $A = C = 1$.

Case 2: $F(\xi) = \cot \xi$ when $B = 0$ and $A = C = -1$.

Case 3: $F(\xi) = \tanh \xi$ or $F(\xi) = \coth \xi$ when $B = 0, A = 1$, and $C = -1$.

Case 4: $F(\xi) = \tanh \xi \pm i \operatorname{sech} \xi$ or $F(\xi) = \coth \xi \pm \operatorname{csch} \xi$ or $F(\xi) = \frac{\tanh \xi}{1 \pm \operatorname{sech} \xi}$ or

$F(\xi) = \frac{\coth \xi}{1 \pm i \operatorname{csch} \xi}, i^2 = -1$, when $B = 0, A = \frac{1}{2}$, and $C = -\frac{1}{2}$.

Case 5: $F(\xi) = \tan \xi \pm \sec \xi$ or $F(\xi) = \csc \xi - \cot \xi$ or $F(\xi) = \frac{\tan \xi}{1 \pm \sec \xi}$ when $A = C = \frac{1}{2}$ and $B = 0$.

Case 6: $F(\xi) = \cot \xi \pm \csc \xi$ or $F(\xi) = \sec \xi - \tan \xi$ or $F(\xi) = \frac{\cot \xi}{1 \pm \csc \xi}$ when $A = C = \frac{-1}{2}$ and $B = 0$.

Case 7: $F(\xi) = \frac{\tanh \xi}{1 + \tanh^2 \xi}$ when $A = 1, C = -4$ and $B = 0$.

Case 8: $F(\xi) = \frac{\tan \xi}{1 - \tan^2 \xi}$ If $A = 1, C = 4$ and $B = 0$.

Case 9: $F(\xi) = \frac{\cot \xi}{1 - \cot^2 \xi}$ If $A = -1, C = -4$ and $B = 0$.

Case 10: $F(\xi) = \frac{\tan \xi}{1 + \tan \xi}$ where $A = 1, B = -2$ and $C = 2$.

Case 11: $F(\xi) = \frac{\tan \xi}{1 - \tan \xi}$ where $A = 1, B = 2$ and $C = 2$.

Case 12: $F(\xi) = \frac{\cot \xi}{1 + \cot \xi}$ where $A = -1, B = 2$ and $C = -2$.

Case 13: $F(\xi) = \frac{\cot \xi}{1 - \cot \xi}$ where $A = -1, B = -2$ and $C = -2$.

Case 14: $F(\xi) = \frac{-1}{C\xi + C_0}$ where $A=B=0$ and $C \neq 0$.

Case 15: $F(\xi) = \frac{1}{B}(\exp(B\xi) - A)$ where $C=0$ and $B \neq 0$.

Currently, we smear the above generalized technique to the Burger's model (3), substituting (5) into equation (3) yields

$$-\omega u' + \alpha u u' - k u'' = 0, \quad (9)$$

by equating the degrees of u'' with uu' we have $s = 1$. thus, solution of equation (3) take the form

$$u = a_0 + a_1 F, \quad (10)$$

putting (10) into (9) and using (8), we have an algebraic equations of F^i . Putting the coefficients of $F^i (i = 0, 1, 2, 3)$ in the obtained equation to zero. Solving this set of equations, we have from (9)

$$a_0 = \frac{1}{\alpha}(\omega + Bk), \quad a_1 = \frac{2Ck}{\alpha}, \quad \alpha \neq 0, \quad (11)$$

with ω and k are optimal constants. Putting (11) into (10) and using special solutions of equation (8), we have soliton solution and periodic triangular solutions of the Burger's model:

$$u_1 = \frac{1}{\alpha}(\omega + 2k \tan \xi), \quad (12)$$

$$u_2 = \frac{1}{\alpha}(\omega - 2k \cot \xi), \quad (13)$$

$$u_3 = \frac{1}{\alpha}(\omega - 2k \tanh \xi), \quad (14)$$

$$u_4 = \frac{1}{\alpha}(\omega - 2k \coth \xi), \quad (15)$$

$$u_5 = \frac{1}{\alpha}(\omega - k(\tanh \xi \pm i \operatorname{sech} \xi)), \quad (16)$$

$$u_6 = \frac{1}{\alpha}(\omega - k(\coth \xi \pm \operatorname{csch} \xi)), \quad (17)$$

$$u_7 = \frac{1}{\alpha}(\omega + k(\tan \xi \pm \sec \xi)), \quad (18)$$

$$u_8 = \frac{1}{\alpha}(\omega - k(\cot \xi \pm \csc \xi)), \quad (19)$$

$$u_9 = \frac{1}{\alpha}\left(\omega - \frac{8k \tanh \xi}{1 + \tanh^2 \xi}\right), \quad (20)$$

$$u_{10} = \frac{1}{\alpha}\left(\omega + \frac{8k \tan \xi}{1 - \tan^2 \xi}\right), \quad (21)$$

$$u_{11} = \frac{1}{\alpha} \left(\omega - \frac{8k \cot \xi}{1 - \cot^2 \xi} \right), \quad (22)$$

$$u_{12} = \frac{1}{\alpha} \left(\omega - \frac{k \tanh \xi}{1 + \operatorname{sech} \xi} \right), \quad (23)$$

$$u_{13} = \frac{1}{\alpha} \left(\omega - \frac{k \coth \xi}{1 + i \operatorname{csch} \xi} \right), \quad (24)$$

$$u_{14} = \frac{1}{\alpha} \left(\omega + \frac{k \tan \xi}{1 + \sec \xi} \right), \quad (25)$$

$$u_{15} = \frac{1}{\alpha} \left(\omega - \frac{k \cot \xi}{1 + \csc \xi} \right), \quad (26)$$

$$u_{16} = \frac{1}{\alpha} \left(\omega - 2\sigma k + \frac{4k \tan \xi}{1 + \tan \xi} \right), \quad (27)$$

$$u_{17} = \frac{1}{\alpha} \left(\omega + 2k - \frac{4k \cot \xi}{1 + \cot \xi} \right), \quad (28)$$

Also, we can get two new explicit solutions by means of the tanh-coth and tan-cot methods, in the form

$$u_{18} = \frac{1}{\alpha} \left(\omega - 2k (\tanh \xi + \coth \xi) \right), \quad (29)$$

$$u_{19} = \frac{1}{\alpha} \left(\omega + 2k (\tan \xi - \cot \xi) \right), \quad (30)$$

with $\xi = k(x - \omega T) + \varphi$ and $T = \int_0^t \beta(\tau) d\tau$.

In the following section we give the BT of Burger's equation by means of the Painleve' property. Moreover, we use the obtained solutions (10)-(28) to generate new abundant explicit solutions via BT.

3. BT AND ABUNDANT EXACT SOLUTIONS

Weiss et al show that Burger's equation possesses the Painleve' property and it has a BT in the formulae

$$u = w - \frac{2\varphi_x}{\alpha\varphi}, \quad (31)$$

with $\varphi = \varphi(x, T)$ is singular manifold variable, w is a function of x and T . Also, the function w solves the Burger's model (3) and the function φ satisfies the PDE

$$\varphi_T + \alpha w \varphi_x = \varphi_{xx}.$$

Now, if we take $w = \varphi$ then the function φ satisfies also the Burger's equation

$$\varphi_t + \alpha\varphi\varphi_x = \varphi_{xx},$$

thus the BT for the Burger's equation have this form

$$u_{n+1} = u_n - \frac{2}{\alpha u_n} \frac{\partial u_n}{\partial x}. \quad (32)$$

We refer to BT's of integrable models. Their main strength lies in which we can use it to obtain further solutions of the desired models. The u_{n+1} amounts designate to new solution and u_n amounts designate to old solution. We take advantage of this that, based on the known solution to Burger's model, we are able to obtain new solution of Burger's model. To construct new solutions to the Burger's equation one may begin with the solution u_1 given in (12) and using equation (32) we get the following new solution :

$$u_{20} = \frac{-4k^2 + \omega^2 + 4k\omega \tan \xi}{\alpha(\omega + 2k \tan \xi)}, \quad (33)$$

Inserting u_{20} into the BT equation (32) we have

$$u_{21} = \frac{12k^2\omega - \omega^3 + 2k(4k^2 - 3\omega^2) \tan \xi}{\alpha(4k^2 - \omega^2 - 4k\omega \tan \xi)}, \quad (34)$$

furthermore, using u_{21} and BT equation (32) we get

$$u_{22} = \frac{-16k^4 + 24k^2\omega^2 - \omega^4 + (32k^3\omega - 8\omega^3k) \tan \xi}{\alpha(12k^2\omega - \omega^3 + 2k(4k^2 - 3\omega^2) \tan \xi)}, \quad (35)$$

and so on, we can obtain a several analytical solutions constructed based on the known tan-function solution (12) of the Burger's equation. We start from u_2 given in (13) and by using equation (32) we have

$$u_{23} = \frac{-4k^2 + \omega^2 - 4k\omega \cot \xi}{\alpha(\omega - 2k \cot \xi)}, \quad (36)$$

Inserting u_{23} into the BT equation (32) we have

$$u_{24} = \frac{-12k^2\omega + \omega^3 - 2k(4k^2 - 3\omega^2) \cot \xi}{\alpha(-4k^2 + \omega^2 + 4k\omega \cot \xi)}, \quad (37)$$

furthermore, using u_{24} and BT equation (32) we get

$$u_{25} = \frac{16k^4 - 24k^2\omega^2 + \omega^4 + (32k^3\omega - 8\omega^3k) \cot \xi}{\alpha(-12k^2\omega + \omega^3 + 2k(4k^2 - 3\omega^2) \cot \xi)}, \quad (38)$$

and so on, we can obtain a several analytical solutions constructed based on the known cot-function solution (13) of the Burger's equation. Based on u_3 given in (14) and using equation (32) we get

$$u_{26} = \frac{-4k^2 - \omega^2 + 4k\omega \tanh \xi}{\alpha(-\omega + 2k \tanh \xi)}, \quad (39)$$

Inserting u_{26} into the BT equation (32) we have

$$u_{27} = \frac{12k^2\omega + \omega^3 - 2k(4k^2 + 3\omega^2) \tanh \xi}{\alpha(4k^2 + \omega^2 - 4k\omega \tanh \xi)}, \quad (40)$$

furthermore, using u_{27} and BT equation (32) we get

$$u_{28} = \frac{16k^4 + 24k^2\omega^2 + \omega^4 - (32k^3\omega + 8\omega^3k) \tanh \xi}{\alpha(12k^2\omega + \omega^3 - 2k(4k^2 + 3\omega^2) \tanh \xi)}, \quad (41)$$

and so on, we can obtain a several analytical solutions constructed based on the known tanh-function solution (14) of the Burger's equation. Based on u_4 given in (15) and using equation (32) we get

$$u_{29} = \frac{-4k^2 - \omega^2 + 4k\omega \coth \xi}{\alpha(-\omega + 2k \coth \xi)}, \quad (42)$$

Inserting u_{29} into the BT equation (32) we have

$$u_{30} = \frac{12k^2\omega + \omega^3 - 2k(4k^2 + 3\omega^2) \coth \xi}{\alpha(4k^2 + \omega^2 - 4k\omega \coth \xi)}, \quad (43)$$

furthermore, using u_{30} and BT equation (32) we get

$$u_{31} = \frac{16k^4 + 24k^2\omega^2 + \omega^4 - (32k^3\omega + 8\omega^3k) \coth \xi}{\alpha(12k^2\omega + \omega^3 - 2k(4k^2 + 3\omega^2) \coth \xi)}, \quad (44)$$

and so on, we can obtain a several analytical solutions constructed based on the known coth-function solution (15) of the Burger's equation. Based on u_5 given in (16) and using equation (32) we get

$$u_{32} = \frac{k^2 + \omega^2 - 2k\omega(\tanh \xi \pm i \operatorname{sech} \xi)}{\alpha(\omega - k(\tanh \xi \pm i \operatorname{sech} \xi))}, \quad (45)$$

and so on. Starting from u_6 obtained in (17) and using equation (32) we get

$$u_{33} = \frac{k^2 + \omega^2 - 2k\omega(\coth \xi \pm \operatorname{csch} \xi)}{\alpha(\omega - k(\coth \xi \pm \operatorname{csch} \xi))}, \quad (46)$$

and so on. Starting from u_7 obtained in (18) and using equation (32) we get

$$u_{34} = \frac{k^2 - \omega^2 - 2k\omega(\tan \xi \pm \sec \xi)}{\alpha(-\omega - k(\tan \xi \pm \sec \xi))}, \quad (47)$$

and so on. Starting from u_8 obtained in (19) and using equation (32) we get

$$u_{35} = \frac{-k^2 + \omega^2 - 2k\omega(\cot \xi \pm \csc \xi)}{\alpha(\omega - k(\cot \xi \pm \csc \xi))}, \quad (48)$$

and so on. Starting from u_9 obtained in (20) and using equation (32) we get

$$u_{36} = \frac{(16k^2 + \omega^2) \tanh^2 \xi - 16k\omega \tanh \xi + 16k^2 + \omega^2}{\alpha(\omega \tanh^2 \xi - 8k \tanh \xi + \omega)}, \quad (49)$$

$$u_{37} = \frac{(48k^2\omega + \omega^3) \tanh^2 \xi - (24k\omega^2 + 128k^3) \tanh \xi + 48k^2\omega + \omega^3}{\alpha((16k^2 + \omega^2) \tanh^2 \xi - 16k\omega \tanh \xi + 16k^2 + \omega^2)}, \quad (50)$$

and so on. Starting from u_{10} obtained in (21) and using equation (32) we get

$$u_{38} = \frac{(16k^2 - \omega^2) \tan^2 \xi + 16k\omega \tan \xi - 16k^2 + \omega^2}{\alpha(\omega - \omega \tan^2 \xi + 8k \tan \xi)}, \quad (51)$$

$$u_{39} = \frac{(48k^2\omega - \omega^3) \tan^2 \xi + (24k\omega^2 - 128k^3) \tan \xi - 48k^2\omega + \omega^3}{\alpha((16k^2 - \omega^2) \tan^2 \xi + 16k\omega \tan \xi - 16k^2 + \omega^2)}, \quad (52)$$

and so on. Starting from u_{11} obtained in (22) and using equation (32) we get

$$u_{40} = \frac{(16k^2 - \omega^2) \cot^2 \xi - 16k \omega \cot \xi - 16k^2 + \omega^2}{\alpha(\omega - \omega \cot^2 \xi - 8k \cot \xi)}, \quad (53)$$

$$u_{41} = \frac{(\omega^3 - 48k^2 \omega) \cot^2 \xi + (24k \omega^2 - 128k^3) \cot \xi + 48k^2 \omega - \omega^3}{\alpha((\omega^2 - 16k^2) \cot^2 \xi + 16k \omega \cot \xi + 16k^2 - \omega^2)}, \quad (54)$$

and so on. Starting from u_{12} obtained in (23) and using equation (32) we get

$$u_{42} = \frac{k^2 + \omega^2 + (k^2 + \omega^2) \operatorname{sech} \xi - 2k \omega \tanh \xi}{\alpha(\omega + \omega \operatorname{sech} \xi - k \tanh \xi)}, \quad (55)$$

and so on. Starting from u_{13} obtained in (24) and using equation (32) we get

$$\begin{aligned} u_{43} = & -[(k^2 + \omega^2) \cosh^2 \xi - 2(k^2 + \omega^2) \\ & + 2i(k^2 + \omega^2) \sinh \xi - 2k \omega \cosh \xi \sinh \xi - 2ik \omega \cosh \xi] \\ & / [\alpha(k \cosh \xi - \omega \sinh \xi - i \omega)(i + \sinh \xi)], \end{aligned} \quad (56)$$

and so on. Starting from u_{14} obtained in (25) and using equation (32) we get

$$u_{44} = -\frac{(k^2 - \omega^2) \cos \xi + k^2 - \omega^2 - 2k \omega \sin \xi}{\alpha(\omega + \omega \cos \xi + k \sin \xi)}, \quad (57)$$

and so on. Starting from u_{15} obtained in (26) and using equation (32) we get

$$\begin{aligned} u_{45} = & -[(k^2 - \omega^2) \cos^2 \xi - 2(k^2 - \omega^2) \\ & - 2(k^2 - \omega^2) \sin \xi - 2k \omega \cos \xi \sin \xi - 2k \omega \cos \xi] \\ & / [\alpha(k \cos \xi - \omega \sin \xi - \omega)(1 + \sin \xi)], \end{aligned} \quad (58)$$

and so on. Starting from u_{16} obtained in (27) and using equation (32) we get

$$u_{46} = \frac{(\omega^2 - 4k^2 + 4k \omega) \tan \xi - 4k^2 - 4k \omega + \omega^2}{\alpha((\omega + 2k) \tan \xi + \omega - 2\sigma k)}, \quad (59)$$

$$u_{47} = \frac{\omega^3 + 8k^3 - 6k \omega^2 - 12k^2 \omega + (\omega^3 - 8k^3 + 6k \omega^2 - 12k^2 \omega) \tan \xi}{\alpha((\omega^2 - 4k^2 + 4k \omega) \tan \xi - 4k^2 - 4k \omega + \omega^2)}, \quad (60)$$

and so on. Starting from u_{17} obtained in (28) and using equation (32) we get

$$u_{48} = \frac{(\omega^2 - 4k^2 - 4k \omega) \cot \xi - 4k^2 + 4k \omega + \omega^2}{\alpha((\omega - 2k) \cot \xi + \omega + 2k)}, \quad (61)$$

$$u_{49} = \frac{\omega^3 - 8k^3 + 6k \omega^2 - 12k^2 \omega + (\omega^3 + 8k^3 - 6k \omega^2 - 12k^2 \omega) \cot \xi}{\alpha((\omega^2 - 4k^2 - 4k \omega) \cot \xi - 4k^2 + 4k \omega + \omega^2)}, \quad (62)$$

and so on. Starting from u_{18} obtained in (29) and using equation (32) we get

$$u_{50} = \frac{-(\omega^2 + 16k^2) \cosh \xi \sinh \xi + 8k \omega \cosh^2 \xi - 4k \omega}{\alpha(-\omega \cosh \xi \sinh \xi + 4k \cosh^2 \xi - 2k)}, \quad (63)$$

and so on, we can obtain a several analytical solutions constructed based on the addition of two functions tanh and coth-function solution of the Burger's model. Based on u_{19} obtained in (30) and using equation (32) we get

$$u_{51} = -\frac{(\omega^2 - 16k^2) \cos \xi \sin \xi - 8k \omega \cos^2 \xi + 4k \omega}{\alpha(-\omega \cos \xi \sin \xi + 4k \cos^2 \xi - 2k)}, \quad (64)$$

and so on, we can obtain a several analytical solutions constructed based on the addition of two functions tan and cot-function solution of Burger's model.

Through simple calculation of (3) we can see that

$$u_{52} = \frac{1}{\alpha} \left(\omega + \frac{2}{\omega T - \omega} \right), \quad x \neq \omega T, \quad (65)$$

is rational solution of Burger's model (3). Using u_{52} given in (65) and (32), we can get a new sequences of rational solutions of Burger's model generated by the rational solution (65) in the form

$$u_{53} = \frac{\omega(\omega^2 T + 4 - \omega x)}{\alpha(\omega^2 T + 2 - \omega x)}, \quad \omega x - 2 \neq \omega^2 T, \quad (66)$$

$$u_{54} = \frac{\omega(\omega^2 T + 6 - \omega x)}{\alpha(\omega^2 T + 4 - \omega x)}, \quad \omega x - 4 \neq \omega^2 T, \quad (67)$$

$$u_{55} = \frac{\omega(\omega^2 T + 8 - \omega x)}{\alpha(\omega^2 T + 6 - \omega x)}, \quad \omega x - 6 \neq \omega^2 T, \quad (68)$$

and so on. Also, we can see from (1) that

$$u_{56} = \frac{x}{\alpha T}, \quad T \neq 0, \quad (69)$$

is a rational solution for the Burger's equation. Using u_{56} given in (69) and (32), we can get a new sequences of rational solutions of the Burger's model generated by the rational solution (69) in the formulae

$$u_{57} = -\frac{2T - x^2}{\alpha T x}, \quad x \neq 0, T \neq 0, \quad (70)$$

$$u_{58} = \frac{(6T - x^2)x}{\alpha(2T - x^2)T}, \quad x \neq \pm\sqrt{2T}, T \neq 0, \quad (71)$$

and so on. By using of the variational iteration technique and the Adomian decomposition procedure the solution of the Burger's model in closed form is

$$u_{59} = \frac{(\omega + k) + (\omega - k) \exp(\xi)}{\alpha(1 + \exp(\xi))}, \quad (73)$$

From u_{59} obtained in (73) we can get a new sequences of explicit solution for the Burger's model by using BT given in (32) in the form

$$u_{60} = \frac{(\omega - k)^2 \exp(\xi) + (k + \omega)^2}{\alpha[(\omega - k) \exp(\xi) + (k + \omega)]}, \quad (74)$$

$$u_{61} = \frac{(\omega - k)^3 \exp(\xi) + (k + \omega)^3}{\alpha((\omega - k)^2 \exp(\xi) + (k + \omega)^2)}, \quad (75)$$

$$u_{62} = \frac{(\omega - k)^4 \exp(\xi) + (k + \omega)^4}{\alpha((\omega - k)^3 \exp(\xi) + (k + \omega)^3)}, \quad (76)$$

$$u_{63} = \frac{(\omega - k)^5 \exp(\xi) + (k + \omega)^5}{\alpha((\omega - k)^4 \exp(\xi) + (k + \omega)^4)}, \quad (77)$$

and so on, we can get a new sequences of explicit solution of the Burger's model.

4 SOLITON INTERACTIONS

Several applications of solitons interaction can be obtain from fiber optics. It is therefore significant to control how nearby two solitons can be without affecting each other. The two soliton interaction has been deliberate lengthily over the past years. We know that based on physical interpretation of two soliton can affect each other only when they are near enough to overlap their tails. So the two solitons possibly will attract (arise nearby) or repel (go away) each other dependent on the parameters (wave numbers, phases, and velocity) or initial conditions.

To discuss the interaction of two soliton, we suppose two solitons whose peaks are separated in time. Generally speaking we suppose that their wave numbers, phases, and velocity different as

$$u_i = \frac{1}{\alpha} \left[\omega_i - 2k_i \tanh(k_i(x - \omega_i T) + \varphi_i) \right], \quad T = \int_0^t \beta(\tau) d\tau. \quad (78)$$

where φ_i represented the phase difference with $i = 1$ or 2 . We will study the interaction of two soliton by using the simple linear sum of two wave solitons u_1 and u_2 which satisfies the Burger's equation individually. It is clear that the total field $u = u_1 + u_2$ satisfies the perturbed Burger's equation which satisfied by each soliton. When $i = 1$, we have

$$u_{1,T} - u_{1,xx} + \alpha u_1 u_{1,x} = -\alpha u_2 u_{1,x} - \alpha u_1 u_{2,x},$$

when $i = 2$, we have

$$u_{2,T} - u_{2,xx} + \alpha u_2 u_{2,x} = -\alpha u_1 u_{2,x} - \alpha u_2 u_{1,x}.$$

So the interaction among two nearby solitons occur based on the two expressions on the right side which performance as a perturbation terms.

Doing the same manner, we can discuss the interaction of three soliton, we suppose that their wave numbers, phases, and velocity different as

$$u_i = \frac{1}{\alpha} \left[\omega_i - 2k_i \tanh(k_i(x - \omega_i T) + \varphi_i) \right], \quad T = \int_0^t \beta(\tau) d\tau. \quad (79)$$

where φ_i represented the phase difference with $i = 1, 2$ or 3 . Since u_1 , u_2 and u_3 satisfies the Burger's equation individually. The total field $u = u_1 + u_2 + u_3$ satisfies the perturbed Burger's equation which satisfied by each soliton. When $i = 1$, we have

$$u_{1,T} - u_{1,xx} + \alpha u_1 u_{1,x} = -\alpha(u_3 + u_2 + u_1)(u_{2,x} + u_{3,x}) - \alpha(u_3 + u_2)u_{1,x},$$

when $i = 2$, we have

$$u_{2,T} - u_{2,xx} + \alpha u_2 u_{2,x} = -\alpha(u_3 + u_2 + u_1)(u_{1,x} + u_{3,x}) - \alpha(u_3 + u_1)u_{2,x},$$

when $i = 3$, we have

$$u_{3,T} - u_{3,xx} + \alpha u_3 u_{3,x} = -\alpha(u_3 + u_2 + u_1)(u_{1,x} + u_{2,x}) - \alpha(u_2 + u_1)u_{3,x},$$

So the nonlinear interaction between three neighboring solitons occur based on the two expressions on the right side which performance as a perturbation terms.

The evolutionary behavior of soliton wave represented by u_1 given by equation (14) with the selection $\alpha = 1, \omega = 2, k = 3$ and $\varphi = 0$, so u_1 take the form

$$u_1 = 2 - 6 \tanh \left[3 \left(x - 2 \int \beta(t) dt \right) \right],$$

Fig (1-a) when $\beta(t) = 1$, Fig (1-b) when $\beta(t) = 10t$, Fig (1-c) when $\beta(t) = \sinh t$, Fig (1-d) when $\beta(t) = \sin t$, Fig (1-e) when $\beta(t) = t \sin t$, Fig (1-f) when $\beta(t) = \tanh t$, Fig (1-g) when $\beta(t) = \text{sect}$, Fig (1-h) when $\beta(t) = t e^t$ and Fig (1-k) when $\beta(t) = \cos(t) e^{\sin(t)}$. figure (1) shows different types of shaped one kink soliton.

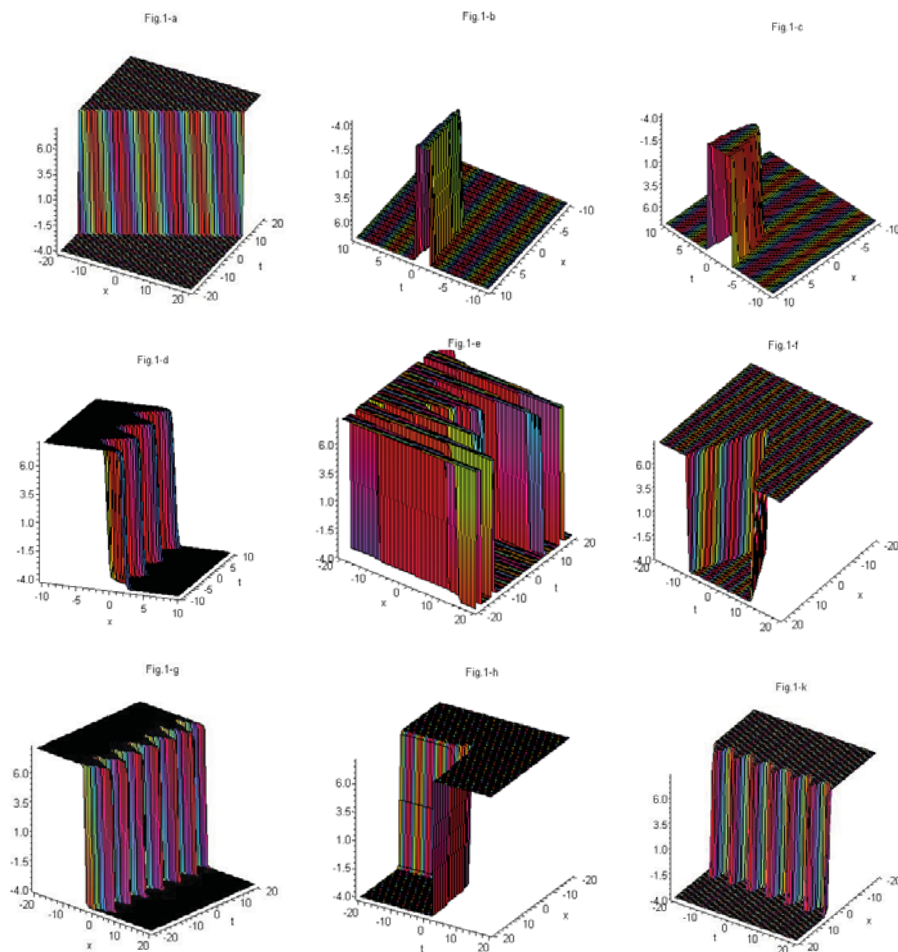


Figure 1. Structures of one soliton solution (14) with $\alpha = 1, \omega = 2, k = 3$ and $\varphi_1 = 0$ (a) $\beta(t) = 1$ (b) $\beta(t) = 10t$ (c) $\beta(t) = \sinh t$ (d) $\beta(t) = \sin t$ (e) $\beta(t) = t \sin t$ (f) $\beta(t) = \tanh t$ (g) $\beta(t) = \text{sect}$ (h) $\beta(t) = t e^t$ (k) $\beta(t) = \cos(t) e^{\sin(t)}$.

The evolutionary behavior of two soliton interaction represented $u = u_1 + u_2$ with u_1, u_2 are given by equation (78) with the selection $\alpha = 1, \omega_1 = 2, \omega_2 = -2, k_1 = k_2 = -1, \varphi_1 = -5$ and $\varphi_2 = 5$, so u take the form

$$u = 2 \tanh(-(x - 2T) - 5) + 2 \tanh(-(x + 2T) + 5), \quad T = \int_0^t \beta(\tau) d\tau. \text{ Fig (2-}$$

a) when $\beta(t) = 1$, Fig (2-b) when $\beta(t) = 10t$, Fig (2-c) when $\beta(t) = \sinh t$, Fig (2-d) when $\beta(t) = \sin t$, Fig (2-e) when $\beta(t) = t \sin t$, Fig (2-f) when $\beta(t) = \tanh t$, Fig (2-g) when $\beta(t) = \sec t$, Fig (2-h) when $\beta(t) = t e^t$ and Fig (2-k) when $\beta(t) = \cos(t) e^{\sin(t)}$. figure (2) shows different types of interaction of shaped two kink soliton.

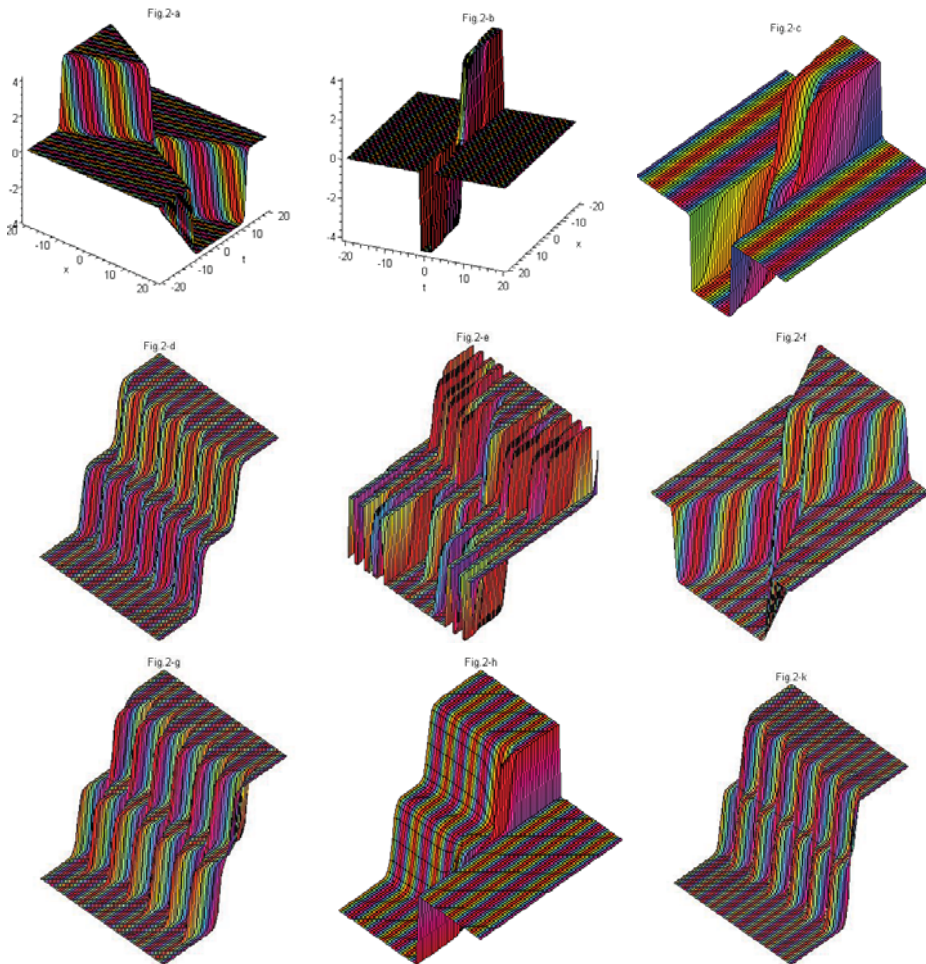


Figure 2. interaction of two kink soliton solution (78) with $\alpha = 1, \omega_1 = 2, \omega_2 = -2, k_1 = k_2 = -1, \varphi_1 = -5$ and $\varphi_2 = 5$, (a) $\beta(t) = 1$ (b) $\beta(t) = 10t$ (c) $\beta(t) = \sinh t$ (d) $\beta(t) = \sin t$ (e) $\beta(t) = t \sin t$ (f) $\beta(t) = \tanh t$ (g) $\beta(t) = \sec t$

$$(h) \beta(t) = t e^t \quad (k) \beta(t) = \cos(t) e^{\sin(t)}.$$

The evolutionary behavior of three kink soliton interaction represented by $u = u_1 + u_2 + u_3$ with u_1, u_2, u_3 are given by equation (79) with the selection $\alpha = 1, \omega_1 = 2, \omega_2 = -2, \omega_3 = 1, k_1 = k_2 = k_3 = -1, \varphi_1 = -5, \varphi_2 = 5$ and $\varphi_3 = 0$, so u take the form

$$u = 1 + 2 \tanh(-(x - 2T) - 5) + 2 \tanh(-(x + 2T) + 5) + 2 \tanh(-(x - T)), \quad T = \int_0^t \beta(\tau) d\tau.$$

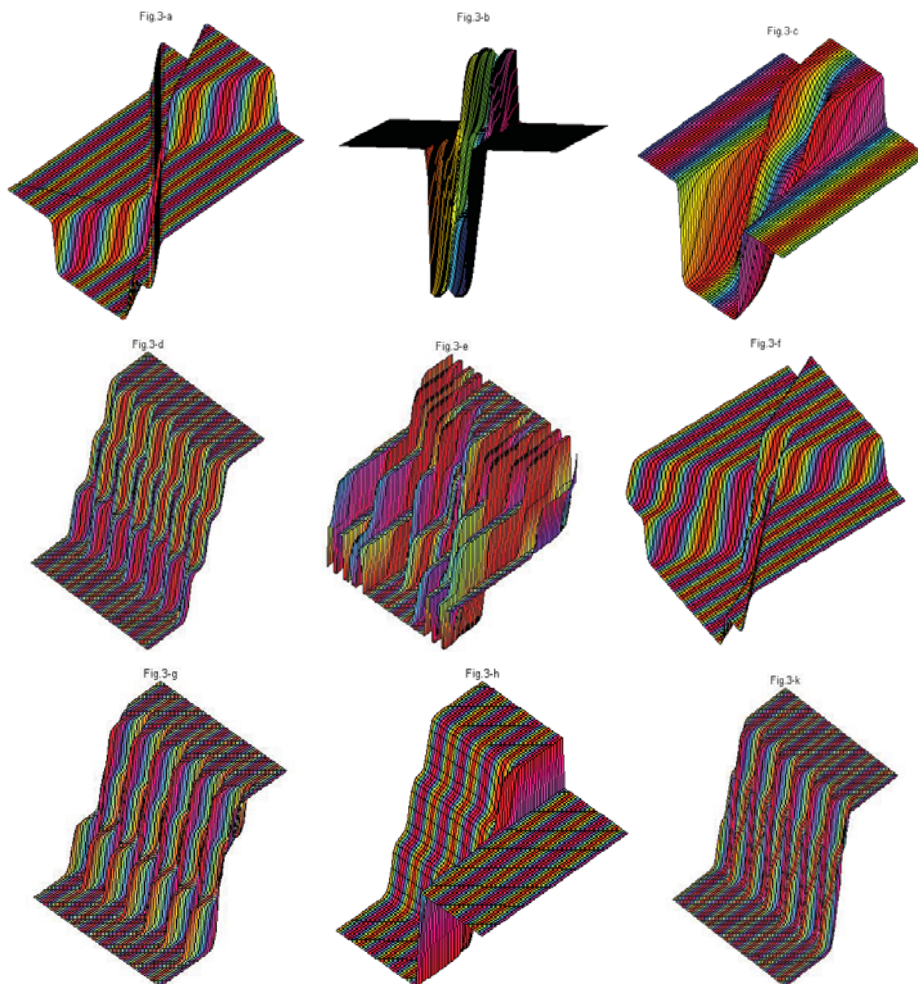


Figure 3. interaction of three kink soliton solution (79) with $\alpha = 1, \omega_1 = 2, \omega_2 = -2, \omega_3 = 1, k_1 = k_2 = k_3 = -1, \varphi_1 = -5, \varphi_2 = 5$ and $\varphi_3 = 0$, (a) $\beta(t) = 1$ (b) $\beta(t) = 10t$ (c) $\beta(t) = \sinh t$ (d) $\beta(t) = \sin t$ (e) $\beta(t) = t \sin t$ (f) $\beta(t) = \tanh t$ (g) $\beta(t) = \sec t$ (h) $\beta(t) = t e^t$ (k) $\beta(t) = \cos(t) e^{\sin(t)}.$

Fig (3-a) when $\beta(t) = 1$, Fig (3-b) when $\beta(t) = 10t$, Fig (3-c) when $\beta(t) = \sinh t$, Fig (3-d) when $\beta(t) = \sin t$, Fig (3-e) when $\beta(t) = t \sin t$, Fig (3-f) when $\beta(t) = \tanh t$, Fig (3-g) when $\beta(t) = \sec t$, Fig (3-h) when $\beta(t) = t e^t$ and Fig (3-k) when $\beta(t) = \cos(t) e^{\sin(t)}$. figure (3) shows different types of interaction of shaped three kink soliton.

5 APPLICATIONS TO FIND NEW EXACT SOLUTIONS

5.1 KdV-Burger's equation

The known and general form of the Burger's and KdV as well as KdVBurger's equations are respectively

$$u_t + uu_x - \alpha u_{xx} = 0, \quad (80)$$

$$u_t + uu_x + \beta u_{xxx} = 0, \quad (81)$$

$$u_t + uu_x - \alpha u_{xx} + \beta u_{xxx} = 0, \quad (82)$$

with α and β are random constants when $\alpha\beta \neq 0$.

These three equations are the greatest standard nonlinear differential models of physical attention, that is not only come from representative physical phenomena, but can likewise be extensively applied to many physical important fields such as quantum mechanics, plasma physics, hydro dynamics, astronomy and astrophysics. Important observations of ours are the study of exact and explicit soliton and solitary wave solution of the KdV-Burger's model that is rises in some different physical settings as a nonlinear model that include the dissémination and distraction besides nonlinearity effects. KdV-Burger's model are used to reprsent the opposée energy effondrement and intermittent turbulence with dissémination effect is considered by using the superposition technique. In the beginning, let us examine carefully the features of the Burger's and KdV models besides KdV-Burger's model. Looking at equations (80) to (82), we notice that it is not difficult to see that they have the similar nonlinear term uu_x and of different linear terms $-\alpha u_{xx} + \beta u_{xxx}$ are precisely equivalent to the superposition of Burger's and KdV models. Based on those reasons, we might construct the solutions of KdV-Burger's model by the linear superposition of those to Burger's and KdV models so we can suppose that KdV-Burger's model (82) has the subsequent ansatz solution:

$$u = au_B + bu_K, \quad (83)$$

with a and b are parameters can be determined, u_B is the Burger's solution and u_K is the KdV solution. Equation (83) is the superposition formulae to obtain the solutions of the KdV-Burger's model (83).

5.2 The linear heat model

The heat model take the form

$$u_t - u_{xx} = 0, \quad (84)$$

the potential Burger's model is

$$v_t - v_x^2 - v_{xx} = 0, \quad (85)$$

which is related to the heat model (84) by taking the following transformation $e^v = u$. Also, The Burger's model

$$w_t - 2ww_x - w_{xx} = 0, \quad (86)$$

which is connected to the potential form of Burgers model (86) through the transformation $w = v_x$, this means that any novel solutions of the Burger's model (86) gives novel solutions of potential formulae of the Burger's model (85). Similarly, any new solutions of the potential formulae of the Burger's model (85) gives new solutions to the linear heat model (84).

5.3 Coupled Burger's equations

The coupled Burger's equations

$$\begin{aligned} u_t + 2uu_x + 2cvv_x + u_{xx} &= 0, \\ v_t + 2(uv)_x + v_{xx} &= 0, \end{aligned} \quad (87)$$

where c is random constant, when $v = 0$, the coupled Burger's equations (87) reduces to the usual Burger's model. Similarly to the single Burger's model, the coupled Burger's system (87) can be used in several physical fields. This model can be derived from the incompressible inviscid two layer fluid. To construct some novel solutions of the system (87), first of all we notice that

$$v = au, \quad a^2 = \frac{1}{c}, \quad (88)$$

can make the coupled Burger's system (87) reduce to a single one

$$u_t + 4uu_x + u_{xx} = 0, \quad (89)$$

thus any solutions of the Burger's model (89) and using the ansatz (88) gives a new solution to the coupled Burger's system (87). If we have

$$u = w + z, \quad v = \pm(z - w)/\sqrt{c}. \quad (90)$$

It is straightforward to see that under the ansatz (90), w and z can be decoupled,

$$w_t + 4ww_x + w_{xx} = 0, \quad z_t + 4zz_x + z_{xx} = 0. \quad (91)$$

In other words, w and z are two different solutions of the same single Burger's equation (91). Using the ansatz (90) gives new solutions of the coupled Burger's system (87).

5.4 Symmetric coupled Burger's equations

The symmetric coupled Burger's equations are

$$\begin{aligned} u_t &= \frac{1}{2}u_{xx} + \frac{1}{2}v_{xx} + 5uu_x + vu_x + 2vv_x, \\ v_t &= \frac{1}{2}v_{xx} + \frac{1}{2}u_{xx} + 5vv_x + uv_x + 2uu_x, \end{aligned} \quad (92)$$

The symmetric coupled Burger's system (92) can be reduced back to the normal Burger's model

$$q_t = q_{xx} + \frac{1}{2} q q_x, \quad (93)$$

with $u = v = q/4$. Obviously, we obtain the solution of equation (93) from the solution of Burger's model (1) by taking $\alpha = -1/2$.

6. DISCUSSION AND CONCLUSION

In this paper, we have got more travelling wave solutions stated by trigonometric and hyperbolic functions for the Burger's equation with time dependent variable coefficient based on the generalized tanh function method. The essential benefit of this manuscript is to construct several sequences of analytical solutions of the Burger's model generated by the known solution of the Burger's model using the BT (32). Also, we set two rational solutions of the Burger's model and use these solutions to obtain two new sequences of rational solutions of desire model. In addition, we use the solution of the Burger's model constructed by the variational iteration technique and the Adomian decomposition method to get a new sequences of explicit solution for the Burger's model. The exact solutions expressed by trigonometric functions, hyperbolic functions, exponential function and other type of function for the Burger's equation are obtained by using this way. The obtained solutions include periodic wave, solitary wave and shockwave solutions. The abundant exact solutions of the Burger's equation may be describe various features of waves and may be helpful in understanding the problems in mathematics, engineering, physics and other fields. Also, these results have not been reported in the literatures with the best of our knowledge. Moreover, this technique can be applied to construct new solutions for several nonlinear evolution equations.

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