

Some Properties and Applications of a Weighted Continuous Distribution

Brijesh P. Singh and Utpal Dhar Das*

Department of Statistics, Institute of Science
 Banaras Hindu University, Varanasi-221005, India

Email : brijesh@bhu.ac.in ;

**Email(corresponding Author)* : utpal.statmath@gmail.com

ABSTRACT

In this article we introduce a two-parameter continuous distribution which is generated from gamma distribution as well as exponential distribution using weight function. Its moments, hazard and mean residual life function have been obtained. Also order statistics and stochastic ordering are derived. Maximum likelihood estimation method has been employed to estimate the parameters. Some real data sets have been applied to access the performance of the distribution. Various goodness of fit criterion shows quite satisfactory presentation over others competitive distributions.

Keywords: Gamma distribution, generalization, weight function, MLE, entropy, MRLF.

1 Introduction

A well known and useful two parameter Gamma distribution having parameter α and θ is a weighted version of exponential distribution with weight function $x^{\alpha-1}$, where $\alpha > 1$. The pdf and cdf is given below

$$\begin{aligned} f_1(x; \theta, \alpha) &= \frac{\theta^\alpha}{\Gamma\alpha} e^{-\theta x} x^{\alpha-1}; \quad x > 0, \alpha > 0, \theta > 0 \\ F_1(x; \theta, \alpha) &= 1 - \frac{\Gamma(\alpha, \theta x)}{\Gamma\alpha}; \quad x > 0, \alpha > 0, \theta > 0 \end{aligned} \quad (1.1)$$

Here parameters α and θ are shape and scale parameter respectively. It is easy to shown that for the value $\alpha = 1$ the two parameter gamma distribution reduces as classical exponential distribution. (Kiche, Ngesa and Orwa, 2019) discussed generalized gamma distribution and applied on survival data. (Nadarajah and Kotz, 2006) proposed various exponentiated distribution including exponentiated gamma distribution and discussed their statistical properties. (Nadarajah and Gupta, 2007) proposed another generalized gamma distribution and discussed its properties and application to real data. (Weibull and Sweden, 1951), introduced two parameter Weibull distribution with pdf and cdf as given below

$$\begin{aligned} f_2(x; \theta, \alpha) &= \theta \alpha x^{\alpha-1} e^{-\theta x^\alpha}; \quad x > 0, \alpha > 0, \theta > 0 \\ F_2(x; \theta, \alpha) &= 1 - e^{-\theta x^\alpha}; \quad x > 0, \alpha > 0, \theta > 0 \end{aligned} \quad (1.2)$$

The gamma distribution is used in a variety of applications including queuing, financial and weather models. It is can naturally be considered as the distribution of the waiting time between events distributed according to a Poisson process. (Balakrishnan and Peng, 2006) applied this distribution to

develop generalized gamma frailty model. (Cordeiro, Ortega and Silva, 2011) derived another generalization of Stacy's generalized gamma distribution using exponentiated method, and applied it to life time and survival analysis. (Stacy, 1962) introduced a generalization of gamma distribution using power transformation $y = x^\beta$ in gamma distribution and after that (Stacy and Mihram, 1965) discussed its parametric estimation.

$$f(x; \alpha, \beta, \theta) = \frac{\beta \theta^\alpha}{\Gamma(\alpha)} x^{\beta \alpha - 1} e^{-\theta x^\beta}; \quad \alpha > 0, \beta > 0, \theta > 0. \quad (1.3)$$

Now if $\beta = 1$ then (1.3) becomes Gamma distribution and if $\beta = 1$ and $\alpha = 1$ then (1.3) becomes classical exponential distribution. If $\beta = 1$ and $\alpha = 2$ the distribution transformed into length biased exponential distribution (LBED) (Singh and Das, 2020). It is also known that Weighted distribution provides a flexible models as discussed by (Patil and Rao, 1978) is defined as follows

$$f^*(x) = \frac{w(x)f(x)}{E[w(x)]} \quad (1.4)$$

Where $w(x)$ is the weight function, $f(x)$ is the baseline distribution. The generalize Gamma distribution can also be obtained by weighted version of exponential distribution with weight function $w(x) = e^{-\theta x^\beta + \theta x} x^{\beta \alpha - 1}$. Here if $\beta = 1$ the weight function become $x^{\alpha-1}$ which generate for gamma distribution. Also if $\alpha = \frac{1}{2}$ and $\beta = 2$ in the weight function we obtained quasi exponential distribution (QED) which is discussed by (Shanker, Shukla, Sharma and Shanker, 2018a). If we take only $\beta = 2$ in the weight function, then we get quasi gamma distribution (QGD) earlier discussed by (Shanker, Shukla, Sharma and Shanker, 2018b). Several authors has been discussed the generalized gamma distribution like (Khodabina and Ahmadabadib, 2010), (Singh, Singh and Das, 2020), (Singh, Das and Singh, 2021), (Lienhard and Meyer, 1967), (Hager and Bain, 1970).

2 Generation of the Distribution

Let $Y \sim \text{Gamma}(\alpha, \theta)$ and $y = x^3$.

$$f_Y(y) = \frac{\theta^\alpha}{\Gamma(\alpha)} y^{\alpha-1} e^{-\theta y}; \quad y > 0, \alpha > 0, \theta > 0. \quad (2.1)$$

If $X = g(Y)$, then $g_X(x) = f_Y(g^{-1}(x)) \left| \frac{d}{dx} g^{-1}(x) \right|$ which is reduce to

$$g_X(x) = \frac{\theta^\alpha}{\Gamma(\alpha)} x^{3(\alpha-1)} e^{-\theta x^3} \cdot 3x^2 = \frac{3\theta^\alpha}{\Gamma(\alpha)} x^{3\alpha-1} e^{-\theta x^3}$$

Therefore the above distribution is a two parameter distribution and named as apparent gamma distribution (AGD) with parameters α and θ is defined and its pdf is given as

$$f(x; \theta, \alpha) = \frac{3\theta^\alpha}{\Gamma(\alpha)} x^{3\alpha-1} e^{-\theta x^3}; \quad x > 0, \alpha > 0, \theta > 0 \quad (2.2)$$

Here, it is necessary to note that AGD can be obtained using a suitable weight function and base line distribution as exponential distribution. The weight function is as $e^{-\theta x^3 + \theta x} x^{3\alpha-1}$

$$\frac{w(x)f(x)}{\int_0^\infty w(x)f(x)dx} = \frac{e^{-\theta x^3 + \theta x} x^{3\alpha-1} \theta e^{-\theta x}}{\theta \int_0^\infty x^{3\alpha-1} e^{-\theta x^3} dx} = \frac{3\theta^\alpha}{\Gamma(\alpha)} x^{3\alpha-1} e^{-\theta x^3} \quad (2.3)$$

where parameters θ is scale parameter and α is a shape parameter.

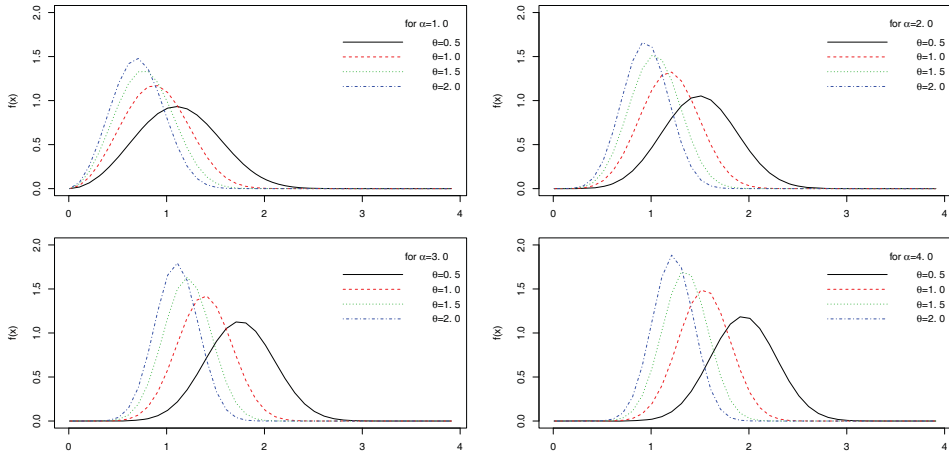


Figure 1: plot of pdf of AGD for different values of α and θ

The corresponding cdf of the AGD is given by

$$F(x) = \int_0^x f(t)dt = 1 - \int_x^\infty f(t)dt = 1 - \frac{3\theta^\alpha}{\Gamma(\alpha)} \int_x^\infty t^{3\alpha-1} e^{-\theta t^3} dt$$

Now let $\theta t^3 = z$; $3t^2 dt = dz$ hence $dt = \frac{dz}{3\theta(\frac{z}{\theta})^{\frac{2}{3}}}$

$$F(x) = 1 - \frac{1}{\Gamma(\alpha)} \int_{\theta x^3}^\infty z^{\alpha-1} e^{-z} dz = 1 - \frac{\Gamma(\alpha, \theta x^3)}{\Gamma(\alpha)}; x > 0, \theta > 0, \alpha > 0 \quad (2.4)$$

and survival function of AGD is obtained as

$$S(x; \theta, \alpha) = \frac{\Gamma(\alpha, \theta x^3)}{\Gamma(\alpha)}; x > 0, \theta > 0, \alpha > 0 \quad (2.5)$$

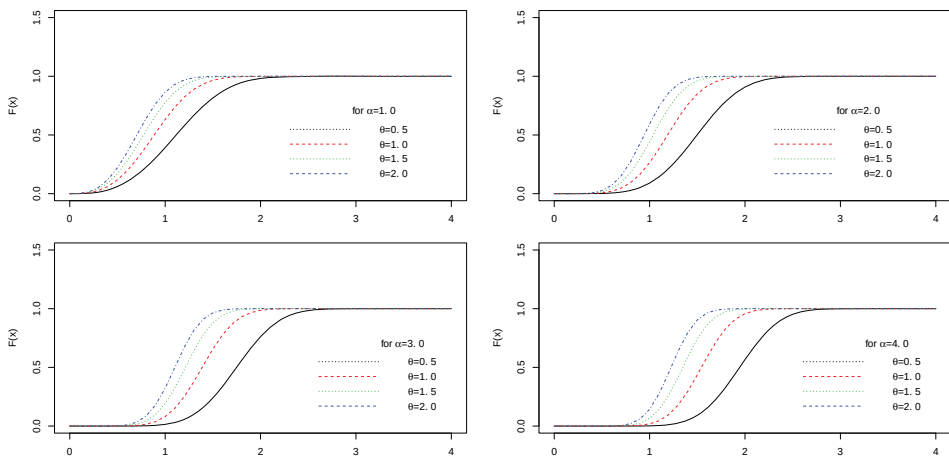
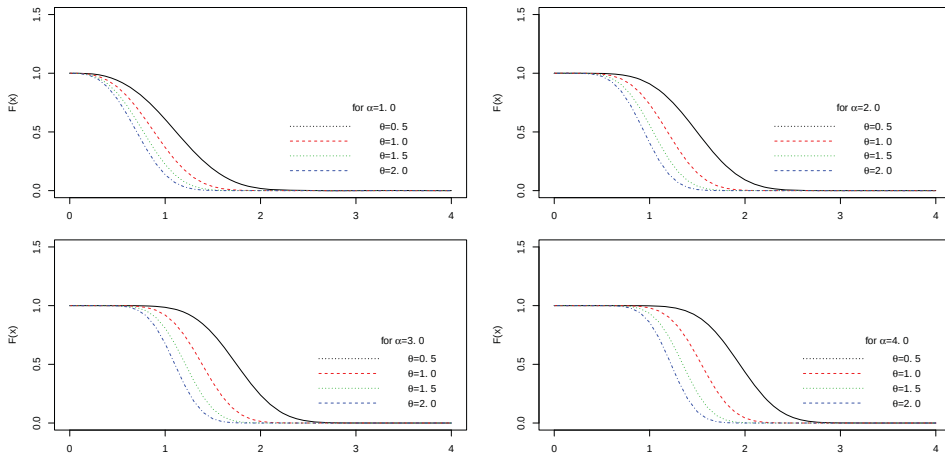


Figure 2: plot of cdf of AGD for different values of α and θ


 Figure 3: plot of survival function of AGD for different values of α and θ

3 Moments

The r^{th} moment about origin of AGD can be obtained as

$$\mu'_r = E(X^r) = \int_0^\infty x^r f(x) dx = \frac{3\theta^\alpha}{\Gamma(\alpha)} \int_0^\infty e^{-\theta x^3} x^{3\alpha+r-1} dx = \frac{\Gamma(\alpha + \frac{r}{3})}{\theta^{\frac{r}{3}} \Gamma(\alpha)}; \quad r \in \mathbb{N}$$

taking $r = 1, 2, 3$, we get the moments about origin of AGD are obtained as

$$\mu'_1 = \frac{\Gamma(\alpha + \frac{1}{3})}{\theta^{\frac{1}{3}} \Gamma(\alpha)}; \quad \mu'_2 = \frac{\Gamma(\alpha + \frac{2}{3})}{\theta^{\frac{2}{3}} \Gamma(\alpha)}; \quad \mu'_3 = \frac{\alpha}{\theta}; \quad \mu'_4 = \frac{\Gamma(\alpha + \frac{4}{3})}{\theta^{\frac{4}{3}} \Gamma(\alpha)} \quad (3.1)$$

and the variance of the AGD is obtained as

$$\mu_2 = \frac{\Gamma(\alpha) \Gamma(\alpha + \frac{2}{3}) - (\Gamma(\alpha + \frac{1}{3}))^2}{\theta^{\frac{2}{3}} (\Gamma(\alpha))^2} \quad (3.2)$$

4 Moment and Characteristic Generating Function

The moment generating function $M_x(t)$ and characteristic generating function $\Phi_x(t)$ of AGD is obtained as equation number (4.4) and (4.5).

$$M_x(t) = \int_0^\infty e^{tx} f(x) dx = \frac{3\theta^\alpha}{\Gamma(\alpha)} \int_0^\infty e^{tx} x^{3\alpha-1} e^{-\theta x^3} dx \quad (4.1)$$

Now applying exponential series expansion in equation (4.1), $e^{tx} = \sum_{k=0}^\infty \frac{(tx)^k}{k!}$ we get ,

$$M_x(t) = \frac{3\theta^\alpha}{\Gamma(\alpha)} \int_0^\infty \sum_{k=0}^\infty \frac{(tx)^k}{k!} x^{3\alpha-1} e^{-\theta x^3} dx \quad (4.2)$$

Now applying Fubini's theorem over integration under summation we get

$$M_x(t) = \frac{3\theta^\alpha}{\Gamma(\alpha)} \sum_{k=0}^\infty \frac{(t)^k}{k!} \int_0^\infty x^{3\alpha+k-1} e^{-\theta x^3} dx \quad (4.3)$$

Making $\theta x^3 = z$, as $x \rightarrow 0, z \rightarrow 0$ and $x \rightarrow \infty, z \rightarrow \infty$, we get, from (4.3),

$$M_x(t) = \frac{1}{\Gamma\alpha} \sum_{k=0}^{\infty} \frac{t^k}{k! \theta^{\frac{k}{3}}} \Gamma\left(\alpha + \frac{k}{3}\right) \quad (4.4)$$

Now replacing it for t in equation number (4.4) we get the characteristic function in equation number (4.5).

$$\Phi_x(t) = \frac{1}{\Gamma\alpha} \sum_{k=0}^{\infty} \frac{(it)^k}{k! \theta^{\frac{k}{3}}} \Gamma\left(\alpha + \frac{k}{3}\right) \quad (4.5)$$

5 Hazard Rate Function and Mean Residual Life Function

Let X be a continuous random variable with pdf (2.2) and cdf (2.4). The hazard function and mean residual life function is defined of random variable is defined as equation number (5.1) and (5.4).

$$\begin{aligned} h(x) &= \lim_{\delta x \rightarrow 0} \frac{P(X < x + \delta x | X > x)}{\delta x} = \frac{f(x)}{1 - F(x)} \\ &= \frac{3\theta^\alpha x^{3\alpha-1} e^{-\theta x^3}}{\Gamma(\alpha, \theta x^3)} \end{aligned} \quad (5.1)$$

$$\begin{aligned} m(x) &= E[X - x | X > x] = \frac{1}{1 - F(x)} \int_x^{\infty} [1 - F(t)] dt \\ &= \frac{1}{S(x)} \int_x^{\infty} t f(t) dt - x = \frac{3\theta^\alpha}{S(x) \Gamma\alpha} \int_x^{\infty} t^{3\alpha-1} e^{-\theta t^3} dt - x \end{aligned} \quad (5.2)$$

Making $\theta t^3 = z$; $3\theta t^2 dt = dz$ now $dt = \frac{dz}{3\theta(\frac{z}{\theta})^{\frac{2}{3}}}$ as $t \rightarrow x, z \rightarrow \theta x^3$ and $t \rightarrow \infty, z \rightarrow \infty$, we get

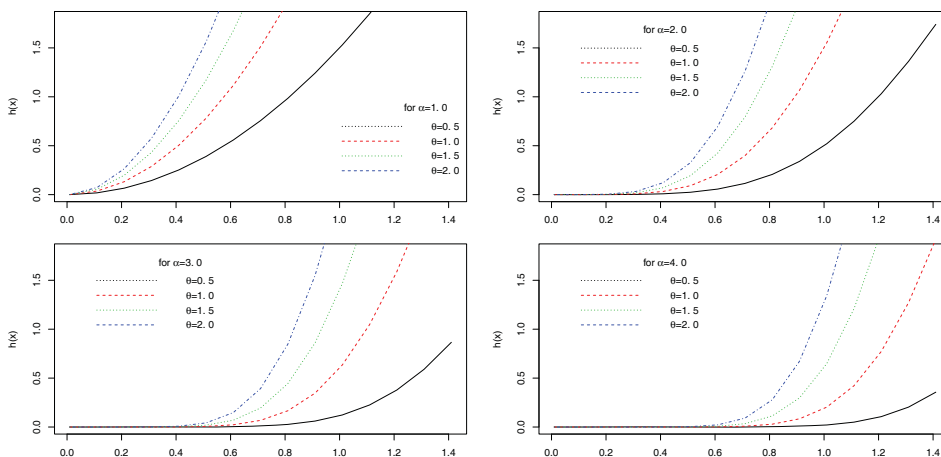
$$\frac{3\theta^\alpha}{S(x) \Gamma\alpha} \int_{\theta x^3}^{\infty} \left(\frac{z}{\theta}\right)^\alpha e^{-z} \frac{dz}{3\theta \left(\frac{z}{\theta}\right)^{\frac{2}{3}}} - x = \frac{\theta^{\alpha-1}}{S(x) \Gamma\alpha} \int_{\theta x^3}^{\infty} \left(\frac{z}{\theta}\right)^{\alpha-\frac{2}{3}} e^{-z} dz - x \quad (5.3)$$

After putting the value of $S(x)$ from the equation (2.5) and simplifying we get the required results in equation number (5.4).

$$m(x) = \frac{\Gamma\left(\alpha + \frac{1}{3}, \theta x^3\right)}{\theta^{\frac{1}{3}} \Gamma(\alpha, \theta x^3)} - x \quad (5.4)$$

Now if we put $x = 0$ in equation number (5.4) we get the mean of AGD as

$$m(0) = \frac{\Gamma\left(\alpha + \frac{1}{3}\right)}{\theta^{\frac{1}{3}} \Gamma(\alpha)} \quad (5.5)$$


 Figure 4: plot of hazard function of AGD for different values of α and θ

6 Bonferroni and Lorenz Curve

The Bonferroni (Bonferroni, 1933) and Lorenz curves are used not only in economics to study income and poverty, but also in other fields like reliability, demography, insurance and medicine. The Bonferroni and Lorenz curve are defined as

$$B(p) = \frac{1}{p\mu} \int_0^q x f(x) dx \quad \text{and} \quad L(p) = \frac{1}{\mu} \int_0^q x f(x) dx \quad (6.1)$$

Respectively where, $\mu = E(x)$ and $q = F^{-1}(p)$. The Bonferroni and Gini indices are defined by

$$B = 1 - \int_0^1 B(p) dp \quad \text{and} \quad G = 1 - 2 \int_0^1 L(p) dp \quad (6.2)$$

Here

$$B(p) = \frac{3\theta^\alpha}{p\mu\Gamma\alpha} \int_0^q x^{3\alpha} e^{-\theta x^3} dx \quad (6.3)$$

Now by taking $\theta x^3 = z$, the above equation reduces to

$$\begin{aligned} B(p) &= \frac{3\theta^{\alpha+\frac{1}{3}}}{p\Gamma(\alpha+\frac{1}{3})} \int_0^{\theta q^3} \left(\frac{z}{\theta}\right)^\alpha e^{-z} \frac{dz}{3\theta \left(\frac{z}{\theta}\right)^{\frac{2}{3}}} \\ &= \frac{\theta^{\alpha+\frac{1}{3}-1}}{p\Gamma(\alpha+\frac{1}{3})} \int_0^{\theta q^3} \left(\frac{z}{\theta}\right)^{\alpha+\frac{1}{3}-1} e^{-z} dz \\ &= \frac{\gamma(\alpha+\frac{1}{3}, \theta q^3)}{p\Gamma(\alpha+\frac{1}{3})} \end{aligned} \quad (6.4)$$

where $\int_0^{\theta q^3} \left(\frac{z}{\theta}\right)^{\alpha+\frac{1}{3}-1} e^{-z} dz = \gamma(\alpha+\frac{1}{3}, \theta q^3)$ is the lower incomplete gamma function

Similarly,

$$L(p) = \frac{1}{\mu} \int_0^q x f(x) dx = \frac{\gamma(\alpha+\frac{1}{3}, \theta q^3)}{\Gamma(\alpha+\frac{1}{3})}; \quad q = F^{-1}(p) \quad (6.5)$$

and the Bonferroni and Gini indices are defined as

$$B = 1 - \int_0^1 B(p)dp \quad \text{and} \quad G = 1 - 2 \int_0^1 L(p)dp \quad (6.6)$$

7 Rényi Entropy

We know that the entropy is a measure of uncertainty of a random variable X . A most popular measure of entropy is Rényi entropy given by (Rényi, 1961). If X is continuous random variable with pdf (2.2) then Rényi entropy is given by the equation (7.2).

$$\begin{aligned} e(\eta) &= \frac{1}{1-\eta} \log \left[\int_0^\infty f^\eta(x) dx \right] = \frac{1}{1-\eta} \log \left[\int_0^\infty \left(\frac{3\theta^\alpha}{\Gamma\alpha} \right)^\eta e^{-\eta\theta x^3} x^{\eta(3\alpha-1)} dx \right] \\ &= \frac{1}{1-\eta} \log \left[\left(\frac{3\theta^\alpha}{\Gamma\alpha} \right)^\eta \int_0^\infty e^{-\eta\theta x^3} x^{\eta(3\alpha-1)} dx \right] \end{aligned} \quad (7.1)$$

let $\theta\eta x^3 = z$; $3\eta\theta x^2 dx = dz$ now $dx = \frac{dz}{3\theta\eta\left(\frac{z}{\theta\eta}\right)^{\frac{2}{3}}}$

$$\begin{aligned} &= \frac{1}{1-\eta} \log \left[\left(\frac{3\theta^\alpha}{\Gamma\alpha} \right)^\eta \int_0^\infty \left(\frac{z}{\theta\eta} \right)^{\frac{\eta(3\alpha-1)}{3}} e^{-z} \frac{dz}{3\theta\eta\left(\frac{z}{\theta\eta}\right)^{\frac{2}{3}}} \right] \\ &= \frac{1}{1-\eta} \log \left[\left(\frac{3\theta^\alpha}{\Gamma\alpha} \right)^\eta \frac{1}{(\theta\eta)^{\frac{\eta(3\alpha-1)-2}{3}+1}} \int_0^\infty (z)^{\frac{\eta(3\alpha-1)-2}{3}} e^{-z} dz \right] \\ &= \frac{1}{1-\eta} \log \left[\left(\frac{3\theta^\alpha}{\Gamma\alpha} \right)^\eta \frac{\Gamma\left(\frac{\eta(3\alpha-1)+1}{3}\right)}{3(\theta\eta)^{\frac{\eta(3\alpha-1)+1}{3}}} \right] \end{aligned} \quad (7.2)$$

8 Order Statistics

Let $X_1, X_2, \dots, X_n$ be random sample of size n from AGD distribution. Let $X_{(1)}, X_{(2)}, \dots, X_{(n)}$ denote the corresponding order statistics. The pdf and cdf of m^{th} order statistics say $Y = X_{(m)}$ are given by

$$f_Y(y) = \frac{n!}{(m-1)!(n-m)!} F^{m-1}(y) \{1 - F(y)\}^{n-m} f(y)$$

or

$$f_Y(y) = \frac{n!}{(m-1)!(n-m)!} \sum_{l=0}^{n-m} \binom{n-m}{l} (-1)^l F^{m+l-1}(y) f(y) \quad (8.1)$$

and

$$F_Y(y) = \sum_{j=m}^n \binom{n}{j} F^j(y) \{1 - F(y)\}^{n-j}$$

or

$$F_Y(y) = \sum_{j=m}^n \sum_{l=0}^{n-j} \binom{n}{j} \binom{n-j}{l} (-1)^l F^{j+l}(y) \quad (8.2)$$

where $m = 1(1)n$

Now, using equation number (2.2) and (2.4) in equation (8.1) and (8.2) we get the corresponding pdf

and the cdf of m^{th} order statistics of the AGD distribution are obtained in the equation number (8.3) and (8.4).

$$f_Y(y) = \frac{3n!\theta^\alpha y^{3\alpha-1} e^{-\theta y^3}}{(m-1)!(n-m)!\Gamma\alpha} \sum_{l=0}^{n-m} \binom{n-m}{l} (-1)^l \left[1 - \frac{\Gamma(\alpha, \theta y^3)}{\Gamma\alpha} \right]^{r+l-1} \quad (8.3)$$

and

$$F_Y(y) = \sum_{j=m}^n \sum_{l=0}^{n-j} \binom{n}{j} \binom{n-i}{l} (-1)^l \left[1 - \frac{\Gamma(\alpha, \theta y^3)}{\Gamma\alpha} \right]^{j+l} \quad (8.4)$$

where $m = 1(1)n$.

9 Stochastic Orderings

In probability theory and statistics the stochastic ordering is a measure of judging between two random variable X and Y . A random variable X is said to be smaller than a random variable Y .

(i) Stochastic order $X \leq_{st} Y$ if $F_X(x) \geq F_Y(x)$ for all x .

(ii) Hazard rate order $X \leq_{hr} Y$ if $h_X(x) \geq h_Y(x)$ for all x .

(iii) Mean residual life order $X \leq_{mrl} Y$ if $m_X(x) \geq m_Y(x)$ for all x .

(iv) Likelihood ratio order $X \leq_{lr} Y$ if $\frac{f_X(x)}{f_Y(x)}$ decreases in x .

The following results by (Shaked and Shanthikumar, 2007) are well known for introducing stochastic ordering of distributions.

$$X \leq_{lr} Y \implies X \leq_{hr} Y \implies X \leq_{mrl} Y \quad i.e. \quad X \leq_{st} Y$$

with the help of following theorem we claim that AGD distribution is ordered with respect to strongest likelihood ratio ordering.

Theorem 1. Let $X \sim AGD(\theta_1)$ distribution and $Y \sim AGD(\theta_2)$ distribution. If $\theta_1 > \theta_2$ then $X \leq_{lr} Y$ and therefore $X \leq_{hr} Y$, $X \leq_{mrl} Y$ and $X \leq_{st} Y$.

Proof: we have

$$\frac{f_X(x; \theta_1, \alpha_1)}{f_Y(x; \theta_2, \alpha_2)} = \frac{\frac{3\theta_1^{\alpha_1}}{\Gamma\alpha_1} x^{3\alpha_1-1} e^{-\theta_1 x^3}}{\frac{3\theta_2^{\alpha_2}}{\Gamma\alpha_2} x^{3\alpha_2-1} e^{-\theta_2 x^3}} = \frac{\theta_1^{\alpha_1} \Gamma\alpha_2}{\theta_2^{\alpha_2} \Gamma\alpha_1} e^{-(\theta_1-\theta_2)x^3} x^{3(\alpha_1-\alpha_2)}; \quad x > 0 \quad (9.1)$$

Now taking log both side we get,

$$\log \left[\frac{f_X(x; \theta_1, \alpha_1)}{f_Y(x; \theta_2, \alpha_2)} \right] = \log \left[\frac{\theta_1^{\alpha_1} \Gamma\alpha_2}{\theta_2^{\alpha_2} \Gamma\alpha_1} \right] - (\theta_1 - \theta_2)x^3 + 3(\alpha_1 - \alpha_2) \log x \quad (9.2)$$

Now differentiating with respect to x both side we get,

$$\frac{d}{dx} \left\{ \log \left[\frac{f_X(x; \theta_1, \alpha_1)}{f_Y(x; \theta_2, \alpha_2)} \right] \right\} = \frac{3(\alpha_1 - \alpha_2)}{x} - 3(\theta_1 - \theta_2)x^2 \quad (9.3)$$

Now from above expression if $\alpha_1 \leq \alpha_2$ and $\theta_1 > \theta_2$ or $\alpha_1 < \alpha_2$ and $\theta_1 \geq \theta_2$, $\frac{d}{dx} \left\{ \log \left[\frac{f_X(x; \theta_1, \alpha_1)}{f_Y(x; \theta_2, \alpha_2)} \right] \right\} < 0$. This proves that $X \leq_{lr} Y$ and therefore $X \leq_{hr} Y$, $X \leq_{mrl} Y$ and $X \leq_{st} Y$.

10 Maximum Likelihood Estimation of Parameters

Suppose $X = (X_1, X_2, X_3, \dots, X_n)$ be an independently and identically distributed (iid) random variables of size n with pdf (2.2) from AGD(α, θ). Then, the likelihood function based on observed sample $X = (x_1, x_2, x_3, \dots, x_n)$ is defined as

$$L(x; \theta, \alpha) = \left(\frac{3\theta^\alpha}{\Gamma\alpha} \right)^n e^{-\theta \sum_{i=1}^n x_i^3} \prod_{i=1}^n x_i^{3\alpha-1} \quad (10.1)$$

The log-likelihood function corresponding to (10.1) is given by

$$\log L = n \log 3 + n\alpha \log \theta - n \log(\Gamma\alpha) - \theta \sum_{i=1}^n x_i^3 + (3\alpha - 1) \sum_{i=1}^n \log(x_i) \quad (10.2)$$

By differentiating (10.2) partially, with respect to α and θ respectively we get,

$$\frac{\partial \log L}{\partial \alpha} = n \log \theta - n\xi(\alpha) + 3 \sum_{i=1}^n \log(x_i) \quad (10.3)$$

and

$$\frac{\partial \log L}{\partial \theta} = \frac{n\alpha}{\theta} - \sum_{i=1}^n x_i^3 \quad (10.4)$$

where $\bar{x}$ is the sample mean and $\xi(\alpha) = \frac{\partial}{\partial \alpha} (\log(\Gamma\alpha))$ is the digamma function. Above equations (10.3) and (10.4) are not solvable analytically for θ and α . Thus numerical iteration technique is used to get its desired solutions.

11 Goodness of Fit

To check the suitability of Apparent Gamma Distribution (AGD), two real data sets are used, the first data set is based on from 69 carbon fiber's tensile strength (in GPa), tested under tension at gauge length of 20mm (Bader and Priest, 1982) in Table (1). The second data set (Xu, Xie, Tang and Ho, 2003) shown in Table (2) is the time of failure of turbocharger. AGD is compared with other distribution viz. Exponential, Gamma, Weibull and Quasi Exponential and Gamma distribution (Shanker et al., 2018a; Shanker et al., 2018b) using these two data sets discussed. The parameters are estimated with maximum likelihood estimate (MLE) and these estimates are used to calculate -2LL, AIC, BIC, K-S statistics and p -value. Here among the considered distributions AGD is observed best with respect to lower value of -2LL, AIC, BIC, K-S statistics and associated higher p -value. Therefore, the AGD is considering as an alternative to the other distributions and further useful in the medical and engineering science.

Table 1: Comparison of Goodness of fit for AGD, QGD, QED, Weibull and Gamma on carbon fibres data.

Distribution	Estimate	Standard Errors	-2ll	AIC	BIC	K-S	p-value
AGD	$\theta = 0.1794$	0.0316	97.790	101.79	106.26	0.0388	0.9999
	$\alpha = 2.9605$	0.4792					
QGD	$\theta = 0.9955$	0.1719	98.463	102.46	106.93	0.0454	0.9989
	$\alpha = 6.2229$	1.0323					
Gamma	$\theta = 9.5384$	1.6376	100.07	104.07	108.54	0.0583	0.9732
	$\alpha = 23.381$	3.9713					
Weibull	$\theta = 0.0046$	0.0063	99.192	103.31	107.66	0.0562	0.9814
	$\alpha = 5.5067$	1.2185					
QED	$\theta = 0.0799$	0.0136	226.62	228.62	230.85	0.4489	0.0000
Exponential	$\theta = 0.4079$	0.0491	261.73	263.73	265.97	0.4482	0.0000

Table 2: Comparison of Goodness of fit for AGD, QGD, QED, Weibull and Gamma on turbocharger data.

Distribution	Estimate	Standard Errors	-2ll	AIC	BIC	K-S	p-value
AGD	$\theta = 0.0042$	0.0011	166.73	170.73	174.11	0.1126	0.6907
	$\alpha = 1.2909$	0.2665					
QGD	$\theta = 0.0558$	0.0131	170.04	174.04	177.42	0.1157	0.6579
	$\alpha = 2.3917$	0.5041					
Gamma	$\theta = 1.2351$	0.2799	174.82	178.82	182.19	0.1277	0.5311
	$\alpha = 7.7225$	1.6935					
Weibull	$\theta = 0.0064$	0.0086	171.87	175.87	179.25	0.1519	0.3146
	$\alpha = 2.6494$	0.6435					
QED	$\theta = 0.0116$	0.0026	208.35	210.35	212.03	0.6466	0.0000
Exponential	$\theta = 0.1599$	0.0253	226.64	228.64	230.33	0.3631	0.0000

12 Conclusion

Apparent Gamma distribution (AGD), a simple continuous distribution is proposed and discussed in this study. The moments, moment generating function, characteristics generating function, survival function, hazard rate function, mean residual life function have been obtained and discussed. It is noticed that the third, sixth, ninth and twelfth moment about origin of AGD is same as the first, second, third and fourth moment about origin of Gamma distribution respectively. Also these are equal to the second, fourth, sixth and eighth moment of the Quasi Gama distribution respectively. The stochastic ordering, order statistics, Bonferroni and Lorenz curve, Rényi entropy have also examined. The AGD may be use as an alternative distribution to explain real life data related to medical and engineering science etc.

Acknowledgment

No funding was received to conduct this study.

References

- Bader, M. G. and Priest, A. M. 1982. Statistical aspects of fibre and bundle strength in hybrid composites, *Progress in science and engineering of composites, ICCM-IV, Tokyo*, pp. 1129–1136.
- Balakrishnan, N. and Peng, Y. 2006. Generalized gamma frailty model, *Statistics in medicine* **25**(16): 2797–2816.
- Bonferroni, C. E. 1933. *Elementi di statistica generale, (ristampa con aggiunte): anno accademico 1932/33; Bari, R. Istit. super. di scienze economiche, Gili.*
- Cordeiro, G. M., Ortega, E. M. and Silva, G. O. 2011. The exponentiated generalized gamma distribution with application to lifetime data, *Journal of statistical computation and simulation* **81**(7): 827–842.
- Hager, H. W. and Bain, L. J. 1970. Inferential procedures for the generalized gamma distribution, *Journal of the American Statistical Association* **65**(332): 1601–1609.
- Khodabina, M. and Ahmadabadib, A. 2010. Some properties of generalized gamma distribution, *Tech-nometrics* **4**(1): 9–28.

- Kiche, J., Ngesa, O. and Orwa, G. 2019. On generalized gamma distribution and its application to survival data, *International Journal of Statistics and Probability* **8**(5): 85–102.
- Lienhard, J. H. and Meyer, P. L. 1967. A physical basis for the generalized gamma distribution, *Quarterly of Applied Mathematics* **25**(3): 330–334.
- Nadarajah, S. and Gupta, A. K. 2007. The exponentiated gamma distribution with application to drought data, *Calcutta Statistical Association Bulletin* **59**(1-2): 29–54.
- Nadarajah, S. and Kotz, S. 2006. The exponentiated type distributions, *Acta Applicandae Mathematica* **92**(2): 97–111.
- Patil, G. P. and Rao, C. R. 1978. Weighted distributions and size-biased sampling with applications to wildlife populations and human families, *Biometrics* pp. 179–189.
- Rényi, A. 1961. On measures of entropy and information, *Proceedings of the Fourth Berkeley Symposium on Mathematical Statistics and Probability, Volume 1: Contributions to the Theory of Statistics*, The Regents of the University of California.
- Shaked, M. and Shanthikumar, J. G. 2007. *Stochastic orders*, Springer Science & Business Media.
- Shanker, R., Shukla, K. K., Sharma, S. and Shanker, R. 2018a. A quasi exponential distribution, *Biometrics & Biostatistics International Journal* **7**(2): 90–94.
- Shanker, R., Shukla, K. K., Sharma, S. and Shanker, R. 2018b. A quasi gamma distribution, *International Journal of Statistics and Applied Mathematics* **3**(4): 208–217.
- Singh, B. P. and Das, U. D. 2020. Some statistical properties of a weighted distribution and its application to waiting time data, *Asian Journal of Probability and Statistics* **9**(1): 1–12.
- Singh, B. P., Das, U. D. and Singh, S. 2021. A compounded probability model for decreasing hazard and its inferential properties, *Reliability: Theory & Applications* **16**(2(62)): 230–246.
- Singh, B. P., Singh, S. and Das, U. D. 2020. A general class of new continuous mixture distribution and application, *J. Math. Comput. Sci.* **11**(1): 585–602.
- Stacy, E. W. 1962. A generalization of the gamma distribution, *The Annals of mathematical statistics* **33**(3): 1187–1192.
- Stacy, E. W. and Mihram, G. A. 1965. Parameter estimation for a generalized gamma distribution, *Mathematical Sciences* **7**(3): 349–358.
- Weibull, W. and Sweden, S. 1951. A statistical distribution function of wide applicability, *Journal of applied mechanics* **18**(3): 293–297.
- Xu, K., Xie, M., Tang, L. C. and Ho, S. L. 2003. Application of neural networks in forecasting engine systems reliability, *Applied Soft Computing* **2**(4): 255–268.