

An Analysis of Two-State Retrial Queueing Model with Feedback having Impatient Customers

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ABSTRACT

Present paper considers a two dimensional retrial queueing model with feedback having impatient customers. If the server is idle on customer's arrival it is served immediately else it may balk because of impatience. If the server is busy the customer joins the orbit and retries for service until it finds server idle or may renege from the orbit. After service the unsatisfied customer may join the orbit as a feedback customer. The primary as well as secondary calls follow Poisson process. Service times follow Exponential distribution. The transient probabilities for exact number of arrivals and departures when the server is idle or busy are obtained. Some performance measures, numerical and graphical solutions are also computed

Keywords: queueing, retrial, feedback, impatience.

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1 Introduction

Recently, significant contribution has been provided to retrial queueing systems. In contrast to standard queueing systems there is no waiting space in retrial queueing systems. Infact, on encountering a busy server in retrial queueing systems, the customers instead of leaving the system forever, joins the orbit (virtual queue) and retries for service from orbit after a random period of time. These systems are highly applicable in restaurant reservations, telecommunication systems, computer systems. The simplest and most commonly referred example of such system is when a customer trying to connect to a call gets a busy line, he retries to connect after a random interval of time. (Falin and Templeton, 1997), (Artalejo and Gómez-Corral, 2008) are the books available as a great source for retrial queues.

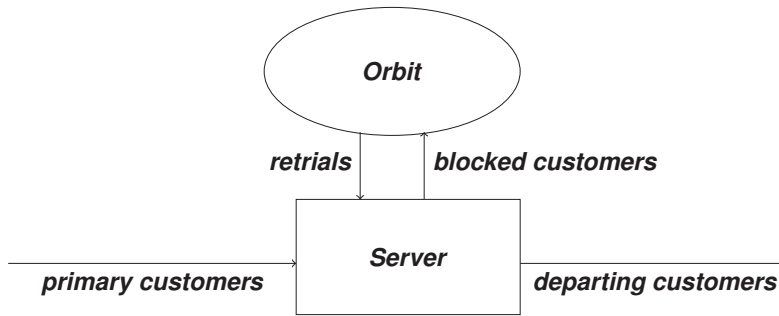


Figure 1: Basic Structure of a Retrial Queueing System

(Pegden and Rosenshine, 1982) analyzed 'Some new results for the $M/M/1$ queue' in which a closed form solution with finite sums is obtained for the probability that exactly i number of arrivals, j services occur over a time interval t . (Singla and Kalra, 2018) worked on 'Performance analysis of a two-state queueing model with retrials'.

In case of higher demand for any service, reneged and balked i.e., impatient customers are usually observed. A wide study has been done in this direction for both standard and re-trial queueing systems. (Abou-El-Ata and Hariri, 1992) analyzed 'The $M/M/c/N$ queue with balking and reneging' where the steady-state distribution of number of units in the system is derived. (Garg and Kumar, 2012) worked on 'A Single Server Retrial Queue with Impatient Customers' in which difference-differential equations are solved recursively to find the transient probabilities of exact number of arrivals and departures from the orbit. (Singla and Kalra, 2020) published 'A two-state multiserver retrial queueing model with balking'.

Dissatisfaction is a common phenomenon that comes into picture in many real life situations. For example: when a message faces a failed transmission it can be sent again. (Kumar and Raja, 2006) published 'On multiserver feedback retrial queues with balking and control retrial rate' in which the optimization of retrial rate and specific probabilistic descriptors of the system are obtained. (Chang, Liu and Ke, 2018) analyzed 'On an unreliable-server retrial queue with customer feedback and impatience' where the system is analyzed as a process of quasi-birth-and-death (QBD) and the quasi-progression algorithm is applied to compute the rate matrix of QBD model.

The paper is structured in sections as: In section 2 the model is described both theoretically and mathematically. The solution of the model is given in section 3 where the time-dependent probabilities are obtained. Section 4 gives verification of results. The numerical and graphical representations of the system are illustrated in section 5 where effect of various parameters is studied. In section 6 busy period probabilities are obtained including the numerical and graphical results. Finally, the paper is concluded in section 7 and followed by references at the end.

2 Model Description

We considered a two-state retrial queueing model with feedback having impatient customers. The primary or fresh customers arrive at system following Poisson process. On arrival if the entering customer finds the system free it is served immediately else it may balk from the system or may join the orbit and retry for service as a repeated call or secondary customer. The secondary customers also follow Poisson process. The customers retrying from orbit in case getting busy server may renege from the orbit. Also, if the customer after service feels unsatisfied may join the orbit in order to obtain satisfied service. Service times follow exponential distribution.

- **Arrival Process:** The primary calls arrive at the system following Poisson process with mean arrival rate λ .
- **Retrial Process:** The secondary calls from the orbit also arrive at the system according to Poisson process with mean retrial rate θ .
- **Impatience:** The fresh customers on encountering busy server may balk with probability $(1 - a_1)$, $a_1 > 0$. Also, the customers retrying for service from orbit as secondary customers on encountering busy server may renege with probability $(1 - a_2)$, $a_2 > 0$.
- **Feedback Rule:** After receiving service, the customer joins the orbit with probability γ (i.e., when unsatisfied) and departs from the system with probability $1 - \gamma$.
- **Service Process:** The service times follow Exponential distribution with parameter μ .

The input flow of primary calls, intervals between repetitions, service times are statistically independent.

Laplace Transformation of $\bar{f}(s)$ of $f(t)$ is given by:

$$\bar{f}(s) = \int_0^{\infty} e^{-st} f(t) dt; \quad \text{Re}(s) > 0$$

The Laplace Inverse of

$$\frac{Q(p)}{P(p)} = \sum_{k=1}^n \sum_{l=1}^{m_k} \frac{t^{m_k-l} e^{a_k t}}{(m_k - l)!(l - 1)!} \times \left(\frac{d}{dp} \right)^{l-1} \frac{Q(p)}{P(p)} \quad \forall p = a_k, a_i \neq a_k \text{ for } i \neq k$$

where

$$P(p) = (p - a_1)^{m_1} (p - a_2)^{m_2} \dots (p - a_n)^{m_n}$$

$Q(p)$ is a polynomial of degree $< m_1 + m_2 + m_3 + \dots + m_n - 1$

The Laplace Inverse of

$$\bar{N}_{n_1, n_2, n_3}^{a, b, c}(s) = \frac{1}{(s+a)^{n_1} + (s+b)^{n_2} + (s+c)^{n_3}} is$$

$$N_{n_1, n_2, n_3}^{a, b, c}(t) = \sum_{l=1}^{n_3} \sum_{m=1}^l \frac{e^{-at} t^{n_3-l} (-1)^{m+1} \binom{l-1}{m-1} \left(\prod_{g_1=0}^{l-m-1} (n_1 + g_1) \right) \left(\prod_{g_2=0}^{m-2} (n_2 + g_2) \right)}{(n_3-l)! (m-1)! (b-a)^{n_2+m-1} (c-a)^{n_1+l-m}}$$

$$+ \sum_{l=1}^{n_2} \sum_{m=1}^l \frac{e^{-bt} t^{n_2-l} (-1)^{m+1} \binom{l-1}{m-1} \left(\prod_{g_1=0}^{l-m-1} (n_1 + g_1) \right) \left(\prod_{g_2=0}^{m-2} (n_3 + g_2) \right)}{(n_2-l)! (m-1)! (a-b)^{n_3+m-1} (c-b)^{n_1+l-m}}$$

$$+ \sum_{l=1}^{n_1} \sum_{m=1}^l \frac{e^{-ct} t^{n_1-l} (-1)^{m+1} \binom{l-1}{m-1} \left(\prod_{g_1=0}^{l-m-1} (n_2 + g_1) \right) \left(\prod_{g_2=0}^{m-2} (n_3 + g_2) \right)}{(n_1-l)! (m-1)! (a-c)^{n_3+m-1} (b-c)^{n_2+l-m}}$$

If $L^{-1}\{f(s)\} = F(t)$ and $L^{-1}\{g(s)\} = G(t)$, then

$$L^{-1}\{f(s)g(s)\} = \int_0^1 F(u)G(t-u)du = F * G,$$

$F * G$ is called the convolution of F and G .

2.1 The Two-Dimensional State Model

2.1.1 Definitions

$P_{i,j,0}(t)$ = Probability that there are exactly i number of arrivals, j number of departures from the system by time t when server is idle.

$P_{i,j,1}(t)$ = Probability that there are exactly i number of arrivals, j number of departures from the system by time t when server is busy.

$P_{i,j}(t)$ = Probability that there are exactly i number of arrivals, j number of departures from the system by time t .

$$P_{i,j}(t) = P_{i,j,0}(t) + P_{i,j,1}(t) \quad \forall i, j; \quad i \geq j$$

$$P_{i,j,0}(t) = 0; \quad i < j \quad P_{i,j,1}(t) = 0; \quad i \leq j$$

Initially

$$P_{0,0,0}(0) = 1; \quad P_{i,j,0}(0) = 0 \text{ and } P_{i,j,1}(0) = 0; \quad i \neq j$$

2.1.2 Difference-Differential Equations Governing the System

$$\frac{d}{dt}P_{i,j,0}(t) = -(\lambda + (i-j)\theta)P_{i,j,0}(t) + \mu(1-\gamma)P_{i,j-1,1}(t) + \mu\gamma P_{i,j,1}(t); \quad i \geq j \geq 0 \quad (2.1)$$

$$\frac{d}{dt}P_{1,0,1}(t) = -(\lambda a_1 + \mu)P_{1,0,1}(t) + \lambda P_{0,0,0}(t) + \theta P_{1,0,0}(t) \quad (2.2)$$

$$\begin{aligned} \frac{d}{dt}P_{i,j,1}(t) = & -(\lambda a_1 + \mu + (i-j-1)\theta(1-a_2))P_{i,j,1}(t) + \lambda P_{i-1,j,0}(t) + \\ & \lambda a_1(1-\delta_{i-1,j})P_{i-1,j,1}(t) + (i-j)\theta P_{i,j,0}(t) + (i-j)\theta(1-a_2)P_{i,j-1,1}(t); \\ & i > j \geq 0 \end{aligned} \quad (2.3)$$

Using Laplace Transform $\bar{f}(s)$ of $f(t)$ given by

$$\bar{f}(s) = \int_0^\infty e^{-st} f(t) dt; \quad \operatorname{Re}(s) > 0$$

and using initial condition in equations (2.1) to (2.3), we have

$$(s + \lambda)\bar{P}_{0,0,0}(s) = \bar{P}(0) \quad (2.4)$$

$$(s + \lambda + (i-j)\theta)\bar{P}_{i,j,0}(s) = \mu(1-\gamma)\bar{P}_{i,j-1,1}(s) + \mu\gamma\bar{P}_{i,j,1}(s); \quad i \geq j \geq 0$$

$$(s + \lambda a_1 + \mu)\bar{P}_{1,0,1}(s) = \lambda\bar{P}_{0,0,0}(s) + \theta\bar{P}_{1,0,0}(s) \quad (2.5)$$

$$\begin{aligned} (s + \lambda a_1 + \mu + (i-j-1)\theta(1-a_2))\bar{P}_{i,j,1}(s) = & \lambda\bar{P}_{i-1,j,0}(s) + \lambda a_1(1-\delta_{i-1,j})\bar{P}_{i-1,j,1}(s) \\ & (i-j)\theta\bar{P}_{i,j,0}(s) + (i-j)\theta(1-a_2)\bar{P}_{i,j-1,1}(s); \quad i > j \geq 0 \end{aligned} \quad (2.6)$$

where

$$\delta_{i-1,j} = \begin{cases} 1; & i-1=j \\ 0; & \text{otherwise} \end{cases}$$

3 Solution of the Problem

Solving equations (2.4) -(2.6) recursively, we get:

$$\bar{P}_{0,0,0}(s) = \frac{1}{s + \lambda} \quad (3.1)$$

$$\bar{P}_{1,0,1}(s) = \frac{\lambda}{s + \lambda a_1 + \mu} \left(\frac{1}{s + \lambda} \right) + \frac{\theta}{s + \lambda a_1 + \mu} \bar{P}_{1,0,0}(s) \quad (3.2)$$

$$\bar{P}_{1,1,0}(s) = \frac{\mu(1-\gamma)}{s + \lambda} \bar{P}_{1,0,1}(s) \quad (3.3)$$

$$\bar{P}_{i,0,0}(s) = \frac{\mu\gamma}{s + \lambda + i\theta} \bar{P}_{i,0,1}(s); \quad i \geq 1 \quad (3.4)$$

$$\bar{P}_{i,i,0}(s) = \frac{\mu(1-\gamma)}{s+\lambda} \left[\frac{\lambda}{s+\lambda a_1+\mu} \bar{P}_{i-1,i-1,0}(s) + \frac{\theta}{s+\lambda a_1+\mu} \bar{P}_{i,i-1,0}(s) + \frac{\theta(1-a_2)}{s+\lambda a_1+\mu} \bar{P}_{i,i-2,1}(s) \right];$$

$$i \geq 1 \quad (3.5)$$

$$\bar{P}_{i,1,0}(s) = \frac{\mu(1-\gamma)}{s+\lambda+(i-1)\theta} \bar{P}_{i,0,1}(s) + \frac{\mu\gamma}{s+\lambda+(i-1)\theta} \bar{P}_{i,1,1}(s); i \geq 2 \quad (3.6)$$

$$\bar{P}_{i,2,0}(s) = \frac{\mu(1-\gamma)}{s+\lambda+(i-2)\theta} \bar{P}_{i,1,1}(s) + \frac{\mu\gamma}{s+\lambda+(i-2)\theta} \bar{P}_{i,2,1}(s); i \geq 3 \quad (3.7)$$

$$\bar{P}_{i,1,1}(s) = \frac{\lambda}{s+\lambda a_1+\mu+(i-2)\theta(1-a_2)} \bar{P}_{i-1,1,0}(s) + \frac{\lambda a_1}{s+\lambda a_1+\mu+(i-2)\theta(1-a_2)} \bar{P}_{i-1,1,1}(s)$$

$$+ \frac{(i-1)\theta}{s+\lambda a_1+\mu+(i-2)\theta(1-a_2)} \bar{P}_{i,1,0}(s) + \frac{(i-1)\theta(1-a_2)}{s+\lambda a_1+\mu+(i-2)\theta(1-a_2)} \bar{P}_{i,0,1}(s);$$

$$i \geq 3 \quad (3.8)$$

$$\bar{P}_{i,i-1,1}(s) = \frac{\lambda}{s+\lambda a_1+\mu} \bar{P}_{i-1,i-1,0}(s) + \frac{\theta}{s+\lambda a_1+\mu} \bar{P}_{i,i-1,0}(s) + \frac{\theta(1-a_2)}{s+\lambda a_1+\mu} \bar{P}_{i,i-2,1}(s);$$

$$i \geq 2 \quad (3.9)$$

$$\bar{P}_{i,j,1}(s) = \sum_{k=1}^{i-j} \left\{ \prod_{m=k}^{i-j-1} \left(\frac{1}{s+\lambda a_1+\mu+m\theta(1-a_2)} \right)^{\psi'_k(s)} \lambda^{\psi'_k(s)} (\lambda a_1)^{(i-j-k-1)\psi'_k(s)} \right.$$

$$\left. \eta'_k(s) \bar{P}_{j+k,j,0}(s) \right\} + \sum_{k=1}^{i-j} \left\{ \prod_{m=k}^{i-j-1} (k+1)\theta(1-a_2)(\lambda a_1)^{i-j-k-1} \right.$$

$$\left. \left(\frac{1}{s+\lambda a_1+\mu+m\theta(1-a_2)} \right) \bar{P}_{j+k+1,j-1,1}(s) \right\} + \prod_{p=1}^{i-j-1} \left(\frac{1}{s+\lambda a_1+\mu+p\theta(1-a_2)} \right)$$

$$(\lambda a_1)^{i-j-1} \bar{P}_{j+1,j,1}(s); \quad i \geq j+2, \quad j > 1 \quad (3.10)$$

where

$$\eta'_k(s) = \begin{cases} 1 & k=1 \\ 1 + \frac{k\theta a_1}{s+\lambda a_1+\mu+(k-1)\theta(1-a_2)} & k=2 \text{ to } i-j \\ \frac{k\theta}{s+\lambda a_1+\mu+(k-1)\theta(1-a_2)} & k=i-j+1 \end{cases}$$

$$\psi'_k = \begin{cases} 1 & k=1 \text{ to } i-j \\ 0 & k=i-j+1 \end{cases}$$

$$\begin{aligned}
 \bar{P}_{i,j,0}(s) &= \frac{\mu(1-\gamma)}{s+\lambda+(i-j)\theta} \sum_{k=1}^{i-j+1} \left\{ \prod_{m=k}^{i-j} \left(\frac{1}{s+\lambda a_1+\mu+m\theta(1-a_2)} \right)^{\psi'_k(s)} \lambda^{\psi'_k(s)} (\lambda a_1)^{(i-j-k)\psi'_k(s)} \right. \\
 &\quad \left. \eta'_k(s) \bar{P}_{j+k-1,j-1,0}(s) \right\} + \sum_{k=1}^{i-j} \left\{ \prod_{m=k}^{i-j} (k+1)\theta(1-a_2)(\lambda a_1)^{i-j-k} \right. \\
 &\quad \left(\frac{1}{s+\lambda a_1+\mu+m\theta(1-a_2)} \right) \bar{P}_{j+k,j-2,1}(s) \Big\} + \prod_{p=1}^{i-j} \left(\frac{1}{s+\lambda a_1+\mu+p\theta(1-a_2)} \right) \\
 &\quad (\lambda a_1)^{i-j} \bar{P}_{j,j-1,1}(s) + \frac{\mu\gamma}{s+\lambda+(i-j)\theta} \sum_{k=1}^{i-j} \left\{ \prod_{m=k}^{i-j-1} \left(\frac{1}{s+\lambda a_1+\mu+m\theta(1-a_2)} \right)^{\psi''_k(s)} \right. \\
 &\quad \left. \lambda^{\psi''_k(s)} (\lambda a_1)^{(i-j-k-1)\psi''_k(s)} \phi'_k(s) \bar{P}_{j+k,j,0}(s) \right\} + \sum_{k=1}^{i-j-1} \left\{ \prod_{m=k}^{i-j-1} (k+1)\theta(1-a_2)(\lambda a_1)^{i-j-k-1} \right. \\
 &\quad \left(\frac{1}{s+\lambda a_1+\mu+m\theta(1-a_2)} \right) \bar{P}_{j+k,j-1,1}(s) \Big\} + \prod_{p=1}^{i-j-1} \left(\frac{1}{s+\lambda a_1+\mu+p\theta(1-a_2)} \right) \\
 &\quad (\lambda a_1)^{i-j-1} \bar{P}_{j+1,j,1}(s); \qquad \qquad \qquad i > j > 1 \qquad \qquad \qquad (3.11)
 \end{aligned}$$

where

$$\eta'_k(s) = \begin{cases} 1 & k = 1 \\ 1 + \frac{k\theta a_1}{s+\lambda a_1+\mu+(k-1)\theta(1-a_2)} & k = 2 \text{ to } i-j \\ \frac{k\theta}{s+\lambda a_1+\mu+(k-1)\theta(1-a_2)} & k = i-j+1 \end{cases}$$

$$\psi'_k = \begin{cases} 1 & k = 1 \text{ to } i-j \\ 0 & k = i-j+1 \end{cases}$$

$$\phi'_k(s) = \begin{cases} 1 & k = 1 \\ 1 + \frac{k\theta a_1}{s+\lambda a_1+\mu+(k-1)\theta(1-a_2)} & k = 2 \text{ to } i-j-1 \\ \frac{k\theta}{s+\lambda a_1+\mu+(k-1)\theta(1-a_2)} & k = i-j \end{cases}$$

$$\psi''_k = \begin{cases} 1 & k = 1 \text{ to } i-j-1 \\ 0 & k = i-j \end{cases}$$

Taking the Laplace inverse of (3.1)-(3.11), we get the transient state probabilities as:

$$P_{0,0,0}(t) = e^{-\lambda t} \quad (3.12)$$

$$P_{1,0,1}(t) = \lambda e^{-\lambda t} \left\{ \frac{1}{\frac{\mu}{a_1}} - \frac{e\left(-\frac{\mu}{a_1}\right)t}{\frac{\mu}{a_1}} \right\} \quad (3.13)$$

$$P_{1,1,0}(t) = \mu(1-\gamma)e^{-\lambda t} * P_{1,0,1}(t) \quad (3.14)$$

$$P_{i,0,0}(t) = \mu\gamma e^{-(\lambda+i\theta)t} * P_{i,0,1}(t); \quad i \geq 1 \quad (3.15)$$

$$\begin{aligned} P_{i,i,0}(t) = & \mu(1-\gamma)\lambda e^{-\lambda t} \left\{ \frac{1}{\frac{\mu}{a_1}} - \frac{e\left(-\frac{\mu}{a_1}\right)t}{\frac{\mu}{a_1}} \right\} * P_{i-1,i-1,0}(t) \\ & + \mu(1-\gamma)\theta e^{-\lambda t} \left\{ \frac{1}{\frac{\mu}{a_1}} - \frac{e\left(-\frac{\mu}{a_1}\right)t}{\frac{\mu}{a_1}} \right\} * P_{i,i-1,0}(t) \\ & + \mu(1-\gamma)\theta(1-a_2)e^{-\lambda t} \left\{ \frac{1}{\frac{\mu}{a_1}} - \frac{e\left(-\frac{\mu}{a_1}\right)t}{\frac{\mu}{a_1}} \right\} * P_{i,i-2,1}(t); i \geq 2 \end{aligned} \quad (3.16)$$

$$P_{i,1,0}(t) = \mu(1-\gamma)e^{-(\lambda+(i-1)\theta)t} * P_{i,0,1}(t) + \mu\gamma e^{-(\lambda+(i-1)\theta)t} * P_{i,1,1}(t); i \geq 2 \quad (3.17)$$

$$P_{i,2,0}(t) = \mu(1-\gamma)e^{-(\lambda+(i-1)\theta)t} * P_{i,1,1} + \mu\gamma e^{-(\lambda+(i-1)\theta)t} * P_{i,2,1}(t); \quad i \geq 3 \quad (3.18)$$

$$\begin{aligned} P_{i,1,1}(t) = & \lambda e^{-(\lambda a_1 + \mu + (i-2)\theta(1-a_2))t} * P_{i-1,1,0}(t) + \lambda a_1 e^{-(\lambda a_1 + \mu + (i-2)\theta(1-a_2))t} * P_{i-1,1,1}(t) \\ & + (i-1)\theta e^{-(\lambda a_1 + \mu + (i-2)\theta(1-a_2))t} * P_{i,1,0}(t) + (i-1)\theta(1-a_2)e^{-(\lambda a_1 + \mu + (i-2)\theta(1-a_2))t} * P_{i,0,1}(t); \\ & i \geq 3 \end{aligned} \quad (3.19)$$

$$\begin{aligned} P_{i,i-1,1}(t) = & \lambda e^{-(\lambda+\mu)t} * P_{i-1,i-1,0}(t) + \theta e^{-(\lambda+\mu)t} * P_{i,i-1,0}(t) + \theta(1-a_2)e^{-(\lambda+\mu)t} * P_{i,i-2,1}(t); \\ & i \geq 2 \end{aligned} \quad (3.20)$$

$$\begin{aligned} P_{i,j,1}(t) = & \lambda(\lambda a_1)^{i-j-2} \left\{ \prod_{m=1}^{i-j-1} e^{-(\lambda a_1 + \mu + m\theta(1-a_2))t} \frac{t^{m-1}}{(m-1)!} \right\} * P_{j+1,j,0}(t) + \sum_{k=2}^{i-j-1} \lambda(\lambda a_1)^{i-j-k-1} \\ & \left\{ \prod_{m=k}^{i-j-1} e^{-(\lambda a_1 + \mu + m\theta(1-a_2))t} \right\} \frac{t^{m-k}}{(m-k)!} * P_{j+k,j,0}(t) + \sum_{k=2}^{i-j-1} (k\theta)(\lambda a_1)^{i-j-k} \\ & \left\{ \prod_{m=k-1}^{i-j-1} e^{-(\lambda a_1 + \mu + m\theta(1-a_2))t} \right\} \frac{t^{m-k}}{(m-k)!} * P_{j+k,j,0}(t) + (i-j)\theta e^{-(\lambda a_1 + \mu + m\theta(1-a_2))t} \\ & * P_{i,j,0}(t) + \sum_{k=1}^{i-j-1} (k+1)\theta(1-a_2)(\lambda a_1)^{i-j-k-1} \left\{ \prod_{m=k}^{i-j-1} e^{-(\lambda a_1 + \mu + m\theta(1-a_2))t} \right\} \\ & \frac{t^{m-k}}{(m-k)!} * P_{j+k+1,j-1,1}(t) + (\lambda a_1)^{i-j-k-1} \left\{ \prod_{p=1}^{i-j-1} e^{-(\lambda a_1 + \mu + p\theta(1-a_2))t} \right\} \\ & \frac{t^{p-1}}{(p-1)!} * P_{j+1,j,1}(t); \quad i \geq j+2, j > 1 \end{aligned} \quad (3.21)$$

$$\begin{aligned}
 P_{i,j,0}(t) &= \mu(1-\gamma)\lambda(\lambda a_1)^{i-j-1}e^{-(\lambda+(i-j)\theta)t} \left\{ \prod_{m=1}^{i-j} \frac{1}{\left(\frac{\mu}{a_1} + \frac{m\theta a_2}{a_1}\right)^m} - e^{\left(\frac{\mu}{a_1} + \frac{m\theta(1-a_2)}{a_1}\right)t} \right. \\
 &\quad \left. \sum_{r=0}^{m-1} \frac{t^r}{r!} \frac{1}{\left(\frac{\mu}{a_1} + \frac{m\theta(1-a_2)}{a_1}\right)^{m-r}} \right\} * P_{j,j-1,0}(t) + \mu(1-\gamma)\lambda e^{-(\lambda+(i-j)\theta)t} \sum_{k=2}^{i-j} (\lambda a_1)^{i-j-k} \\
 &\quad \left\{ \prod_{m=k}^{i-j} \frac{1}{\left(\frac{\mu}{a_1} + \frac{m\theta a_2}{a_1}\right)^{m-k+1}} - e^{\left(\frac{\mu}{a_1} + \frac{m\theta(1-a_2)}{a_1}\right)t} \sum_{r=0}^{m-k} \frac{t^r}{r!} \frac{1}{\left(\frac{\mu}{a_1} + \frac{m\theta(1-a_2)}{a_1}\right)^{m-k+1-r}} \right\} * P_{j+k-1,j-1,0}(t) \\
 &\quad + \mu(1-\gamma)e^{-(\lambda+(i-j)\theta)t} \sum_{k=2}^{i-j} k\theta(\lambda a_1)^{i-j-k+1} \left\{ \prod_{m=k-1}^{i-j} \frac{1}{\left(\frac{\mu}{a_1} + \frac{m\theta a_2}{a_1}\right)^{m-k+2}} - e^{\left(\frac{\mu}{a_1} + \frac{m\theta(1-a_2)}{a_1}\right)t} \right. \\
 &\quad \left. \sum_{r=0}^{m-k+1} \frac{t^r}{r!} \frac{1}{\left(\frac{\mu}{a_1} + \frac{m\theta(1-a_2)}{a_1}\right)^{m-k+2-r}} \right\} * P_{j+k-1,j-1,0}(t) + \mu(1-\gamma)(i-j+1)\theta e^{-(\lambda+(i-j)\theta)t} \\
 &\quad \left\{ \frac{1}{\left(\frac{\mu}{a_1} + \frac{m\theta(1-a_2)}{a_1}\right)} - \frac{e^{-\left(\frac{\mu}{a_1} + \frac{m\theta(1-a_2)}{a_1}\right)t}}{\left(\frac{\mu}{a_1} + \frac{m\theta(1-a_2)}{a_1}\right)} \right\} * P_{i,j-1,0}(t) + \mu(1-\gamma)e^{-(\lambda+(i-j)\theta)t} \sum_{k=1}^{i-j} (k+1)\theta(1-a_2) \\
 &\quad (\lambda a_1)^{i-j-k} \left\{ \prod_{m=k}^{i-j} \frac{1}{\left(\frac{\mu}{a_1} + \frac{m\theta a_2}{a_1}\right)^{m-k+1}} - e^{\left(\frac{\mu}{a_1} + \frac{m\theta(1-a_2)}{a_1}\right)t} \sum_{r=0}^{m-k} \frac{t^r}{r!} \frac{1}{\left(\frac{\mu}{a_1} + \frac{m\theta(1-a_2)}{a_1}\right)^{m-k+1-r}} \right\} \\
 &\quad * P_{j+k,j-2,1}(t) + \mu(1-\gamma)(\lambda)^{i-j}e^{-(\lambda+(i-j)\theta)t} \left\{ \prod_{p=1}^{i-j} \frac{1}{\left(\frac{\mu}{a_1} + \frac{p\theta a_2}{a_1}\right)^p} - e^{\left(\frac{\mu}{a_1} + \frac{p\theta(1-a_2)}{a_1}\right)t} \right. \\
 &\quad \left. \sum_{r=0}^{p-1} \frac{t^r}{r!} \frac{1}{\left(\frac{\mu}{a_1} + \frac{p\theta(1-a_2)}{a_1}\right)^{p-r}} \right\} * P_{j,j-1,1}(t) + \mu\gamma\lambda(\lambda a_1)^{i-j-2}e^{-(\lambda+(i-j)\theta)t} \left\{ \prod_{m=1}^{i-j-1} \frac{1}{\left(\frac{\mu}{a_1} + \frac{m\theta a_2}{a_1}\right)^m} - \right. \\
 &\quad \left. e^{\left(\frac{\mu}{a_1} + \frac{m\theta(1-a_2)}{a_1}\right)t} \sum_{r=0}^{m-1} \frac{t^r}{r!} \frac{1}{\left(\frac{\mu}{a_1} + \frac{m\theta(1-a_2)}{a_1}\right)^{m-r}} \right\} * P_{j+1,j,0}(t) + \mu\gamma\lambda e^{-(\lambda+(i-j)\theta)t} \sum_{k=2}^{i-j-1} (\lambda a_1)^{i-j-k-1} \\
 &\quad \left\{ \prod_{m=k}^{i-j-1} \frac{1}{\left(\frac{\mu}{a_1} + \frac{m\theta a_2}{a_1}\right)^{m-k+1}} - e^{\left(\frac{\mu}{a_1} + \frac{m\theta(1-a_2)}{a_1}\right)t} \sum_{r=0}^{m-k} \frac{t^r}{r!} \frac{1}{\left(\frac{\mu}{a_1} + \frac{m\theta(1-a_2)}{a_1}\right)^{m-k+1-r}} \right\} * P_{j+k,j,0}(t) \\
 &\quad + \mu\gamma e^{-(\lambda+(i-j)\theta)t} \sum_{k=2}^{i-j-1} k\theta(\lambda a_1)^{i-j-k} \left\{ \prod_{m=k-1}^{i-j-1} \frac{1}{\left(\frac{\mu}{a_1} + \frac{m\theta a_2}{a_1}\right)^{m-k+2}} - e^{\left(\frac{\mu}{a_1} + \frac{m\theta(1-a_2)}{a_1}\right)t} \sum_{r=0}^{m-k+1} \frac{t^r}{r!} \right. \\
 &\quad \left. \frac{1}{\left(\frac{\mu}{a_1} + \frac{m\theta(1-a_2)}{a_1}\right)^{m-k+2-r}} \right\} * P_{j+k,j,0}(t) + \mu\gamma(i-j)\theta e^{-(\lambda+(i-j)\theta)t} \left\{ \frac{1}{\left(\frac{\mu}{a_1} + \frac{m\theta(1-a_2)}{a_1}\right)} - \right. \\
 &\quad \left. \frac{e^{-\left(\frac{\mu}{a_1} + \frac{m\theta(1-a_2)}{a_1}\right)t}}{\left(\frac{\mu}{a_1} + \frac{m\theta(1-a_2)}{a_1}\right)} \right\} * P_{i,j,0}(t) + \mu\gamma e^{-(\lambda+(i-j)\theta)t} \sum_{k=1}^{i-j-1} (k+1)\theta(1-a_2)(\lambda a_1)^{i-j-k-1} \\
 &\quad \left\{ \prod_{m=k}^{i-j-1} \frac{1}{\left(\frac{\mu}{a_1} + \frac{m\theta a_2}{a_1}\right)^{m-k+1}} - e^{\left(\frac{\mu}{a_1} + \frac{m\theta(1-a_2)}{a_1}\right)t} \sum_{r=0}^{m-k} \frac{t^r}{r!} \frac{1}{\left(\frac{\mu}{a_1} + \frac{m\theta(1-a_2)}{a_1}\right)^{m-k+1-r}} \right\} \\
 &\quad * P_{j+k+1,j-1,1}(t) + \mu\gamma(\lambda)^{i-j-1}e^{-(\lambda+(i-j)\theta)t} \left\{ \prod_{p=1}^{i-j-1} \frac{1}{\left(\frac{\mu}{a_1} + \frac{p\theta a_2}{a_1}\right)^p} - e^{\left(\frac{\mu}{a_1} + \frac{p\theta(1-a_2)}{a_1}\right)t} \right. \\
 &\quad \left. \sum_{r=0}^{p-1} \frac{t^r}{r!} \frac{1}{\left(\frac{\mu}{a_1} + \frac{p\theta(1-a_2)}{a_1}\right)^{p-r}} \right\} * P_{j+1,j,1}(t); \quad i > j > 1 \quad (3.22)
 \end{aligned}$$

4 Verification of Results

- The Laplace transform of $\bar{P}_i(s)$ is given as:

$$\bar{P}_i(s) = \sum_{j=0}^i \bar{P}_{i,j}(s) = \frac{\lambda^i}{(s + \lambda)^{i+1}}; \quad i > 0$$

and its Inverse Laplace is:

$$P_i(t) = \frac{e^{-\lambda t} (\lambda t)^i}{i!}$$

confirms that primary arrivals follow Poisson process.

- The probability that exactly j customers depart from the system by time t is given as:

$$P_{.j}(t) = \sum_{i=j}^{\infty} P_{i,j}(t)$$

- Summing equations (2.4)-(2.6) over i and j we get:

$$\sum_{i=0}^{\infty} \sum_{j=0}^i \{ \bar{P}_{i,j,0}(s) + \bar{P}_{i,j,1}(s) \} = \frac{1}{s}$$

and hence

$$\sum_{i=0}^{\infty} \sum_{j=0}^i \{ P_{i,j,0}(t) + P_{i,j,1}(t) \} = 1$$

- Define $Q_{n,m}(t)$ = Probability that there are n customers in the orbit where $m = (0, 1)$ i.e., either the server is idle or busy at time t .

For idle server we represent it by probability $Q_{n,0}(t)$.

$$Q_{n,0}(t) = \sum_{j=0}^{\infty} P_{j+n,j,0}(t)$$

The number of customers in the orbit in this case are calculated with the following formula:

$$n = (\text{number of arrivals} - \text{number of departures})$$

When the server is busy, it is represented by the probability $Q_{n,1}(t)$

$$Q_{n,1}(t) = \sum_{j=0}^{\infty} P_{j+n+1,j,1}(t)$$

The number of customers in the orbit in this case is calculated by the following formula:

$n = (\text{number of arrivals} - \text{number of departures} - 1)$

Using above definitions in (2.1)-(2.3) and letting $\gamma=0$, the equations we get under statistical equilibrium are:

$$(\lambda + n\theta)Q_{n,0}(t) = \mu Q_{n,1}; \quad n \geq 0 \quad (4.1)$$

$$(\lambda a_1 + \mu + n\theta(1 - a_2))Q_{n,1} = \lambda Q_{n,0} + \lambda a_1 Q_{n-1,1} + (n+1)\theta Q_{n+1,0} + (n+1)\theta(1 - a_2)Q_{n+1,1}; \quad n \geq 2 \quad (4.2)$$

which coincides with the result (3.68) of (Falin and Templeton, 1997).

5 Numerical Solution and Graphical Representation

MATLAB programming is used to generate numerical solution for the case $\rho=0.7$, $\eta=0.6$, $\gamma=0.5$, $a_1 = 0.6$ and $a_2 = 0.5$. Observing the tables below for various time instants t , it is concluded that the sum of probabilities at any instant approaches to 1.

Table 1: At $t=1$

$P_{0,0,0}$	$P_{1,0,0}$	$P_{1,1,0}$	$P_{2,1,0}$	$P_{5,2,0}$	$P_{1,0,1}$	$P_{2,0,1}$	$P_{2,1,1}$	$P_{6,1,1}$	Sum
0.496	0.058	0.0708	0.0111	0	0.2585	0.0565	0.021	0	0.9719

Table 2: At $t=5$

$P_{0,0,0}$	$P_{1,0,0}$	$P_{1,1,0}$	$P_{2,0,0}$	$P_{2,1,0}$	$P_{2,2,0}$	$P_{3,0,0}$	$P_{3,1,0}$	$P_{3,2,0}$	$P_{3,3,0}$
0.0302	0.0291	0.0697	0.0146	0.0566	0.0586	0.0049	0.0255	0.045	0.0273

$P_{4,1,0}$	$P_{4,2,0}$	$P_{4,3,0}$	$P_{4,4,0}$	$P_{5,2,0}$	$P_{5,3,0}$	$P_{5,4,0}$	$P_{5,5,0}$	$P_{6,3,0}$	$P_{6,4,0}$
0.008	0.0193	0.0208	0.0086	0.0058	0.0087	0.0066	0.002	0.0026	0.0027

$P_{1,0,1}$	$P_{2,0,1}$	$P_{2,1,1}$	$P_{3,0,1}$	$P_{3,1,1}$	$P_{3,2,1}$	$P_{4,0,1}$	$P_{4,1,1}$	$P_{4,2,1}$	$P_{4,3,1}$
0.046	0.0445	0.0806	0.0221	0.0721	0.0576	0.0074	0.0338	0.05	0.0243

$P_{5,1,1}$	$P_{5,2,1}$	$P_{5,3,1}$	$P_{5,4,1}$	$P_{6,2,1}$	$P_{6,3,1}$	$P_{6,4,1}$	Sum
0.0108	0.0228	0.0209	0.0071	0.0072	0.0094	0.0061	0.9693

Table 3: At $t=15$

$P_{3,2,0}$	$P_{3,3,0}$	$P_{4,4,0}$	$P_{5,4,0}$	$P_{5,5,0}$	$P_{6,4,0}$	$P_{6,5,0}$	$P_{6,6,0}$	$P_{7,5,0}$	$P_{7,6,0}$
0.0029	0.0058	0.0119	0.0154	0.0179	0.0102	0.0222	0.021	0.0142	0.0252

$P_{7,7,0}$	$P_{8,5,0}$	$P_{8,6,0}$	$P_{8,7,0}$	$P_{8,8,0}$	$P_{4,3,1}$	$P_{5,3,1}$	$P_{5,4,1}$	$P_{6,3,1}$	$P_{6,4,1}$
0.02	0.0158	0.0544	0.1254	0.1535	0.0099	0.0135	0.0186	0.0092	0.0246

$P_{6,5,1}$	$P_{7,4,1}$	$P_{7,5,1}$	$P_{7,6,1}$	$P_{8,4,1}$	$P_{8,5,1}$	$P_{8,6,1}$	$P_{8,7,1}$	Sum
0.0262	0.0164	0.0337	0.0291	0.0118	0.042	0.0912	0.1005	0.9425

 Table 4: At $t=25$

$P_{6,5,0}$	$P_{6,6,0}$	$P_{7,6,0}$	$P_{7,7,0}$	$P_{8,6,0}$	$P_{8,7,0}$	$P_{8,8,0}$	$P_{8,6,1}$	$P_{8,7,1}$	Sum
0.0011	0.0019	0.0027	0.004	0.0122	0.1112	0.7444	0.0194	0.0863	0.9832

 Table 5: At $t=35$

$P_{0,0,0}$	$P_{5,5,0}$	$P_{8,7,0}$	$P_{8,8,0}$	$P_{5,4,1}$	$P_{8,7,1}$	Sum
0	0	0.0201	0.9629	0	0.0154	0.99984

The probabilities against time are plotted in following figures.

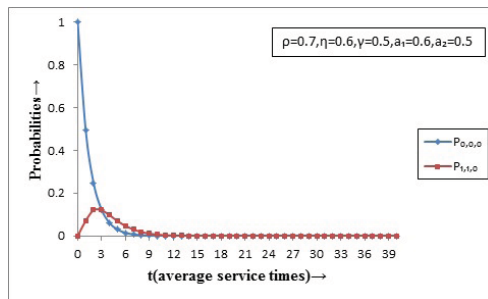


Figure 2

The probabilities $P_{0,0,0}$ and $P_{1,1,0}$ are plotted against time t (average service times) in figure 2. for the case when $\rho=0.7$, $\eta=0.6$, $\gamma=0.5$, $a_1=0.6$, $a_2=0.5$. It is clearly visible from the graph that after starting from value 1 at time $t = 0$ the probability $P_{0,0,0}$ decreases with time and approaches 0. But the probability $P_{1,1,0}$ increases from initial value 0 at $t = 0$ and then decreases with time.

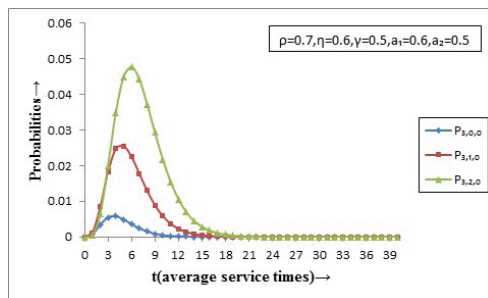


Figure 3

Figure 3 depicts the probabilities $P_{3,0,0}$, $P_{3,1,0}$ and $P_{3,2,0}$ plotted against time t for the case when $\rho=0.7$, $\eta=0.6$, $\gamma=0.5$, $a_1=0.6$, $a_2=0.5$. It is observed that all the probabilities increase from initial value 0 at $t = 0$ and then decrease. The highest value achieved by probability $P_{3,2,0}$ is greater than $P_{3,1,0}$ which in turn is larger than $P_{3,0,0}$.

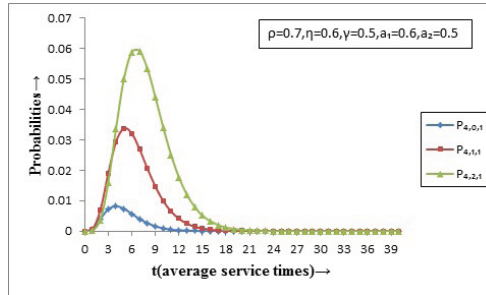


Figure 4

Figure 4 shows the comparison between the probabilities $P_{4,0,1}$, $P_{4,1,1}$ and $P_{4,2,1}$ when plotted against time t for the case $\rho=0.7$, $\eta=0.6$, $\gamma=0.5$, $a_1=0.6$, $a_2=0.5$. It is observed that all the probabilities start increasing in the beginning and then decreases. Also, the probabilities attain higher values for greater number of departures.

6 Busy Period Probabilities

The busy period distribution of the system as well as the server is discussed in this section. The probability that server is busy is given by

$$P(\text{Server is busy}) = \sum_{i>j \geq 0} P_{i,j,1}(t) \quad (6.1)$$

The probability that system is busy is given by

$$P(\text{System is busy}) = \sum_{i>j \geq 0} (P_{i,j,0}(t) + P_{i,j,1}(t)) \quad (6.2)$$

6.1 Numerical and Graphical Representation of Busy Period Probabilities

The numerical solution is obtained using MATLAB programming and following (Bunday, 1986) work. The probability for system busy and server busy are studied by varying the value of ρ and keeping other parameters constant at $\eta=0.6$, $\gamma=0.5$, $a_1=0.6$, $a_2=0.5$.

t	Probability(System Busy)			Probability(Server Busy)		
	$\rho=0.5$	$\rho=0.7$	$\rho=0.9$	$\rho=0.5$	$\rho=0.7$	$\rho=0.9$
0	0	0	0	0	0	0
1	0.335	0.4298	0.5131	0.2698	0.3506	0.4195
2	0.4947	0.6126	0.7017	0.3584	0.4484	0.5194
3	0.5907	0.7096	0.7924	0.4034	0.4946	0.5637
4	0.6521	0.767	0.8423	0.4323	0.5232	0.5905
5	0.6933	0.8032	0.8721	0.4525	0.5425	0.6078
6	0.7217	0.827	0.8907	0.467	0.5558	0.6182
7	0.7418	0.8431	0.9022	0.4776	0.5644	0.622
8	0.7562	0.8537	0.9079	0.4851	0.5687	0.6185
9	0.7665	0.8598	0.9075	0.4901	0.5685	0.6067
10	0.7736	0.8614	0.8992	0.4928	0.5632	0.55862

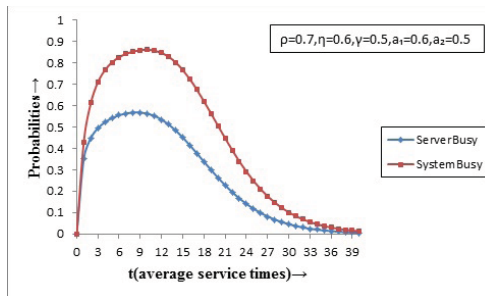


Figure 5

In figure 5 the probability of System busy and probability of Server busy are plotted against time t for the case when $\rho=0.7$, $\eta=0.6$, $\gamma=0.5$, $a_1=0.6$, $a_2=0.5$. It is observed that both the probabilities increase rapidly and then decrease gradually. Also, probability (System busy) remains higher than probability (Server busy) as required.

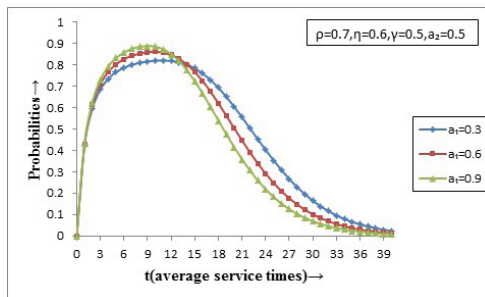


Figure 6

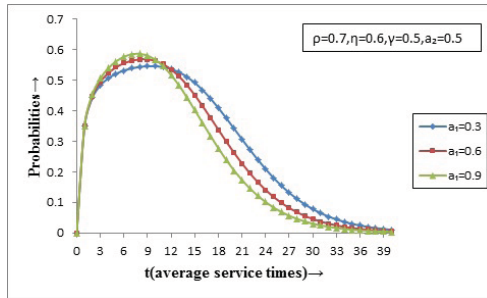


Figure 7

Figure 6 and Figure 7 shows the effect of a_1 on the probability (System busy) and probability (Server busy) respectively keeping other parameters constant. It is interpreted from both the graphs that probabilities are higher for larger values of a_1 for quite some time. After that these probabilities start decreasing and also the trend is reversed i.e., probabilities are smaller for larger values of a_1 .

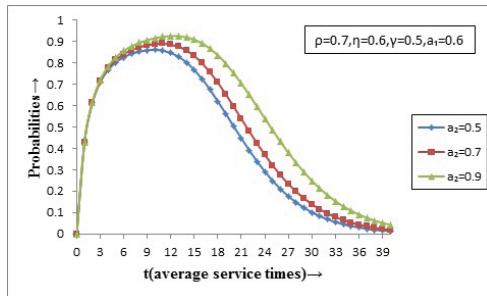


Figure 8

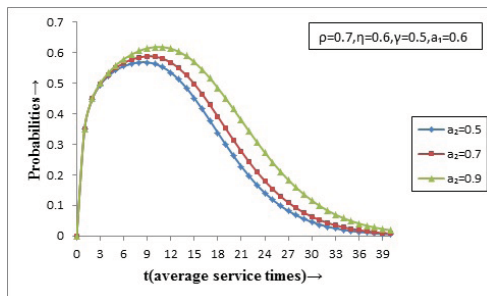


Figure 9

Figure 8 and 9 depict the effect of a_2 on the probability of system busy and probability of server busy respectively when $\rho=0.7$, $\eta=0.6$, $\gamma=0.5$, $a_1=0.6$. It is observed that probabilities increase in the beginning and then decrease gradually with time. Also, the probabilities in both the graphs are higher for larger values of a_2 .

7 Conclusion

In this paper a single server two-state retrial queueing model with the concepts of feedback and impatience is considered. As we know, managing impatient customers can lead to profit in business. The transient state probabilities for exact number of arrivals and departures from the system for busy or idle server are obtained. Numerical and graphical representation are given to study the effect of various parameters. Some performance measures are also obtained.

References

- Abou-El-Ata, M. and Hariri, A. 1992. The M/M/c/N queue with balking and reneging, *Computers & operations research* **19**(8): 713–716.
- Artalejo, J. R. and Gómez-Corral, A. 2008. *Retrial Queueing Systems: A Computational Approach*, Springer-Verlag Berlin Heidelberg.
- Bunday, B. D. 1986. *Basic queueing theory*, Arnold Baltimore (Md).
- Chang, F.-M., Liu, T.-H. and Ke, J.-C. 2018. On an unreliable-server retrial queue with customer feedback and impatience, *Applied Mathematical Modelling* **55**: 171–182.
- Falin, G. I. and Templeton, J. G. C. 1997. *Retrial queues*, CRC Press.
- Garg, P. C. and Kumar, S. 2012. A single server retrial queue with impatient customers, *Mathematical Journal of Interdisciplinary Sciences* **1**(1): 67–82.
- Kumar, B. K. and Raja, J. 2006. On multiserver feedback retrial queues with balking and control retrial rate, *Annals of Operations Research* **141**(1): 211–232.
- Pegden, C. D. and Rosenshine, M. 1982. Some new results for the m/m/1 queue, *Management Science* **28**(7): 821–828.
- Singla, N. and Kalra, S. 2018. Performance analysis of a two-state queueing model with retrials, *Journal of Rajasthan Academy of Physical Sciences* **17**(1): 81–100.
- Singla, N. and Kalra, S. 2020. A two-state multiserver retrial queueing model with balking, *Journal of Physics: Conference Series*, Vol. 1531, IOP Publishing, p. 012061.