

# Three-Dimensional Transformations and the Householder Transformation in Fibonacci and Lucas Three-Vectors

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## ABSTRACT

*In this study, local scaling, overall scaling, shearing, reflection, rotation, translation, and multiple transformations applications in three-dimensional space; by using the properties of Fibonacci and Lucas three-vectors, transformation matrices of Fibonacci and Lucas three-vectors have been obtained. Later, in a Fibonacci and Lucas three-vectors defined in this space, the Householder transform and the matrix corresponding to this transformation are given. By giving examples about each transformation, the figures were drawn in the Maple program.*

**Keywords:** Geometric Transformations, Fibonacci and Lucas three-vectors, Householder transformation.

**Mathematics Subject Classification:** 97G50, 51N25, 14Q65.

## 1. INTRODUCTION

It is not surprising to see that someone who studies with the Fibonacci numbers attain the expected results. Therefore, Fibonacci numbers have been a significant issue, especially for mathematicians who work in this field. In recent years, also geometricians have started to be interested in Fibonacci numbers and then they have aimed at giving meaning to Fibonacci numbers by describing the Fibonacci vectors. In a study carried by K. T. Atanassov, V. Atanassova, A. G. Shannon, and J. C. Turner [Atanassov, et all. 2002], Fibonacci 3-vectors have been defined, and some properties for the Fibonacci 3-vectors have been given by grounding on the properties of Fibonacci numbers. In addition, the area of the triangles formed by Fibonacci vectors has been computed [Çetinberk, 2020]. Then, for the first time, the vector product has been defined in Fibonacci 3-vectors. Like in Fibonacci numbers, in 1878, also Lucas numbers became a math subject which a French mathematician Edouard Lucas worked on it, extensively [Koshy, 2001]. In a study carried by E. Salter [Salter, 2005], he defined the Fibonacci  $r$ -vector (and the Lucas vector) and brought a new perspective to the inner product of these

two vectors. For the first time, he computed the norm of the Fibonacci and the Lucas vectors. Moreover, Fibonacci vectors were firstly analyzed in different dimensions. Then, S. Halıcı worked on Fibonacci p-Numbers [Halıcı, 2009]. O. Kaya, M. Önder have analyzed the Fibonacci and Lucas vectors and their quaternions in a geometric perspective [Kaya et al., 2018].

The geometric transformations have important functions in the design and motion of the objects [Adams and Rogers, 1990]. Moreover, the Householder transformation is a transformation that defines a reflection with respect to a plane or a hyperplane. The Householder transformation was defined in 1958 by Alston Scott Householder and the matrix corresponding to this transformation is applied as the first process for notably numerical linear algebra field and the  $QR$  multiplicative writings of the matrixes [Özdemir, 2016]. The non-degenerate bilinear form of the Householder transformation in various spaces was studied by Mackey et al. (2004). After the theorem (known as Cartan–Dieudonné) which shows that each orthogonal transformation can be denoted as the composite of the transformations which denote a reflection with respect to hyperplanes, reflection transformations have become an important field of study. The structural proof which belongs to the state of this theorem in the generalized reel inner product spaces has been made by Uhlig (2001) and Fuller (2011). In addition, again, the alternative proof of it in the generalized inner product spaces has been given by Rodríguez-Andrade et al. (2011) [Soylu, 2019].

In this study, Fibonacci and the Lucas 3-vectors have been defined, then some properties for these vectors have been given by grounding on the properties of Fibonacci and Lucas numbers. The geometric transformations have been described in the vectors. Then, the Householder transformation and its properties have been given. In addition, the Householder transformation has been defined for Fibonacci and Lucas 3-vectors. By giving an example for each transformation, the drawing of the figures has been done in the Maple program.

## 2. GENERATION OF THE DATA

In this section, Fibonacci and the Lucas numbers, and then, Fibonacci and Lucas 3-vectors and their properties have been given. Also, the Householder transformation and its properties have been given as well.

### 2.1. International Journal of Applied Mathematics and Statistics

**Definition 2.1.1. (Fibonacci-Lucas Numbers)** Assume that for all  $n$  positive integers, the  $n$ th Fibonacci number is  $F_n$  and Lucas number is  $L_n$ . Thus, the second-order iteration relation and initial conditions are defined by  $F_{n+2} = F_{n+1} + F_n$ ,  $F_1 = F_2 = 1$ , and  $L_{n+2} = L_{n+1} + L_n$ ,  $L_1 = 1$ ,  $L_2 = 3$ , [Salter, 2005]. Fibonacci and the Lucas numbers are given as 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, ...,  $F_n$ , ... and 1, 3, 4, 7, 11, 18, 29, 47, ...,  $L_n$ , ...

**Theorem 2.1.1. (Binet Formula)** Let  $\alpha = \frac{1+\sqrt{5}}{2}$ , and  $\beta = \frac{1-\sqrt{5}}{2}$  be the root of the equation  $x^2 - x - 1 = 0$ . Thus, the Binet formulas which give the  $n^{\text{th}}$  term of Fibonacci and Lucas series are defined by  $F_n = \frac{\alpha^n - \beta^n}{\alpha - \beta}$  and  $L_n = \alpha^n + \beta^n$  where the equations  $\alpha + \beta = 1$ ,  $\alpha - \beta = \sqrt{5}$  and  $\alpha\beta = -1$  are satisfied. Besides,  $\alpha$  is called the **golden ratio** [Çetinberk, 2017; Çetinberk&Yüce, 2020].

**Theorem 2.1.2.** The following identities can be given for Fibonacci and Lucas number sequences [Salter, 2005; Çetinberk&Yüce, 2020; Yaşar, 2016];

- 1)  $F_{n+1}^2 + F_n^2 = F_{2n+1}$ ,
- 2)  $F_{n+1}^2 - F_n^2 = F_{2n}$ ,
- 3)  $F_{n-1}F_{n+1} - F_n^2 = (-1)^n$ , (Cassini Formula)
- 4)  $F_nF_m + F_{n+1}F_{m+1} = F_{n+m+1}$ ,
- 5)  $F_nF_{m+r} - F_{n+r}F_m = (-1)^m F_rF_{n-m}$ ,
- 6)  $F_nF_{m+1} - F_{n+1}F_m = (-1)^m F_{n-m}$ ,
- 7)  $F_n = L_{n-1} + L_{n+1}$ ,
- 8)  $F_{2n} = F_nL_n$ ,
- 9)  $F_{-n} = (-1)^{n+1} F_n$ ,
- 10)  $L_{-n} = (-1)^n L_n$ ,

**Definition 2.1.2.** Fibonacci and Lucas vectors for each  $m$  positive integer and each  $n$  integer are defined by

$$\vec{F}_n^m = [F_n \quad F_{n+1} \quad F_{n+2} \dots F_{n+m-1}]^T \quad \text{and} \quad \vec{L}_n^m = [L_n \quad L_{n+1} \quad L_{n+2} \dots L_{n+m-1}]^T$$

where  $F_n$ , and  $L_n$  are the  $n^{\text{th}}$  Fibonacci and Lucas numbers, respectively [Salter, 2005; Atanassov et al., 2002].

**Definition 2.1.3. (Inner Product)** Let's take  $\vec{F}_{n_1}^m, \vec{F}_{n_2}^m$  Fibonacci vectors, and  $\vec{L}_{n_1}^m, \vec{L}_{n_2}^m$  Lucas vectors for each  $m$  positive integer and each  $n$  integer. The Euclidean inner product between the two Fibonacci vectors is

$$\begin{aligned} \langle \vec{F}_{n_1}^m, \vec{F}_{n_2}^m \rangle &= F_{n_1} F_{n_2} + F_{n_1} F_{n_2} + \dots + F_{n_1+m-1} F_{n_2+m-1} \\ &= \begin{cases} F_m F_{n_1+n_2+m-1} & \text{If } m \text{ is even} \\ \frac{1}{5} (L_m L_{n_1+n_2+m-1} - (-1)^{n_1} L_{n_2-n_1}) & \text{If } m \text{ is odd,} \end{cases} \end{aligned}$$

the Euclidean inner product between the two Lucas vectors is

$$\begin{aligned} \langle \vec{L}_{n_1}^m, \vec{L}_{n_2}^m \rangle &= L_{n_1} L_{n_2} + L_{n_1} L_{n_2} + \dots + L_{n_1+m-1} L_{n_2+m-1} \\ &= \begin{cases} 5F_m F_{n_1+n_2+m-1} & \text{If } m \text{ is even} \\ L_m L_{n_1+n_2+m-1} - (-1)^{n_1} L_{n_2-n_1} & \text{If } m \text{ is odd,} \end{cases} \end{aligned}$$

and the Euclidean inner product between a Fibonacci and a Lucas vector is defined as

$$\begin{aligned} \langle \vec{F}_{n_1}^m, \vec{L}_{n_2}^m \rangle &= F_{n_1} L_{n_2} + F_{n_1} L_{n_2} + \dots + F_{n_1+m-1} L_{n_2+m-1} \\ &= \begin{cases} 5F_m F_{n_1+n_2+m-1} & \text{If } m \text{ is even} \\ L_m F_{n_1+n_2+m-1} + (-1)^{n_1+1} F_{n_2-n_1} & \text{If } m \text{ is odd} \end{cases} \end{aligned}$$

, [Salter, 2005].

**Definition 2.1.4. (Norm)** The norm of the  $\vec{F}_n^m$  Fibonacci vector for each  $m$  positive integer and each  $n$  integer is

$$\|\vec{F}_n^m\|^2 = \begin{cases} F_m F_{2n+m-1} & \text{If } m \text{ is even} \\ \frac{1}{5} (L_m L_{2n+m-1} - 2(-1)^n) & \text{If } m \text{ is odd,} \end{cases}$$

and the norm of the  $\vec{L}_n^m$  Lucas vector is defined as

$$\|\vec{L}_n^m\|^2 = \begin{cases} 5F_m F_{2n+m-1} & \text{If } m \text{ is even} \\ L_m L_{2n+m-1} + 2(-1)^n & \text{If } m \text{ is odd} \end{cases}$$

, [Salter, 2005].

**Definition 2.1.5. (Vector Product)** Let  $\{\vec{e}_1, \vec{e}_2, \vec{e}_3\}$  be the orthonormal basis vectors of the Euclidean-3 space and  $\vec{F}_{n_1}, \vec{F}_{n_2}$  are Fibonacci 3 vectors. The vector product of the two Fibonacci vectors is

$$\vec{F}_{n_1} \wedge \vec{F}_{n_2} = \begin{vmatrix} \vec{e}_1 & \vec{e}_2 & \vec{e}_3 \\ F_{n_1} & F_{n_1+1} & F_{n_1+2} \\ F_{n_2} & F_{n_2+1} & F_{n_2+2} \end{vmatrix} = (-1)^{n_1+1} F_{n_2-n_1} (-1, -1, 1)$$

and the vector product of the two Lucas 3-vectors is

$$\vec{L}_{n_1} \wedge \vec{L}_{n_2} = \begin{vmatrix} \vec{e}_1 & \vec{e}_2 & \vec{e}_3 \\ L_{n_1} & L_{n_1+1} & L_{n_1+2} \\ L_{n_2} & L_{n_2+1} & L_{n_2+2} \end{vmatrix} = 5(-1)^{n_1+1} F_{n_2-n_1} (1, 1, -1)$$

then the vector product of a Fibonacci and a Lucas 3-vectors is obtained as

$$\vec{F}_{n_1} \wedge \vec{L}_{n_2} = \begin{vmatrix} \vec{e}_1 & \vec{e}_2 & \vec{e}_3 \\ F_{n_1} & F_{n_1+1} & F_{n_1+2} \\ L_{n_2} & L_{n_2+1} & L_{n_2+2} \end{vmatrix} = (-1)^{n_1} L_{n_2-n_1} (1, 1, -1)$$

, [Atanassov, et al., 2002]

**Corollary 2.1.1.** The Fibonacci and Lucas vectors are always perpendicular to the vector  $(1, 1, -1)$ .

**Corollary 2.1.2.** The Fibonacci and Lucas vectors are involved in the plane with the normal vector  $(1, 1, -1)$ .

**Theorem 2.1.4.** Let  $\vec{F}_{n_1}$ ,  $\vec{F}_{n_2}$  and  $\vec{F}_{n_3}$  be the Fibonacci 3-vectors. The scalar product of these three vectors is zero, i.e.  $\langle \vec{F}_{n_1} \wedge \vec{F}_{n_2}, \vec{F}_{n_3} \rangle = 0$ , [Özdemir, 2020].

**Corollary 2.1.3.** A parallelepiped can not be composed of Fibonacci vectors, [Güven&Nurkan, 2015].

## 2.2. The Householder Transformation in the $E^n$ Space

The Householder transformation is a transformation that makes a reflection movement with respect to a line, a plane, or a hyperplane according to the dimension of the space where it is in its most general form. Let the vectors  $\vec{u}$  and  $\vec{v}$  be the different column vectors in the  $E^n$  space.  $\vec{u}' = (u_1, u_2, \dots, u_n)$  and  $\vec{v}' = (v_1, v_2, \dots, v_n)$ . The product  $\langle \vec{u}, \vec{v} \rangle = u_1 v_1 + u_2 v_2 + \dots + u_n v_n = \vec{u}' I_n \vec{v}$  is defined as the Euclidean inner product [Soylu, 2019]. Thus, it is concluded that a vector that is obtained by the transformation  $H(\vec{u}) = \vec{u} - 2r\vec{v}$  for a transformation  $H$  is a reflection vector of the vector  $\vec{u}$  with respect to the hyperplane  $\vec{v}^\perp$ . The transformation

$$H(\vec{u}) = \vec{u} - 2 \frac{\langle \vec{u}, \vec{v} \rangle}{\langle \vec{v}, \vec{v} \rangle} \vec{v}$$

that is obtained in this way is called Householder transformation and the matrix  $H$  corresponding to this transformation is called the Householder matrix [Özdemir, 2020].

**Theorem 2.2.1.** For the parameters  $\vec{u}, \vec{v} \in E^n$  that are non-zero and different from each other, the Householder matrix corresponds to the Householder transformation which gives the reflection of the vector  $\vec{u}$  with respect to the hyperplane  $\vec{v}^\perp$  is  $H$ . Thus, it is obtained by  $H = I - 2 \frac{\vec{v}\vec{v}'}{\vec{v}'\vec{v}}$ , [Aragón-González, 2009].

## 3. RESULTS

### 3.1. The Transformations in Fibonacci and the Lucas 3-Vectors and the MAPLE Applications

In this section of the study, the transformation matrixes of Fibonacci and the Lucas 3-vectors have been obtained by utilizing the applications of the local scaling, general scaling, shearing, reflection, rotation, translation, and the multiple transformations in the three-dimensional space; and the properties of Fibonacci and the Lucas 3-vectors which their basic concepts have been given in the material and the method section. Then, the Householder transformation and the matrix corresponding to this transformation have been defined in Fibonacci and the Lucas 3-vectors. It has given an

example for each transformation and the drawings of the figures have been done in the Maple program.

### 3.1.1. Scaling in Fibonacci and Lucas 3-Vectors

If a local scaling transformation is applied on Fibonacci 3-vector  $\vec{F}_n^{3T} = [F_n \ F_{n+1} \ F_{n+2}]$ , the effect of the local scaling is

$$\begin{aligned} [\vec{F}_n^{3T}][T] &= [F_n \ F_{n+1} \ F_{n+2} \ 1] \begin{bmatrix} a & 0 & 0 & 0 \\ 0 & e & 0 & 0 \\ 0 & 0 & j & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \\ &= [aF_n \ eF_{n+1} \ jF_{n+2} \ 1] = [F_n^* \ F_{n+1}^* \ F_{n+2}^* \ 1], \end{aligned} \quad (3.1)$$

and when the local scaling transformation is applied on Lucas 3-vector  $\vec{L}_n^{3T} = [L_n \ L_{n+1} \ L_{n+2}]$ , the effect of the local scaling is obtained as

$$\begin{aligned} [\vec{L}_n^{3T}][T] &= [L_n \ L_{n+1} \ L_{n+2} \ 1] \begin{bmatrix} a & 0 & 0 & 0 \\ 0 & e & 0 & 0 \\ 0 & 0 & j & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \\ &= [aL_n \ eL_{n+1} \ jL_{n+2} \ 1] = [L_n^* \ L_{n+1}^* \ L_{n+2}^* \ 1]. \end{aligned} \quad (3.2)$$

Now apply this transformation with an example.

#### Example 3.1.1.1. Scaling in Fibonacci and Lucas 3-vectors

Assume that the local scaling transformation of Fibonacci 3-vector  $\vec{F}_1^{3T}$ , which is known as homogenous position vectors  $\vec{F}_1^{3T} = [F_1 \ F_2 \ F_3 \ 1] = [1 \ 1 \ 2 \ 1]$ , by the scaling factors 2, 2, 3 through the  $x, y, z$  axis, respectively is

$$[T] = \begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

When this transformation is applied to Fibonacci 3-vector  $\vec{F}_1^{3T}$  by using the Equation (3.1);

$$[\vec{F}_1^{3T*}] = [\vec{F}_1^{3T}][T] = [1 \ 1 \ 2 \ 1] \begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = [2 \ 2 \ 6 \ 1]$$

is obtained. If the same transformation is applied to the vector  $\vec{L}_1^{3T} = [L_1 \ L_2 \ L_3 \ 1] = [1 \ 3 \ 4 \ 1]$  by using Equation (3.2), the result is

$$\begin{bmatrix} \vec{L}_1^{3T'} \end{bmatrix} = \begin{bmatrix} \vec{L}_1^{3T} \end{bmatrix} [T] = \begin{bmatrix} 1 & 3 & 4 & 1 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 6 & 12 & 1 \end{bmatrix}.$$

The position vectors of Fibonacci and Lucas 3-vectors  $\vec{F}_1^{3T}$  and  $\vec{L}_1^{3T}$  after the local scaling are  $\vec{F}_1^{3T'} = \begin{bmatrix} 2 & 2 & 6 & 1 \end{bmatrix}$  and  $\vec{L}_1^{3T'} = \begin{bmatrix} 2 & 6 & 12 & 1 \end{bmatrix}$ , and they are shown in the following figures.

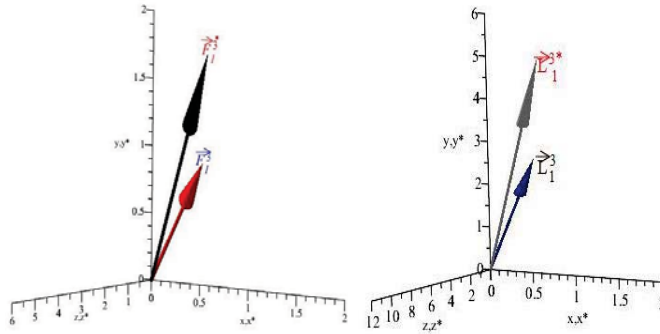


Figure 3.1. The scaling in Fibonacci and Lucas 3-vectors

#### Example 3.1.1.2. General Scaling in Fibonacci 3-Vector

The scaling of the Fibonacci 3-vector  $\vec{F}_1^{3T}$  in a magnitude that equals to its three-fold (increasing the dimension to three-fold) is done by the transformation

$$[T] = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 3 \end{bmatrix}.$$

When this transformation is applied to Fibonacci 3-vector  $\vec{F}_1^{3T}$ , the homogenous position vector

$$\begin{bmatrix} \vec{F}_1^{3T'} \end{bmatrix} = \begin{bmatrix} \vec{F}_1^{3T} \end{bmatrix} [T] = \begin{bmatrix} 1 & 1 & 2 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 2 & 3 \end{bmatrix}$$

is obtained. Note that the homogenous coordinate factor is  $h=3$  for each of the rotated position vectors. Thus, in order to get the ordinary physical coordinates, each position vector has to be divided  $h$ . The result is obtained as  $\begin{bmatrix} \vec{F}_1^{3T'} \end{bmatrix} = \begin{bmatrix} 1/3 & 1/3 & 2/3 & 1 \end{bmatrix}$  and it is shown in the following figure.

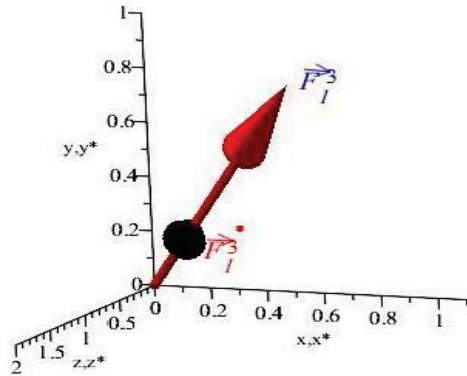


**Figure 3.2.** General scaling in Fibonacci 3-vector

### 3.1.2. Shearing in Fibonacci and Lucas 3-Vectors

The shearing transformation of the Fibonacci 3-vector  $\vec{F}_n^{3T} = [F_n \ F_{n+1} \ F_{n+2}]$  is obtained as

$$\begin{aligned} [\vec{F}_n^{3T}][T] &= [F_n \ F_{n+1} \ F_{n+2} \ 1] \begin{bmatrix} 1 & b & c & 0 \\ d & 1 & f & 0 \\ g & i & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \\ &= [F_n + dF_{n+1} + gF_{n+2} \ bF_n + F_{n+1} + iF_{n+2} \ cF_n + fF_{n+1} + F_{n+2} \ 1]. \end{aligned} \quad (3.3)$$

If the shearing transformation is applied to Lucas 3- vector  $\vec{L}_n^{3T} = [L_n \ L_{n+1} \ L_{n+2}]$ ,

$$\begin{aligned} [\vec{L}_n^{3T}][T] &= [L_n \ L_{n+1} \ L_{n+2} \ 1] \begin{bmatrix} 1 & b & c & 0 \\ d & 1 & f & 0 \\ g & i & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \\ &= [L_n + dL_{n+1} + gL_{n+2} \ bL_n + L_{n+1} + iL_{n+2} \ cL_n + fL_{n+1} + L_{n+2} \ 1]. \end{aligned} \quad (3.4)$$

is obtained.

#### Example 3.1.2.1. Shearing in Fibonacci and Lucas 3-Vectors

The shearing transformation of the Fibonacci 3-vector  $\vec{F}_1^{3T}$ , the transformation is obtained as

$$[T] = \begin{bmatrix} 1 & 0.75 & 0.25 & 0 \\ 0.25 & 1 & 0.8 & 0 \\ 0.5 & 0.75 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

From Equation (3.3), we have

$$[\vec{F}_1^{3T}] = [\vec{F}_1^{3T}][T] = [1 \ 1 \ 2 \ 1] \begin{bmatrix} 1 & 0.75 & 0.25 & 0 \\ 0.25 & 1 & 0.8 & 0 \\ 0.5 & 0.75 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = [2.75 \ 3.25 \ 3.05 \ 1].$$

If the same transformation is applied to the vector  $\vec{L}_1^{3T} = [L_1 \ L_2 \ L_3 \ 1] = [1 \ 3 \ 4 \ 1]$  by using the Equation (3.4);

$$[\vec{L}_1^{3T*}] = [\vec{L}_1^{3T}][T] = [1 \ 3 \ 4 \ 1] \begin{bmatrix} 1 & 0.75 & 0.25 & 0 \\ 0.25 & 1 & 0.8 & 0 \\ 0.5 & 0.75 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = [3.75 \ 6.75 \ 6.65 \ 1]$$

is obtained. The results are;  $[\vec{F}_1^{3T*}] = [2.75 \ 3.25 \ 3.05 \ 1]$  and  $[\vec{L}_1^{3T*}] = [3.75 \ 6.75 \ 6.65 \ 1]$ , and they are shown in the following figure.

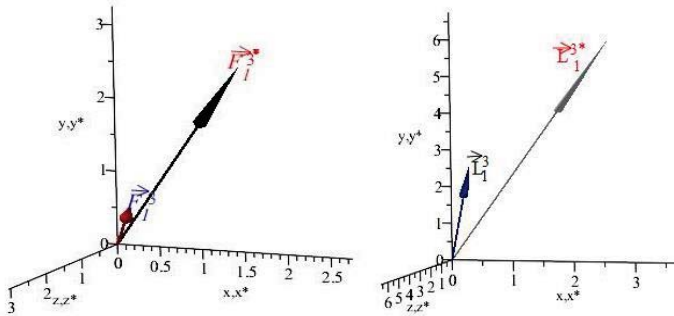


Figure 3.3. Shearing in Fibonacci and Lucas 3-vectors

### 3.1.3. Rotation in Fibonacci and Lucas 3-Vectors

When the rotation transformation is applied to the Fibonacci 3-vector  $\vec{F}_n^{3T} = [F_n \ F_{n+1} \ F_{n+2}]$  and the Lucas 3-vector  $\vec{L}_n^{3T} = [L_n \ L_{n+1} \ L_{n+2}]$  by an angle  $\theta$  about the x-axis; the transformation has become

$$[T] = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos \theta & \sin \theta & 0 \\ 0 & -\sin \theta & \cos \theta & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix},$$

and

$$\begin{aligned} [\vec{F}_n^{3T}][T] &= [F_n \ F_{n+1} \ F_{n+2} \ 1] \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos \theta & \sin \theta & 0 \\ 0 & -\sin \theta & \cos \theta & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \\ &= [F_n \ F_{n+1} \cos \theta - F_{n+2} \sin \theta \ F_{n+1} \sin \theta + F_{n+2} \cos \theta \ 1] \end{aligned} \quad (3.5)$$

and

$$\begin{aligned}
 [\vec{L}_n^{3T}][T] &= [L_n \quad L_{n+1} \quad L_{n+2} \quad 1] \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos \theta & \sin \theta & 0 \\ 0 & -\sin \theta & \cos \theta & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \\
 &= [L_n \quad L_{n+1} \cos \theta - L_{n+2} \sin \theta \quad L_{n+1} \sin \theta + L_{n+2} \cos \theta \quad 1].
 \end{aligned} \tag{3.6}$$

are obtained. Similarly, the transformation matrix for a rotation by an angle  $\Psi$  about the  $z$ -axis is

$$[T] = \begin{bmatrix} \cos \Psi & \sin \Psi & 0 & 0 \\ -\sin \Psi & \cos \Psi & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix},$$

and

$$\begin{aligned}
 [\vec{F}_n^{3T}][T] &= [F_n \quad F_{n+1} \quad F_{n+2} \quad 1] \begin{bmatrix} \cos \Psi & \sin \Psi & 0 & 0 \\ -\sin \Psi & \cos \Psi & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \\
 &= [F_n \cos \Psi + F_{n+1} \sin \Psi \quad -F_n \sin \Psi + F_{n+1} \cos \Psi \quad F_{n+2} \quad 1],
 \end{aligned} \tag{3.7}$$

and

$$\begin{aligned}
 [\vec{L}_n^{3T}][T] &= [L_n \quad L_{n+1} \quad L_{n+2} \quad 1] \begin{bmatrix} \cos \Psi & \sin \Psi & 0 & 0 \\ -\sin \Psi & \cos \Psi & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \\
 &= [L_n \cos \Psi + L_{n+1} \sin \Psi \quad -L_n \sin \Psi + L_{n+1} \cos \Psi \quad L_{n+2} \quad 1]
 \end{aligned} \tag{3.8}$$

are obtained.

The transformation matrix for a rotation by an angle  $\phi$  about the  $y$ -axis is

$$[T] = \begin{bmatrix} \cos \phi & 0 & -\sin \phi & 0 \\ 0 & 1 & 0 & 0 \\ \sin \phi & 0 & \cos \phi & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

and

$$\begin{aligned}
 [\vec{F}_n^{3T}][T] &= [F_n \quad F_{n+1} \quad F_{n+2} \quad 1] \begin{bmatrix} \cos \phi & 0 & -\sin \phi & 0 \\ 0 & 1 & 0 & 0 \\ \sin \phi & 0 & \cos \phi & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \\
 &= [F_n \cos \phi + F_{n+2} \sin \phi \quad F_{n+1} \quad -F_n \sin \phi + F_{n+2} \cos \phi \quad 1],
 \end{aligned} \tag{3.9}$$

and

$$\begin{aligned}
 [\vec{L}_n^{3T}][T] &= [L_n \quad L_{n+1} \quad L_{n+2} \quad 1] \begin{bmatrix} \cos \phi & 0 & -\sin \phi & 0 \\ 0 & 1 & 0 & 0 \\ \sin \phi & 0 & \cos \phi & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \\
 &= [L_n \cos \phi + L_{n+2} \sin \phi \quad L_{n+1} \quad -L_n \sin \phi + L_{n+2} \cos \phi \quad 1]
 \end{aligned} \tag{3.10}$$

are obtained.

### Example 3.1.3.1. Rotation in Fibonacci and Lucas 3-Vectors

The position vector  $[\vec{F}_2^{3T}]$  is obtained as  $\vec{F}_2^{3T} = [F_2 \quad F_3 \quad F_4 \quad 1] = [1 \quad 2 \quad 3 \quad 1]$ . When a rotation is applied to the Fibonacci 3-vector  $\vec{F}_2^{3T}$  by an angle  $\phi = 90^\circ$  about the  $y$ -axis, the transformation

$$[T] = \begin{bmatrix} 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

is obtained. With the application of the transformation to the Fibonacci 3-vector  $\vec{F}_2^{3T}$

$$[\vec{F}_2^{3T*}] = [\vec{F}_2^{3T}][T] = [1 \quad 2 \quad 3 \quad 1] \begin{bmatrix} 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = [3 \quad 2 \quad -1 \quad 1]$$

is obtained.

If the same transformation is applied to the vector  $\vec{L}_2^{3T} = [L_2 \quad L_3 \quad L_4 \quad 1] = [3 \quad 4 \quad 7 \quad 1]$  by using the Equation (3.10), the result is

$$[\vec{L}_2^{3T*}] = [\vec{L}_2^{3T}][T] = [3 \quad 4 \quad 7 \quad 1] \begin{bmatrix} 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = [7 \quad 4 \quad -3 \quad 1].$$

The new position vectors  $[\vec{F}_2^{3T*}] = [3 \quad 2 \quad -1 \quad 1]$  and  $[\vec{L}_2^{3T*}] = [7 \quad 4 \quad -3 \quad 1]$  are shown in the following figure.

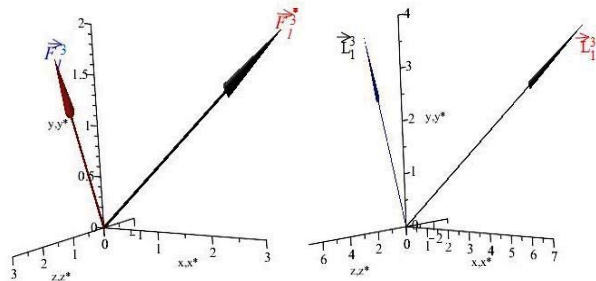


Figure 3.4. Rotation in Fibonacci and Lucas 3-vectors

### 3.1.4. Reflection in Fibonacci and Lucas 3-Vectors

Let's apply the reflection transformation to the Fibonacci 3-vector  $\vec{F}_n^{3T} = [F_n \ F_{n+1} \ F_{n+2}]$  and the

Lucas 3-vector  $\vec{L}_n^{3T} = [L_n \ L_{n+1} \ L_{n+2}]$ .

Thus, the transformation matrix for a reflection through the xy plane is

$$[T_{xy}] = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

The reflection of Fibonacci and Lucas 3-vectors through the xy plane is

$$[\vec{F}_n^{3T}][T] = [F_n \ F_{n+1} \ F_{n+2} \ 1] \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = [F_n \ F_{n+1} \ -F_{n+2} \ 1], \quad (3.11)$$

and

$$[\vec{L}_n^{3T}][T] = [L_n \ L_{n+1} \ L_{n+2} \ 1] \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = [L_n \ L_{n+1} \ -L_{n+2} \ 1]. \quad (3.12)$$

To be reflected on the yz plane, the transformation

$$[T] = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

is used. The reflection of Fibonacci and Lucas 3-vectors through the yz plane is

$$[\vec{F}_n^{3T}][T] = [F_n \ F_{n+1} \ F_{n+2} \ 1] \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = [-F_n \ F_{n+1} \ F_{n+2} \ 1], \quad (3.13)$$

and

$$[\vec{L}_n^{3T}][T] = [L_n \ L_{n+1} \ L_{n+2} \ 1] \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = [-L_n \ L_{n+1} \ L_{n+2} \ 1], \quad (3.14)$$

and to be reflected on the xz plane, the transformation

$$[T] = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

is used. The reflection of Fibonacci and Lucas 3-vectors through the  $xz$  plane is,

is used. The reflection of Fibonacci and Lucas 3-vectors through the  $xz$  plane is,

$$[\vec{F}_n^{3T}][T] = \begin{bmatrix} F_n & F_{n+1} & F_{n+2} & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} F_n & -F_{n+1} & F_{n+2} & 1 \end{bmatrix}, \quad (3.15)$$

and

$$[\vec{L}_n^{3T}][T] = \begin{bmatrix} L_n & L_{n+1} & L_{n+2} & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} L_n & -L_{n+1} & L_{n+2} & 1 \end{bmatrix} \quad (3.16)$$

#### Example 3.1.4.1. Reflection in Fibonacci and Lucas 3-Vectors

Given the fact that the Fibonacci 3-vector  $\vec{F}_3^{3T}$  Fibonacci 3-vector and the Lucas 3-vector  $\vec{L}_3^{3T}$ , the matrix of the position vectors  $[\vec{F}_3^{3T}]$  and  $[\vec{L}_3^{3T}]$  is given as;

$$\vec{F}_3^{3T} = \begin{bmatrix} F_3 & F_4 & F_5 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 3 & 5 & 1 \end{bmatrix}$$

$$\vec{L}_3^{3T} = \begin{bmatrix} L_3 & L_4 & L_5 & 1 \end{bmatrix} = \begin{bmatrix} 4 & 7 & 11 & 1 \end{bmatrix}$$

The transformed position vectors from the Equations (3.15) and (3.16) after the reflection through the  $xz$  plane have become

$$[\vec{F}_3^{3T^*}] = [\vec{F}_3^{3T}][T] = \begin{bmatrix} 2 & 3 & 5 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 2 & -3 & 5 & 1 \end{bmatrix}$$

and

$$[\vec{L}_3^{3T^*}] = [\vec{L}_3^{3T}][T] = \begin{bmatrix} 4 & 7 & 11 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 4 & -7 & 11 & 1 \end{bmatrix}$$

The obtained vectors  $[\vec{F}_3^{3T^*}] = \begin{bmatrix} 2 & -3 & 5 & 1 \end{bmatrix}$  and  $[\vec{L}_3^{3T^*}] = \begin{bmatrix} 4 & -7 & 11 & 1 \end{bmatrix}$  are shown in Figure

3.5.

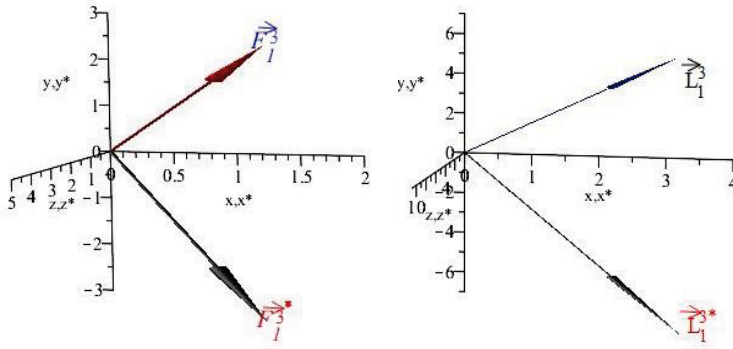


Figure 3.5. Reflection in Fibonacci and Lucas 3-vectors

### 3.1.5. Translation in Fibonacci and Lucas 3-Vectors

The three-dimensional translation matrix of the Fibonacci 3-vector  $\vec{F}_n^{3T} = [F_n \ F_{n+1} \ F_{n+2}]$  and the

Lucas 3-vector  $\vec{L}_n^{3T} = [L_n \ L_{n+1} \ L_{n+2}]$  is given by

$$[T] = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ l & m & n & 1 \end{bmatrix}.$$

When the translation matrix is applied to the Fibonacci 3-vector  $\vec{F}_3^{3T}$  and the Lucas 3-vector  $\vec{L}_3^{3T}$ , the translated homogenous coordinates

$$\begin{bmatrix} F'_n & F'_{n+1} & F'_{n+2} & h \end{bmatrix} = \begin{bmatrix} F_n & F_{n+1} & F_{n+2} & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ l & m & n & 1 \end{bmatrix}$$

and

$$\begin{bmatrix} L'_n & L'_{n+1} & L'_{n+2} & h \end{bmatrix} = \begin{bmatrix} L_n & L_{n+1} & L_{n+2} & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ l & m & n & 1 \end{bmatrix}$$

are obtained. When they are extended, the equation

$$\begin{aligned} \begin{bmatrix} F'_n & F'_{n+1} & F'_{n+2} & h \end{bmatrix} &= \begin{bmatrix} F_n + l & F_{n+1} + m & F_{n+2} + n & 1 \end{bmatrix} \\ \begin{bmatrix} L'_n & L'_{n+1} & L'_{n+2} & h \end{bmatrix} &= \begin{bmatrix} L_n + l & L_{n+1} + m & L_{n+2} + n & 1 \end{bmatrix} \end{aligned} \quad (3.17)$$

is obtained. The transformed physical coordinates are;

$$\begin{aligned} F'_n &= F_n + l & L'_n &= L_n + l \\ F'_{n+1} &= F_{n+1} + m & L'_{n+1} &= L_{n+1} + m \\ F'_{n+2} &= F_{n+2} + n & L'_{n+2} &= L_{n+2} + n \end{aligned}$$

**Example 3.1.5.1. Translation in Fibonacci and Lucas 3-Vectors**

Consider the Fibonacci 3-vector  $\vec{F}_3^{3T}$  and the Lucas 3-vector  $\vec{L}_3^{3T}$ . The position vectors

$\left[ \vec{F}_3^{3T} \right]$  and  $\left[ \vec{L}_3^{3T} \right]$  are obtained as

$$\vec{F}_3^{3T} = [F_3 \quad F_4 \quad F_5 \quad 1] = [2 \quad 3 \quad 5 \quad 1]$$

$$\vec{L}_3^{3T} = [L_3 \quad L_4 \quad L_5 \quad 1] = [4 \quad 7 \quad 11 \quad 1]$$

When the translation transformation is applied to the Fibonacci 3-vector  $\vec{F}_3^{3T}$  and the Lucas 3-vector  $\vec{L}_3^{3T}$ , the result is

$$[T] = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 5 & 0 & 0 & 1 \end{bmatrix}.$$

The translated homogenous coordinates are obtained by the matrix multiplication

$$\left[ \vec{F}_3^{3T'} \right] = \left[ \vec{F}_3^{3T} \right] [T] = [2 \quad 3 \quad 5 \quad 1] \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 5 & 0 & 0 & 1 \end{bmatrix} = [7 \quad 3 \quad 5 \quad 1],$$

and

$$\left[ \vec{L}_3^{3T'} \right] = \left[ \vec{L}_3^{3T} \right] [T] = [4 \quad 7 \quad 11 \quad 1] \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 5 & 0 & 0 & 1 \end{bmatrix} = [9 \quad 7 \quad 11 \quad 1]$$

from Equation (3.17). The obtained position vectors at the end are given by

$$\begin{aligned} F'_3 &= 7 & L'_3 &= 9 \\ F'_4 &= 3 & L'_4 &= 7 \\ F'_5 &= 5 & L'_5 &= 11 \end{aligned}$$

and shown in the following figure.

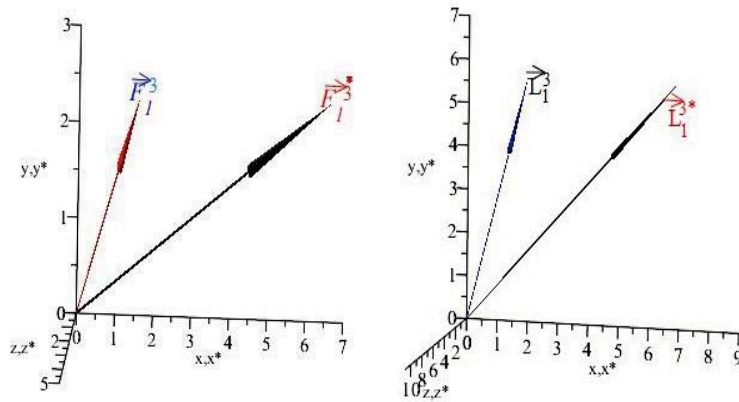


Figure 3.6. Translation in Fibonacci and Lucas 3-vectors

### 3.1.6. Multiple Transformations in Fibonacci and Lucas 3-Vectors

If a combined transformation matrix is used for a general Fibonacci and Lucas 3-vectors, the result is obtained as

$$\begin{aligned}
 \begin{bmatrix} \vec{F}_n^{3T} \end{bmatrix} [T] &= \begin{bmatrix} F_n & F_{n+1} & F_{n+2} & 1 \end{bmatrix} \begin{bmatrix} \cos \phi & 0 & -\sin \phi & 0 \\ \sin \phi \sin \theta & \cos \theta & \cos \phi \sin \theta & 0 \\ \sin \phi \cos \theta & -\sin \theta & \cos \phi \cos \theta & 0 \\ l \cos \phi & m \cos \theta & -l \sin \phi & 1 \\ +m \sin \phi \sin \theta & -n \sin \theta & +m \cos \phi \sin \theta & \\ +n \sin \phi \cos \theta & & +n \cos \phi \cos \theta & \end{bmatrix} \\
 &= \begin{bmatrix} (F_n + l) \cos \phi & (F_{n+1} + m) \cos \theta & -(F_n + l) \sin \phi & 1 \\ +(F_{n+1} + m) \sin \phi \sin \theta & -(F_{n+2} + n) \sin \theta & +(F_{n+1} + m) \cos \phi \sin \theta & \\ +(F_{n+2} + n) \sin \phi \cos \theta & & +(F_{n+2} + n) \cos \phi \cos \theta & \end{bmatrix} \quad (3.18)
 \end{aligned}$$

and

$$\begin{aligned}
 \begin{bmatrix} \vec{L}_n^{3T} \end{bmatrix} [T] &= \begin{bmatrix} L_n & L_{n+1} & L_{n+2} & 1 \end{bmatrix} \begin{bmatrix} \cos \phi & 0 & -\sin \phi & 0 \\ \sin \phi \sin \theta & \cos \theta & \cos \phi \sin \theta & 0 \\ \sin \phi \cos \theta & -\sin \theta & \cos \phi \cos \theta & 0 \\ l \cos \phi & m \cos \theta & -l \sin \phi & 1 \\ +m \sin \phi \sin \theta & -n \sin \theta & +m \cos \phi \sin \theta & \\ +n \sin \phi \cos \theta & & +n \cos \phi \cos \theta & \end{bmatrix} \\
 &= \begin{bmatrix} (L_n + l) \cos \phi & (L_{n+1} + m) \cos \theta & -(L_n + l) \sin \phi & 1 \\ +(L_{n+1} + m) \sin \phi \sin \theta & -(L_{n+2} + n) \sin \theta & +(L_{n+1} + m) \cos \phi \sin \theta & \\ +(L_{n+2} + n) \sin \phi \cos \theta & & +(L_{n+2} + n) \cos \phi \cos \theta & \end{bmatrix}. \quad (3.19)
 \end{aligned}$$

The following example shows this concept clearly.

**Example 3.1.6.1. Multiple Transformations in Fibonacci and Lucas 3-Vectors**

When the Fibonacci 3-vector  $\vec{F}_1$ , which is defined by  $\vec{F}_1 = [F_1 \ F_2 \ F_3 \ 1] = [1 \ 1 \ 2 \ 1]$ , is first translated  $+2, +2, +2$  unit in the direction of  $x, y, z$  and then, in sequence, a  $+60^\circ$  rotation transformation around the  $x$ -axis and a  $+90^\circ$  rotation transformation around the  $y$ -axis are applied, respectively; in other words when it applied by taking the values  $\theta = +60^\circ, \phi = +90^\circ, l = +2, m = +2, n = +2$  from the Equation (3.18),

$$[\vec{F}_1][T] = [1 \ 1 \ 2 \ 1] \begin{bmatrix} 0 & 0 & -1 & 0 \\ 1.155 & 0.577 & 0 & 0 \\ 0.577 & -1.155 & 0 & 0 \\ 3.464 & -1.155 & -2 & 1 \end{bmatrix} = [5.773 \ -2.888 \ -3 \ 1]$$

is obtained. The result is shown in the figure.

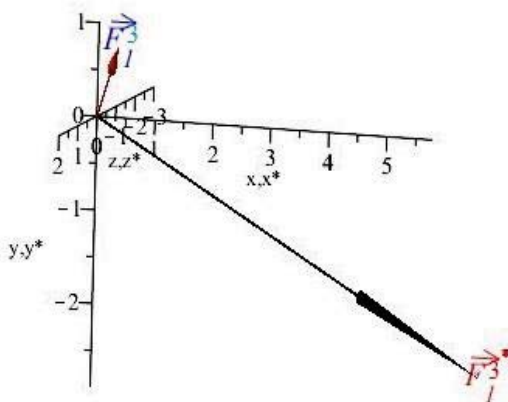


**Figure 3.7.** Multiple transformations in Fibonacci 3-vectors

**3.2. The Householder Transformation and The MAPLE Applications for Fibonacci and Lucas Vectors**

In this part of the thesis, how the Householder transformation will be defined for Fibonacci and Lucas number vectors has been analyzed. Let  $\vec{F}_{n_1}^m$  and  $\vec{F}_{n_2}^m$  be  $m$ -dimensional Fibonacci vectors. The orthogonal projection vector of the vector  $\vec{F}_{n_1}^m$  on the vector  $\vec{F}_{n_2}^m$  has been found. Also, the vertical projection vector of the vector  $\vec{L}_{n_1}^m$  on the vector  $\vec{L}_{n_2}^m$  has been found, as well. The matrixes corresponding to these reflection transformations have been obtained and examined in terms of the reflection properties. Then, as regards the integer  $m$  is even or odd, the Householder transformation and the matrix corresponding to this transformation have been obtained for the  $m$ -dimensional Fibonacci and Lucas vectors by utilizing the inner products of Fibonacci and Lucas vectors.

### 3.2.1. The Householder Transformation for the m-Dimensional Fibonacci Vectors

Let the m-dimensional Fibonacci vectors  $\vec{F}_{n_1}^m$  and  $\vec{F}_{n_2}^m$ , be  $\vec{F}_{n_1}^{mT} = [F_{n_1} \ F_{n_1+1} \ \dots \ F_{n_1+m-1}]$  and  $\vec{F}_{n_2}^{mT} = [F_{n_2} \ F_{n_2+1} \ \dots \ F_{n_2+m-1}]$ . The vertical projection vector of the vector  $\vec{F}_{n_1}^m$  on the vector  $\vec{F}_{n_2}^m$  is given by

$$izd_{\vec{F}_{n_2}^m}(\vec{F}_{n_1}^m) = \frac{\langle \vec{F}_{n_1}^m, \vec{F}_{n_2}^m \rangle}{\langle \vec{F}_{n_2}^m, \vec{F}_{n_2}^m \rangle} \vec{F}_{n_2}^m$$

The Householder transformation which satisfies the reflection of the vector  $\vec{F}_{n_1}^m$  on the vector  $\vec{F}_{n_2}^m$  is defined by the equation

$$H(\vec{F}_{n_1}^m) = \vec{F}_{n_1}^m - 2 \frac{\langle \vec{F}_{n_1}^m, \vec{F}_{n_2}^m \rangle}{\langle \vec{F}_{n_2}^m, \vec{F}_{n_2}^m \rangle} \vec{F}_{n_2}^m \quad (3.20)$$

The Householder transformation by using the Euclidean inner product can be written as

$$H(\vec{F}_{n_1}^m) = \vec{F}_{n_1}^m - 2 \frac{F_{n_1} F_{n_2} + F_{n_1+1} F_{n_2+1} + \dots + F_{n_1+n_2+m-1} F_{n_1+n_2+m-1}}{F_{n_2}^2 + F_{n_2+1}^2 + \dots + F_{n_2+m-1}^2} \vec{F}_{n_2}^m$$

By using the inner product definitions of Fibonacci vectors in Definition 2.1.5 and depending on the integer m is even or odd, the alternative demonstration of the Householder transformation from the Equation (3.20) can be obtained as:

If the integer m is even the result is

$$H(\vec{F}_{n_1}^m) = \vec{F}_{n_1}^m - 2 \frac{F_m F_{n_1+n_2+m-1}}{F_m F_{2n_2+m-1}} \vec{F}_{n_2}^m = \vec{F}_{n_1}^m - 2 \frac{F_{n_1+n_2+m-1}}{F_{2n_2+m-1}} \vec{F}_{n_2}^m, \quad (3.21)$$

if the integer is odd, the result is

$$\begin{aligned} H(\vec{F}_{n_1}^m) &= \vec{F}_{n_1}^m - 2 \frac{\frac{1}{5} (L_m L_{n_1+n_2+m-1} - (-1)^{n_1} L_{n_2-n_1})}{\frac{1}{5} (L_m L_{2n_2+m-1} - 2(-1)^{n_2})} \vec{F}_{n_2}^m \\ &= \vec{F}_{n_1}^m - 2 \frac{L_m L_{n_1+n_2+m-1} - (-1)^{n_1} L_{n_2-n_1}}{\frac{1}{5} (L_m L_{2n_2+m-1} - 2(-1)^{n_2})} \vec{F}_{n_2}^m. \end{aligned} \quad (3.22)$$

The Householder transformation which satisfies the reflection of the vector  $\vec{F}_{n_1}^m$  on the vector  $\vec{F}_{n_2}^m$  is found by

$$H = I - 2 \frac{\vec{F}_{n_2}^m \vec{F}_{n_2}^{mT}}{\vec{F}_{n_2}^{mT} \vec{F}_{n_2}^m}. \quad (3.23)$$

Especially, when Fibonacci vectors are taken in the Euclidean-3 space by using the Equation (3.22), i.e.  $m = 3$ , the Householder transformation is

$$H \left( \vec{F}_{n_1}^3 \right) = \vec{F}_{n_1}^3 - 2 \frac{L_3 L_{n_1+n_2+2} - (-1)^{n_1} L_{n_2-n_1} \vec{F}_{n_2}^3}{L_3 L_{2n_2+2} - 2(-1)^{n_2}} \vec{F}_{n_2}^3,$$

and the Householder matrix from Equation (3.23) corresponding to this transformation is obtained as

$$H = I - 2 \frac{\vec{F}_{n_2}^3 \vec{F}_{n_2}^{3T}}{\vec{F}_{n_2}^{3T} \vec{F}_{n_2}^3}. \quad (3.24)$$

### 3.2.2. The Householder Transformation for the m-Dimensional Lucas Vectors

Let the m-dimensional Lucas vectors  $\vec{L}_{n_1}^m$  and  $\vec{L}_{n_2}^m$ , be  $\vec{L}_{n_1}^{mT} = [L_{n_1} \ L_{n_1+1} \ \dots \ L_{n_1+m-1}]$  and  $\vec{L}_{n_2}^{mT} = [L_{n_2} \ L_{n_2+1} \ \dots \ L_{n_2+m-1}]$ . The vertical projection vector of the vector  $\vec{L}_{n_1}^m$  on the vector  $\vec{L}_{n_2}^m$  is given by

$$izd_{\vec{L}_{n_2}^m}(\vec{L}_{n_1}^m) = \frac{\langle \vec{L}_{n_1}^m, \vec{L}_{n_2}^m \rangle}{\langle \vec{L}_{n_2}^m, \vec{L}_{n_2}^m \rangle} \vec{L}_{n_2}^m.$$

The Householder transformation which satisfies the reflection of the vector  $\vec{L}_{n_1}^m$  on the vector  $\vec{L}_{n_2}^m$  is defined by the equation

$$H \left( \vec{L}_{n_1}^m \right) = \vec{L}_{n_1}^m - 2 \frac{\langle \vec{L}_{n_1}^m, \vec{L}_{n_2}^m \rangle}{\langle \vec{L}_{n_2}^m, \vec{L}_{n_2}^m \rangle} \vec{L}_{n_2}^m. \quad (3.25)$$

The Householder transformation by using the Euclidean inner product can be written as

$$H \left( \vec{L}_{n_1}^m \right) = \vec{L}_{n_1}^m - 2 \frac{L_{n_1} L_{n_2} + L_{n_1+1} L_{n_2+1} + \dots + L_{n_1+n_2+m-1} L_{n_1+n_2+m-1}}{L_{n_2}^2 + L_{n_2+1}^2 + \dots + L_{2n_2+m-1}^2} \vec{L}_{n_2}^m.$$

By using the inner product definitions of Lucas vectors in Definition 2.1.5 and depending on the integer m is even or odd, the alternative demonstrations of the Householder transformation from the Equation (3.25) can be obtained as;

If the integer m is even the result is

$$H \left( \vec{L}_{n_1}^m \right) = \vec{L}_{n_1}^m - 2 \frac{5F_m F_{n_1+n_2+m-1}}{5F_m F_{2n_2+m-1}} \vec{L}_{n_2}^m = \vec{L}_{n_1}^m - 2 \frac{F_{n_1+n_2+m-1}}{F_{2n_2+m-1}} \vec{L}_{n_2}^m, \quad (3.26)$$

if the integer m is odd, the result is

$$H \left( \vec{L}_{n_1}^m \right) = \vec{L}_{n_1}^m - 2 \frac{L_m L_{n_1+n_2+m-1}}{L_m L_{2n_2+m-1}} \vec{L}_{n_2}^m = \vec{L}_{n_1}^m - 2 \frac{L_{n_1+n_2+m-1}}{L_{2n_2+m-1}} \vec{L}_{n_2}^m. \quad (3.27)$$

The matrix corresponding to the Householder transformation which satisfies the reflection of the vector  $\vec{L}_{n_1}^m$  on the vector  $\vec{L}_{n_2}^m$  is found by

$$H = I - 2 \frac{\vec{L}_{n_2} \vec{L}_{n_2}^T}{\vec{L}_{n_2}^T \vec{L}_{n_2}}. \quad (3.28)$$

Especially, when Lucas vectors are taken in the Euclidean 3-space, i.e.  $m=3$ , The Householder transformation from the Equation (3.27) is obtained as

$$H(\vec{L}_{n_1}^3) = \vec{L}_{n_1}^3 - 2 \frac{L_{n_1+n_2+2}}{L_{2n+2}} \vec{L}_{n_2}^3.$$

The Householder matrix corresponding to this transformation is computed by the equation

$$H = I - 2 \frac{\vec{L}_{n_2}^3 \vec{L}_{n_2}^{3T}}{\vec{L}_{n_2}^{3T} \vec{L}_{n_2}^3}. \quad (3.29)$$

### Example 3.2.1.1. The Householder Transformation in Fibonacci 3-Vectors

In order to find the vector which is the symmetry of the Fibonacci 3-vector  $\vec{F}_2^{3T} = [F_2 \ F_3 \ F_4] = [1 \ 2 \ 3]$  with respect to the plane  $x + y + 2z = 0$  in the Euclidean 3-space, the Householder transformation can be applied. In order to get the vector that is the symmetry of the vector  $\vec{F}_2^3$  with respect to the plane  $x + y + 2z = 0$ , an orthogonal vector has to be found, and this vector can be chosen as a normal of the plane in the shape of  $\vec{F}_1^{3T} = [1 \ 1 \ 2]$ . Thus, the Householder matrix is obtained from Equation (3.24) as

$$H(\vec{F}_2^3) = I - 2 \frac{\vec{F}_2^3 \vec{F}_2^{3T}}{\vec{F}_2^{3T} \vec{F}_2^3} = I - 2 \frac{\begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} \begin{bmatrix} 1 & 1 & 2 \end{bmatrix}}{\begin{bmatrix} 1 & 1 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}},$$

$$H(\vec{F}_2^3) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - \frac{2}{6} \begin{bmatrix} 1 & 1 & 2 \\ 1 & 1 & 2 \\ 2 & 2 & 4 \end{bmatrix} = \frac{1}{6} \begin{bmatrix} 4 & -2 & -4 \\ -2 & 4 & -4 \\ -4 & -4 & -2 \end{bmatrix} = \frac{1}{3} \begin{bmatrix} 2 & -1 & -2 \\ -1 & 2 & -2 \\ -2 & -2 & -1 \end{bmatrix}$$

Therefore, it can be easily seen that it is a matrix which is orthogonal, symmetrical and accordingly the inversion of it is equal to itself. Thus, this matrix is obtained as

$$u \ H(\vec{F}_2^3) = \frac{1}{3} \begin{bmatrix} 2 & -1 & -2 \\ -1 & 2 & -2 \\ -2 & -2 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = \frac{1}{3} \begin{bmatrix} -6 \\ -3 \\ -9 \end{bmatrix} = \begin{bmatrix} -2 \\ -1 \\ -3 \end{bmatrix}.$$

The result is given in the following figure.

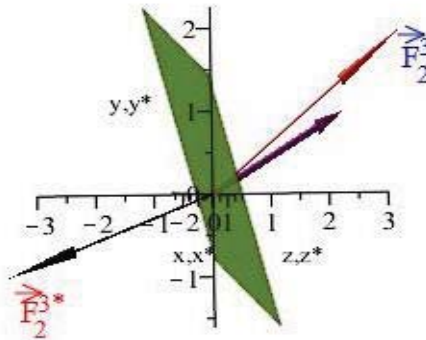


**Figure 3.8.** The Householder transformation in Fibonacci 3-vector

#### 4. DISCUSSION AND CONCLUSION

Fibonacci and Lucas numbers have been used in many disciplines since their discovery. It is one of the number types that allows us to witness the wonder of nature, especially since it has to do with the golden ratio. Therefore, the handling of the Fibonacci and Lucas numbers in our study will give a new perspective in terms of the geometrical design of objects. In our study, three-dimensional transformations of vectors consisting of Fibonacci and Lucas numbers such as rotation, displacement, reflection and scaling were examined. In addition, householder transformation using these vectors has been defined for the first time and explained by numerical examples.

#### 5. ACKNOWLEDGMENTS

This study was produced from the master's thesis of the first author under the supervision of the second author. Also, the authors are grateful to anonymous referees for their careful reading of the manuscript, which helped improve it greatly.

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