

Conformable Discontinuous Sturm-Liouville Problem with Applied Results

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ABSTRACT

In this article, we present some substantial spectral data for discontinuous fractional Sturm-Liouville equation involving conformable fractional derivative. We prove that the discontinuous fractional Sturm-Liouville operator is symmetric, the eigenvalues are real and the eigenfunctions corresponding to different eigenvalues are orthogonal. The given spectral data like representations of the solutions under different initial conditions, the asymptotics for the eigenfunctions and eigenvalues are applied in terms of limit based local derivative. Also, visual results are demonstrated with graphics for different orders and different potentials to check the behaviours of the solutions.

Keywords: discontinuous Sturm-Liouville equation, conformable derivative, asymptotic formula, spectral data.

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1 Introduction

Conformable derivative was presented to the literature by Khalil et al. in study (Khalil, Horani, Yousef and Sababheh, 2014). This new derivative depending on the basic limit definition of the derivative called as "conformable fractional" because of having fractional α power in the shift. Conformable derivative of f of order $0 < \alpha \leq 1$ is given as

$$T_{\alpha}^a f(t) = \lim_{\varepsilon \rightarrow 0} \frac{f\left(t + \varepsilon(t-a)^{1-\alpha}\right) - f(t)}{\varepsilon}.$$

The reason we use the conformable derivative is that it has the same features with the ordinary derivative such as quotient of two functions and the derivative of the product. Also it provides change of order between $0 < \alpha \leq 1$, if $\alpha = 1$, it equals to the ordinary derivative. That's why many scientists have worked on this derivative. Some of those are (Atangana, Baleanu and Alsaedi, 2015; Jarad, Uğurlu, Abdeljawad and Baleanu, 2017; Horani, Hammad and Khalil, 2016; Abdeljawad, 2015; Hammad and Khalil, 2014; Qureshi, Yusuf and Shaikh, 2019; Qureshi and Yusuf, 2019; Qureshi and Atangana, 2019; Yilmazer and Ozturk, 2019).

Katugampola introduced another conformable derivative known as proportional α -derivative or Katugampola derivative in (Katugampola, 2014). This definition includes a proportional ratio as $e^{\varepsilon t^{-\alpha}}$ in place of shifting. It was considered by Anderson and Ulness (Anderson and Ulness, 2015a; Anderson and Ulness, 2015b). The groundwork spectral theory of regular second order operators is composed of Sturm-Liouville (S–L) theory. S–L problems have considerable importance due to their numerous application in mathematics, physics, engineering etc. Some studies about S–L eigenvalue problems are (Ercan and Panakhov, 2020; Levitan and Sargsjan, 1975; Freiling and Yurko, 2001; Kablan, Chisha and Hasan, 2018) . Important results for the classical S–L problem are contained the asymptotic formulas of norming constants, eigenvalues, eigenfunctions, etc. Many studies on the general theory of boundary value problem with continuous coefficients have been considered but the spectral structure of similar problems with discontinuities is known very little.

The concept of boundary-value problems with discontinuities has applications in mathematics, mechanics, geophysics and other branches of natural properties. The studies on the discontinuous S–L problems have been considered in (Amirov, 2006), (Aydemir and Mukhtarov, 2016), (Ercan and Panakhov, 2019) and (Topsakal and Amirov, 2010). In this study, we give the asymptotic behaviors of the eigenfunctions and eigenvalues for conformable discontinuous S–L problems. S–L problems and its basic spectral features including conformable derivatives were studied by (Abdeljawad and Al-Refai, 2017), (Anderson and Ulness, 2015a), (Anderson and Ulness, 2015b). Fractional S–L problems in Riemann-Liouville and Caputo sense and other derivatives were studied by (Ansari, 2015; Bas and Ozarslan, 2018; Bas, Ozarslan, Baleanu and Ercan, 2018; Ozarslan, Ercan and Bas, 2019). Fractional discontinuous S–L problems were considered in study (Yakar and Akdogan, 2017).

In this study, we want to extend general spectral theory of fractional S–L problems in the case of conformable discontinuous S–L problems and with this end, we show the properties of operator and obtain the representations of the solutions under different initial conditions and the asymptotics for the eigenfunctions and the eigenvalues. The main benefit of conformable derivative enabling arbitrary change of order is that it allows us to better deal with the problem at hand. Also, we check the solutions by graphs for different potentials and different orders. Thereby, we can observe the behaviours of solutions.

2 Preliminaries

This section contains some fundamental definitions and lemmas related to conformable calculus theory by quoting from (Khalil et al., 2014; Horani et al., 2016; Hammad and Khalil, 2014; Anderson and Ulness, 2015a; Anderson and Ulness, 2015b).

Definition 2.1. Let $f : [a, \infty) \rightarrow \mathbb{R}$ and $t > 0$. The left fractional conformable derivative of f of order $0 < \alpha \leq 1$ is given by,

$$(T_{\alpha}^a f)(t) = \lim_{\varepsilon \rightarrow 0} \frac{f\left(t + \varepsilon(t-a)^{1-\alpha}\right) - f(t)}{\varepsilon}.$$

The right fractional conformable derivative of order $0 < \alpha \leq 1$ is given by

$$\left({}^b T_{\alpha} f\right)(t) = -\lim_{\varepsilon \rightarrow 0} \frac{f\left(t + \varepsilon (b - t)^{1-\alpha}\right) - f(t)}{\varepsilon}$$

If, in addition, f is differentiable, then $T_{\alpha}^a(f)(t) = t^{1-\alpha} \frac{df}{dt}(t)$.

Definition 2.2. (Khalil et al., 2014) Let $\alpha \in (0, 1]$ and $t > 0$. Then, conformable left and right fractional integrals of order α are defined as follows:

$$\begin{aligned} \left(I_{\alpha}^a f\right)(x) &= \int_a^x f(t) d_{\alpha}(t, a) = \int_a^x f(t) (t - a)^{\alpha-1} dt, \\ \left({}^b I_{\alpha} f\right)(x) &= \int_x^b f(t) d_{\alpha}(t, b) = \int_x^b f(t) (b - t)^{\alpha-1} dt, \end{aligned}$$

respectively.

Definition 2.3. Let $y, z : I \rightarrow \mathbb{R}$ be α -differentiable on I , then the conformable wronskian of y and z is defined by

$$W_{\alpha}(y, z)(t) = \det \begin{pmatrix} y(t) & z(t) \\ T_{\alpha}^a y(t) & T_{\alpha}^a z(t) \end{pmatrix}, \text{ for } t \in I.$$

Lemma 2.1. Suppose that $f : [a, \infty) \rightarrow \mathbb{R}$ is continuous and $0 < \alpha \leq 1$. Then for all $t > a$ we have

$$T_{\alpha}^a I_{\alpha}^a f(t) = f(t)$$

Let $0 < \alpha \leq 1$. If the function $f : (a, b) \rightarrow \mathbb{R}$ is differentiable then we have for all $t > a$

$$I_{\alpha}^a T_{\alpha}^a f(t) = f(t) - f(a).$$

Lemma 2.2. Suppose that $f : (-\infty, b] \rightarrow \mathbb{R}$ is continuous and $0 < \alpha \leq 1$. Then for all $t < b$ we have

$${}^b T_{\alpha} {}^b I_{\alpha} f(t) = f(t).$$

Let $0 < \alpha \leq 1$. If the function $f : (-\infty, b] \rightarrow \mathbb{R}$ is differentiable then we have for all $t < b$

$${}^b I_{\alpha} {}^b T_{\alpha} f(t) = f(t) - f(b).$$

Theorem 2.3. Let $f, g : [a, b] \rightarrow \mathbb{R}$ be functions such that $f \in I_{\alpha}([a, b])$ with $T_{\alpha}^a f$ is continuous and $g \in {}^b I_{\alpha}([a, b])$ with ${}^b T_{\alpha} g$ is continuous and $0 < \alpha \leq 1$. Then

$$\int_a^b (T_{\alpha}^a f)(x) g(x) d_{\alpha}(x, a) = \int_a^b f(x) ({}^b T_{\alpha} g(x)) d_{\alpha}(b, x) + f(x) g(x)|_a^b.$$

Now, we consider conformable discontinuous Sturm-Liouville equation

$$\begin{aligned} -T_{\alpha}^0 T_{\alpha}^0 y(x) + q(x) y(x) &= \lambda y(x), \quad 0 < x < d, \\ -{}^{\pi} T_{\alpha} {}^{\pi} T_{\alpha} y(x) + q(x) y(x) &= \lambda y(x), \quad d < x < \pi, \end{aligned} \quad (2.1)$$

where $\lambda = k^2$ is a spectral parameter, $0 < \alpha \leq 1$, $x \in [0, \pi]$, T_0^α and ${}^\pi T$ are conformable derivative operators, $y(x)$ is 2α -continuously differentiable on $[0, \pi]$, that is $y \in C^{2\alpha}[0, \pi]$, $q(x)$ is a real-valued and continuous function on $[0, \pi]$, $T_\alpha^0 y(x)$ and ${}^\pi T_\alpha$ are continuous on $[0, \pi]$. We use the boundary conditions given by

$$T_\alpha^0 y(0) - hy(0) = 0, \quad (2.2)$$

$${}^\pi T_\alpha y(\pi) + Hy(\pi) = 0, \quad (2.3)$$

and jump conditions

$$\begin{cases} y(d+0) = -\beta y(d-0) \\ {}^\pi T_\alpha y(d+0) = \beta^{-1} T_\alpha^0 y(d-0) \end{cases} \quad (2.4)$$

where β, d, h, H are real constants, $d \in (0, \pi)$, $\beta \neq 1$, $\beta > 0$. Consider the inner product on $L^2(0, \pi)$ Hilbert space, as follows:

$$\langle f, g \rangle = \int_0^{d-0} f(x) \overline{g(x)} d_\alpha x + \int_{d+0}^\pi f(x) \overline{g(x)} d_\alpha x$$

where $f(x), g(x) \in L^2(0, \pi)$ and $d_\alpha x \equiv x^{\alpha-1} dx$. We define the operator L in this space with domain

$$D(L) = \{f : f(x), T_\alpha^0 f(x) \text{ and } {}^\pi T_\alpha f(x) \text{ are absolutely continuous in } [0, d) \cup (d, \pi], \ell_\alpha f \in L^2(0, \pi), T_\alpha^0 f(0) - hy(0) = 0, f(d+0) = -\beta y(d-0), {}^\pi T_\alpha f(d+0) = \beta^{-1} T_\alpha^0 f(d-0)\}$$

and acting law

$$Lf := \ell_\alpha f.$$

Hence, the problem (2.1)-(2.4) can be written as follows

$$Lv = \lambda v.$$

3 Main Results

The useful feature of the conformable derivative is that it has properties similar to the ordinary derivative. It also allows us to observe the change of order between $0 < \alpha \leq 1$, when $\alpha = 1$, it corresponds to ordinary derivative. Spectral theory of Sturm-Liouville problems was analyzed by Levitan and Sargsjan (Levitan and Sargsjan, 1975) in classical case. Differently from (Levitan and Sargsjan, 1975), we consider spectral asymptotics of discontinuous fractional Sturm-Liouville problems making use of the conformable derivative. In this part, we discuss some properties of operator L and asymptotics of the eigenfunctions and the eigenvalues of problem (2.1)-(2.4).

Theorem 3.1. *The linear operator L is symmetric.*

Proof. Let $f, g \in D(L)$.

$$\begin{aligned}
 \langle Lf, g \rangle - \langle f, Lg \rangle &= \\
 &= \int_0^{d-0} Lf(x) \overline{g(x)} d_\alpha x + \int_{d+0}^\pi Lf(x) \overline{g(x)} d_\alpha x - \int_0^{d-0} f(x) \overline{Lg(x)} d_\alpha x - \int_{d+0}^\pi f(x) \overline{Lg(x)} d_\alpha x \\
 &= \int_0^{d-0} (-T_\alpha^0 T_\alpha^0 f(x) + q(x) f(x)) \overline{g(x)} d_\alpha x + \int_{d+0}^\pi (-{}^\pi T_\alpha {}^\pi T_\alpha f(x) + q(x) f(x)) \overline{g(x)} d_\alpha x \\
 &\quad - \int_0^{d-0} (-T_\alpha^0 T_\alpha^0 g(x) + q(x) g(x)) \overline{f(x)} d_\alpha x - \int_{d+0}^\pi (-{}^\pi T_\alpha {}^\pi T_\alpha g(x) + q(x) g(x)) \overline{f(x)} d_\alpha x \\
 &= \int_0^{d-0} \left(\overline{f(x)} T_\alpha^0 T_\alpha^0 g(x) - \overline{g(x)} T_\alpha^0 T_\alpha^0 f(x) \right) d_\alpha x \\
 &\quad + \int_{d+0}^\pi \left(\overline{f(x)} {}^\pi T_\alpha {}^\pi T_\alpha g(x) - \overline{g(x)} {}^\pi T_\alpha {}^\pi T_\alpha f(x) \right) d_\alpha x \\
 &= \left. \overline{f(x)} T_\alpha^0 g(x) - \overline{g(x)} T_\alpha^0 f(x) \right|_0^{d-0} - \left. \overline{f(x)} {}^\pi T_\alpha g(x) - \overline{g(x)} {}^\pi T_\alpha f(x) \right|_{d+0}^\pi
 \end{aligned}$$

By using conditions (2.2)-(2.3) and jump conditions (2.4), we have

$$\langle Lf, g \rangle = \langle f, Lg \rangle.$$

So L is symmetric. Furthermore, it is easily seen that the operator L is self adjoint. □

Lemma 3.2. *All eigenvalue of problem (2.1)-(2.4) are real.*

Lemma 3.3. *Two eigenfunctions $f(x, \lambda_1)$ and $g(x, \lambda_2)$ corresponding to different eigenvalues λ_1 and λ_2 are orthogonal, i.e.*

$$\int_0^\pi f(x, \lambda_1) \overline{g(x, \lambda_2)} d_\alpha x = 0.$$

Now we will give the representations of the solutions by integral equations for conformable discontinuous Sturm-Liouville problems.

Let $\varphi(x, k)$ and $\psi(x, k)$ be the solutions of equation (2.1) satisfying the initial conditions respectively,

$$\varphi(0, k) = 1, \quad T_\alpha^0 \varphi(0, k) = h, \tag{3.1}$$

$$\psi(\pi, k) = -1, \quad {}^\pi T_\alpha \psi(\pi, k) = H, \tag{3.2}$$

and the jump conditions (2.4).

Theorem 3.4. We have the integral equations for solutions of the problems (2.1), (2.4), (3.1) and (2.1), (2.4), (3.2) as follows, respectively,

$$\begin{aligned} \varphi(x, k) = & \cos \frac{kx^\alpha}{\alpha} + \frac{h}{k} \sin \frac{kx^\alpha}{\alpha} \\ & + \frac{1}{k} \int_0^x \sin k \left(\frac{x^\alpha - t^\alpha}{\alpha} \right) q(t) \varphi(t, k) d_\alpha t, \quad x < d. \end{aligned} \quad (3.3)$$

$$\begin{aligned} \varphi(x, k) = & - \left(\beta^+ \cos \frac{kx^\alpha}{\alpha} + \beta^- \cos \frac{k(x^\alpha - 2d^\alpha)}{\alpha} \right) - \frac{h}{k} \left(\beta^+ \sin \frac{kx^\alpha}{\alpha} - \beta^- \sin \frac{k(x^\alpha - 2d^\alpha)}{\alpha} \right) \\ & - \frac{1}{k} \int_0^d \left(\beta^+ \sin k \left(\frac{x^\alpha - t^\alpha}{\alpha} \right) - \beta^- \sin \frac{k(x^\alpha + t^\alpha - 2d^\alpha)}{\alpha} \right) q(t) \varphi(t, k) d_\alpha t \\ & + \frac{1}{k} \int_d^x \sin k \left(\frac{x^\alpha - t^\alpha}{\alpha} \right) q(t) \varphi(t, k) d_\alpha t, \quad x > d. \end{aligned} \quad (3.4)$$

$$\begin{aligned} \psi(x, k) = & - \cos \frac{k(\pi - x)^\alpha}{\alpha} + \frac{H}{k} \sin \frac{k(\pi - x)^\alpha}{\alpha} \\ & + \frac{1}{k} \int_x^\pi \sin k \left(\frac{(\pi - x)^\alpha - (\pi - t)^\alpha}{\alpha} \right) q(t) \psi(t, k) d_\alpha t, \quad x > d. \end{aligned} \quad (3.5)$$

$$\begin{aligned} \psi(x, k) = & d^{\alpha-1} (\pi - d)^{1-\alpha} \left\{ \beta^+ \cos \frac{k(\pi - x)^\alpha}{\alpha} - \beta^- \cos k \frac{(2(\pi - d)^\alpha - (\pi - x)^\alpha)}{\alpha} \right. \\ & - \frac{H}{k} \left(\beta^+ \sin \frac{k(\pi - x)^\alpha}{\alpha} + \beta^- \sin k \frac{(2(\pi - d)^\alpha - (\pi - x)^\alpha)}{\alpha} \right) \\ & + \frac{1}{k} \int_d^\pi \left(\beta^- \sin k \frac{(2(\pi - d)^\alpha - (\pi - x)^\alpha - (\pi - t)^\alpha)}{\alpha} + \beta^+ \sin k \frac{((\pi - t)^\alpha - (\pi - x)^\alpha)}{\alpha} \right) q(t) \varphi(t, k) d_\alpha t \\ & \left. + \frac{1}{k} \int_x^d \sin k \frac{((\pi - x)^\alpha - (\pi - t)^\alpha)}{\alpha} q(t) \varphi(t, k) d_\alpha t \right\}, \quad x < d \end{aligned} \quad (3.6)$$

where

$$\beta^+ = \frac{1}{2} \left(\beta + \frac{(\pi - d)^{\alpha-1} d^{1-\alpha}}{\beta} \right) \text{ and } \beta^- = \frac{1}{2} \left(\beta - \frac{(\pi - d)^{\alpha-1} d^{1-\alpha}}{\beta} \right). \quad (3.7)$$

Theorem 3.5. Let be $k = \sigma + i\tau$. As $|\lambda| \rightarrow \infty$ uniformly in x for $0 \leq x \leq \pi$, the following asymptotics are valid.

$$\begin{aligned} \varphi(x, k) = & \cos \frac{kx^\alpha}{\alpha} + \frac{1}{2k} \sin \frac{kx^\alpha}{\alpha} \int_0^x q(t) d_\alpha t + O \left(\frac{1}{k} \exp |\tau| \frac{x^\alpha}{\alpha} \right), \quad x < d \\ \varphi(x, k) = & - \left(\beta^+ \cos \frac{kx^\alpha}{\alpha} + \beta^- \cos \frac{k(x^\alpha - 2d^\alpha)}{\alpha} \right) \\ & - \frac{1}{2k} \int_d^x \left(\beta^+ \sin \frac{kx^\alpha}{\alpha} + \beta^- \sin \frac{k(x^\alpha - 2d^\alpha)}{\alpha} \right) q(t) d_\alpha t + O \left(\frac{1}{k} \exp |\tau| \frac{x^\alpha}{\alpha} \right), \quad x > d \end{aligned} \quad (3.8)$$

For the proof of Theorem 3.5, use the integral equations (3.3), (3.4). Making classical processes in (Freiling and Yurko, 2001), the proof can be obtain easily.

Now, we will obtain asymptotics for eigenvalues.

Theorem 3.6. *The asymptotic formula for the eigenvalues of the boundary-value problem (2.1), (2.4), (3.1) is given by*

$$\sqrt{\lambda_n} = n\alpha\pi^{1-\alpha} + h_n + \frac{\delta_n}{n} + \frac{\xi_n}{n}, \quad (3.9)$$

where $h_n, \delta_n \in l_\infty$ $\xi_n \in l_2$.

Proof. The characteristic function of the problem L can be defined as follows

$$\Delta(k) := W_\alpha[\varphi, \psi] = V(\varphi) = {}^\pi T_\alpha \varphi(\pi, k) + H\varphi(\pi, k).$$

The eigenvalues $\{\lambda_n\}_{n \geq 0}$ be the zeros of $\Delta(k)$ where

$$\begin{aligned} {}^\pi T_\alpha \varphi(x, k) = & -k(\pi - x)^{1-\alpha} x^{\alpha-1} \left(\beta^+ \sin k \frac{x^\alpha}{\alpha} + \beta^- \sin k \frac{(x^\alpha - 2d^\alpha)}{\alpha} \right) \\ & + \frac{(\pi - x)^{1-\alpha}}{2k} \left\{ \left(\beta^+ \sin k \frac{(x^\alpha - 2d^\alpha)}{\alpha} + \beta^- \sin k \frac{x^\alpha}{\alpha} \right) q(x) \right. \\ & \left. + \int_d^x kx^{\alpha-1} \left\{ \beta^- \cos k \frac{(x^\alpha - 2d^\alpha)}{\alpha} + \beta^+ \cos k \frac{x^\alpha}{\alpha} \right\} q(t) d_\alpha t \right\} + O\left(\exp|\tau| \frac{x^\alpha}{\alpha}\right). \end{aligned} \quad (3.10)$$

Substituting ${}^\pi T_\alpha \varphi(\pi, k)$ and $\varphi(\pi, k)$ from the estimates (3.10), for large $|k|$ we obtain the asymptotic formula (3.9). In here it is easily seen that $\sin \delta_n = O(n^{-1})$, i.e. $\delta_n = O(n^{-1})$. $\square$

Visual Results

This section contains significant data which is comparison of the solutions for discontinuous S-L problem in terms of conformable derivative. The reason why we prefer this derivative is that it has same features with the ordinary derivative and is easier to process. We show the comparisons of the eigenfunctions $\varphi(x, k)$ and $\psi(x, k)$ of the problems (2.1), (2.4), (3.1) and (2.1), (2.4), (3.2) with the graphics in different cases. The right and left-sided translations are observed in all graphics as potentials and orders of α increase. The main purpose in drawing graphs with different values is that one can observe the behavior of representations of solutions curves for discontinuous SL equation in light of conformable derivative. The left and right sides of all figures are considering for $0 < x < \frac{\pi}{2}$ and $\frac{\pi}{2} < x < \pi$, respectively. Assume that $h = 1$, $H = 1$, $d = \frac{\pi}{2}$ and $\beta = 2$ for all figures.

Conclusion

In this article, the spectral construction of conformable S-L equation with jump conditions is investigated. It is a first in the literature to define this kind of spectral problem with jump conditions according to right and left sided derivatives in sense of conformable derivatives. We have obtained the representations of the solutions through variation of parameters method. By

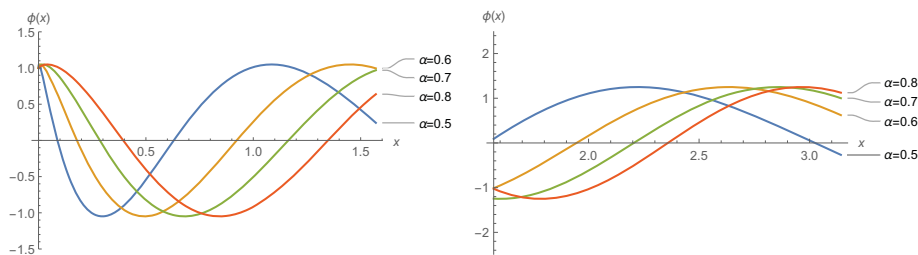


Figure 1: Comparisons of the solutions under different orders, $\lambda = 10$, $q(x) = 0$.

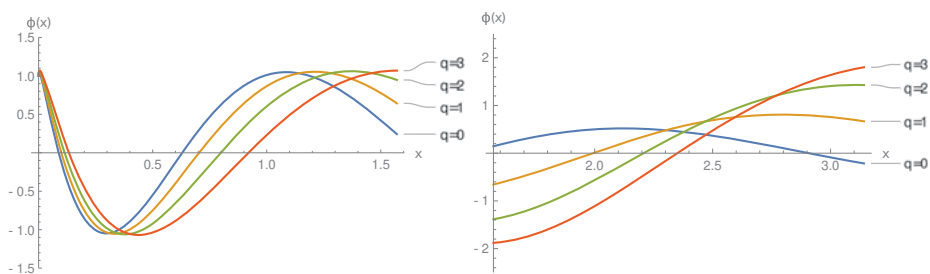


Figure 2: Comparisons of the solutions under different potentials, $\lambda = 10$, $\alpha = 0.5$.

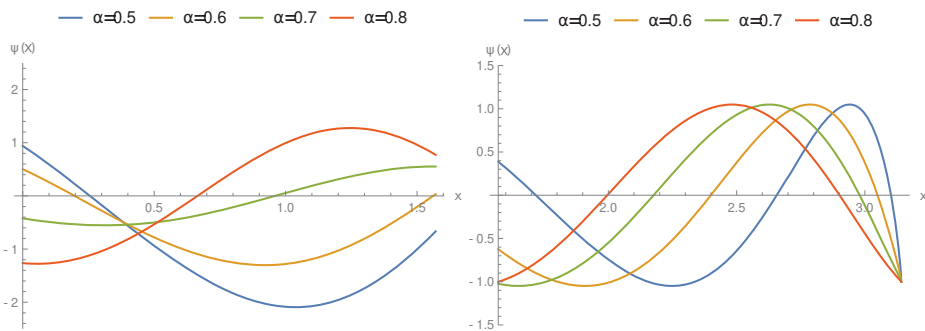


Figure 3: Comparisons of the solutions under different orders, $\lambda = 10$, $q(x) = 0$.

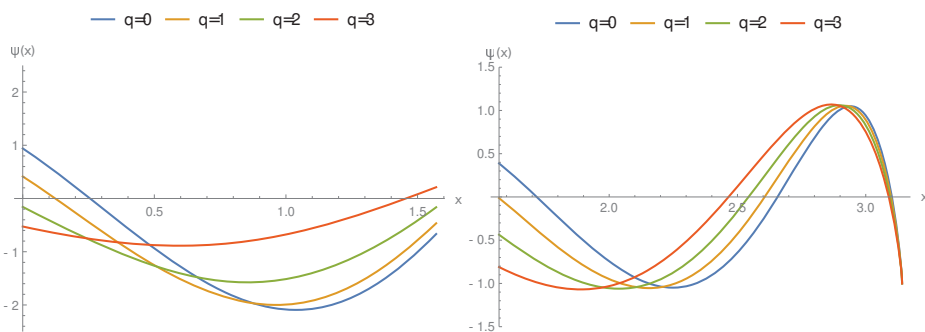


Figure 4: Comparisons of the solutions under different potentials, $\lambda = 10$, $\alpha = 0.5$.

virtue of these representations of solutions we have presented the asymptotics of solutions, eigenvalues and norming constants. Consequently, we inquire spectral features of discontinuous S-L problem by using conformable derivative. In order to calculate the spectral data for considered problem, one relies on a number of indispensable tools which are already well-known in classical analysis. By considering this necessity, limit-based local derivatives create an cognizance to this objective. Because using many features in classical analysis is difficult to handle for non-local derivatives. Hence, in this study we make use of basic local derivative which allows us to use arbitrary order and to observe S-L problem in detail. Also, we compare the solutions $\varphi(x, k)$ and $\psi(x, k)$ with graphics under different orders and different potentials given by Figure 1-Figure 4. The main purpose in drawing graphs with different values is that one can observe the behavior of representations of solutions curves for S-L problem in light of conformable derivative. New applications of conclusions and opinions from this topic shed light on future problems such as inverse problems of spectral theory.

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References

- Abdeljawad, T. 2015. On conformable fractional calculus, *J. Comput. Appl. Math.* **279**: 57–66.
- Abdeljawad, T. and Al-Refai, M. 2017. Fundamental results of conformable sturm-liouville eigenvalue problems, *Complexity* **Article ID 3720471**: 1–7.
- Amirov, R. K. 2006. On sturm-liouville operators with discontinuity conditions inside an interval, *J. Math. Anal. Appl.* **317(1)**: 163–176.
- Anderson, D. R. and Ulness, D. J. 2015a. Newly defined conformable derivatives, *Adv. Dyn. Syst. Appl.* **10(2)**: 109–137.
- Anderson, D. R. and Ulness, D. J. 2015b. Properties of the katugampola fractional derivative with potential application in quantum mechanics, *J. Math. Phys.* **56(6)**: 063502.
- Ansari, A. 2015. On finite fractional sturm-liouville transforms, *Integral Transforms Spec. Funct.* **26(1)**: 51–64.
- Atangana, A., Baleanu, D. and Alsaedi, A. 2015. New properties of conformable derivative, *Open Math* **13(1)**: 889–898.
- Aydemir, K. and Mukhtarov, O. 2016. Asymptotic distribution of eigenvalues and eigenfunctions for a multi-point discontinuous sturm-liouville problem, *Electron. J. Differ. Equ* pp. 1–14.
- Bas, E. and Ozarslan, R. 2018. Real world applications of fractional models by atangana-baleanu fractional derivative, *Chaos, Solitons & Fractals* **116**: 121–125.

- Bas, E., Ozarslan, R., Baleanu, D. and Ercan, A. 2018. Comparative simulations for solutions of fractional sturm–liouville problems with non-singular operators, *Adv. Differ. Equ.* **350**: 1–19.
- Ercan, A. and Panakhov, E. S. 2019. Stability of the reconstruction discontinuous sturm-liouville problem, *Commun. Fac. Sci. Univ. Ank. Ser. A1. Math. Stat.* **68(1)**: 484–499.
- Ercan, A. and Panakhov, E. S. 2020. Spectral analysis for discontinuous conformable sturm-liouville problems with spectral parameter contained in boundary conditions, *Appl. Comput. Math.* **19(2)**: 245–254.
- Freiling, G. and Yurko, V. 2001. *Inverse Sturm-Liouville Problems and Their Applications*, Nova Science Publishers, Huntington, NY.
- Hammad, M. A. and Khalil, R. 2014. Abel's formula and wronskian for conformable fractional differential equations, *Int. J. Differ. Equ. Appl.* **13(3)**: 177–183.
- Horani, M. A., Hammad, M. A. and Khalil, R. 2016. Variation of parameters for local fractional nonhomogenous linear-differential equations, *J Math Comput Sci.* **16**: 147–153.
- Jarad, F., Uğurlu, E., Abdeljawad, T. and Baleanu, D. 2017. On a new class of fractional operators, *Adv. Differ. Equ.* **247**: 16.
- Kablan, A., Chisha, R. and Hasan, S. H. 2018. Green's function for finitely many-interval sturm-liouville problem, *Int. J. Appl. Math. Stat.* **57(3)**: 180–193.
- Katugampola, U. 2014. A new fractional derivative with classical properties, *arxiv preprint* p. arXiv:1410.6535v2.
- Khalil, R., Horani, M. A., Yousef, A. and Sababheh, M. 2014. A new definition of fractional derivative, *J. Comput. Appl. Math.* **264**: 65–70.
- Levitan, B. M. and Sargsjan, I. S. 1975. *Introduction to Spectral Theory: Selfadjoint Ordinary Differential Operators*, American Math. Soc., Pro., R.I.
- Ozarslan, R., Ercan, A. and Bas, E. 2019. β -type fractional sturm-liouville coulomb operator and applied results, *Math. Methods Appl. Sci.* **42(18)**: 6648–6659.
- Qureshi, S. and Atangana, A. 2019. Mathematical analysis of dengue fever outbreak by novel fractional operators with field data, *Physica A: Stat.Mech. Appl.* **526(12)**: 1127.
- Qureshi, S. and Yusuf, A. 2019. Fractional derivatives applied to msei problems: comparative study with real world data, *European Phys J Plus* **134(4)**: 171.
- Qureshi, S., Yusuf, A. and Shaikh, A. e. a. 2019. Fractional modeling of blood ethanol concentration system with real data application, *Chaos* **29**: 013143.
- Topsakal, N. and Amirov, R. 2010. Inverse problem for sturm-liouville operators with coulomb potential which have discontinuity conditions inside an interval, *Math. Phys. Anal. Geom.* **13(1)**: 29–46.

- Yakar, A. and Akdogan, Z. 2017. On the fundamental solutions of a discontinuous fractional boundary value problem, *Adv. Differ. Equ* **378(15)**: 15.
- Yilmazer, R. and Ozturk, O. 2019. Explicit solutions of the whittaker equation via discrete fractional calculus operator, *Int. J. Appl. Math. Stat.* **58(3)**: 73–81.