

On the Archimedean copula constructed from the Bézier polynomial type diagonal section

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ABSTRACT

In this study, we construct a convex Bézier type diagonal section which uniquely characterizes the Archimedean copula. The proposed diagonal section is flexible according to its polynomial degree; hence, we can add the additional parameters to the diagonal section by increasing the polynomial degree. We also present a graphical tool based on the diagonal section in order to visualize the different types of Archimedean copulas in the copula test space. As an illustration, the real data examples are given.

Keywords: Archimedean copula, diagonal section, Bézier polynomial.

2000 Mathematics Subject Classification: 62E10.

1 Introduction

Copula is a popular tool for modelling the dependence structure between random variables. One of the problems investigated in copula theory is the construction of unique copula using diagonal section. A diagonal section is a considerable part in the presence of tail dependence of the copula, as the tail dependence of continuous random variable can be described by using diagonal section of copulas. Therefore, many papers have investigated the construction and generalization of the diagonal section. (Sánchez and Flores, 2018) investigated the construction of bivariate copulas with given diagonal sections. For given diagonal function and opposite diagonal functions, (De Amo, De Meyer, Carrillo and Sánchez, 2016) proposed the constructing method of unique copulas using diagonal (and opposite diagonal) sections. (Wysocki, 2012) introduced a family $\mathbb{F}$ of functions called convex diagonal generators of Archimedean family of copula. They showed that to each diagonal generator f in $\mathbb{F}$ there corresponds an asymptotic Archimedean family of copula C_f .

The main objective of this paper is to introduce a new family of Bézier type convex diagonal section which belongs to the family $\mathbb{F}$ introduced by (Wysocki, 2012). The advantage of the proposed Bézier type convex diagonal section is its flexibility according to its polynomial degree. By increasing the polynomial degree, we have additional parameters in the Bézier type

convex diagonal section. As a result, this procedure increases model freedom.

Organization of the paper is given by following: In Section 2, the main properties of the copula and diagonal section are given. In Section 3, Bézier type convex diagonal section of Archimedean copula and its properties are discussed. In Section 4, we fit real data sets to the proposed diagonal sections using the non-metric multi-dimensional scaling (NMDS) method. Finally, the conclusion is presented in Section 5.

2 Preliminaries

In this section, we give an introduction on the most important copula properties, with a focus on the diagonal copula class with diagonal section δ . Copula is a two-dimensional function described as $C(y, z) : [0, 1]^2 \rightarrow [0, 1]$ which satisfies the following properties

1. $C(0, z) = C(z, 0) = 0, z \in [0, 1]$
2. $C(1, z) = C(z, 1) = z, z \in [0, 1]$
3. for all $y, y', z, z' \in [0, 1]$ with $y < y'$ and $z < z'$

$$V_C([y, y'] \times [z, z']) = C(y, y') - C(y, z') - C(y', z) + C(y, z) \geq 0$$

where $V_C([y, y'] \times [z, z'])$ is the C -volume of $[y, y'] \times [z, z']$. Consider two random variables y, z with cumulative distribution function H and marginals F, G . (Sklar, 1959) states that a copula C can be written as

$$H(y, z) = C(F(y), G(z)) = C(u, v)$$

for all $y, z \in [-\infty, +\infty]$ and $u, v \in [0, 1]$. Copula allow practitioners to model marginal and dependence structures separately. This feature is very practicable in areas such as survival analysis, hydrology, actuarial science and finance. For the applications, see (Dharmawan, Ida Harini and Sumarjaya, 2015), (Resti, Ismail and Jaaman, 2012), (Zaroudi, Behzadi and Faridrohani, 2018), (Nguyen and Thuy, 2018), (Chuang and Tang, 2014), (Sewe, Weke and Mungátu, 2015), (Shams and Rezaee, 2014), (Amphanthong and Busababodhin, 2015), (Yamut, Ucer and Turgutlu, 2021), (Ucer, 2013) and (Ucer and Turgutlu, 2009).

This paper mainly concentrates on the diagonal section of an Archimedean copula C . A copula C is said to be Archimedean if $C(u, v) = \varphi^{[-1]}(\varphi(u) + \varphi(v))$ where $\varphi : [0, 1] \rightarrow [0, \infty)$ is generator function, which is convex and strictly decreasing. For more details about Archimedean copula, see (Susam and Hudaverdi, 2020). A diagonal section is the function $\delta : [0, 1] \rightarrow [0, 1]$ which is given by $\delta(t) = C(t, t)$. (Durante, Fernández-Sánchez and Pappadá, 2015) defined the properties of diagonal section given in the following definition:

Definition 1. Let $\delta(t)$ be the diagonal section of a copula C , then

$$(D11) \quad \delta(1) = 1;$$

$$(D12) \quad \delta(t) \leq t, t \in [0, 1];$$

(D13) $\delta(t)$ is increasing;

(D14) $\delta(t)$ is 2-Lipschitz, i.e. $|\delta(t_2) - \delta(t_1)| \leq 2|t_2 - t_1|$, $t_2, t_1 \in [0, 1]$.

For the diagonal section, there are some copulas with diagonal section which coincide with δ . Let us suppose that δ is a diagonal function satisfied D11-D14. (Nelsen and Fredricks, 1997) introduced diagonal copula $K_\delta(y, z)$ with diagonal section δ . They defined $K_\delta(y, z)$ given by

$$K_\delta(y, z) = \min\left(y, z, \frac{\delta(y)}{2} + \frac{\delta(z)}{2}\right),$$

for all $(y, z) \in [0, 1]^2$. Also, (Fredricks and Nelsen, 2002) proposed Bertino copula

$$B_\delta(y, z) = \min(y, z) - \min_{t \in [y \wedge z, y \vee z]} (t - \delta(t)).$$

It is important to ask the question: “Does a unique copula having diagonal sections exist?”. (De Amo et al., 2016) and (Klement and Kolesárová, 1998) stated that there are infinite copulas having the same diagonal functions. Also, (De Amo et al., 2016)) determined the sufficient conditions on diagonal and opposite diagonal sections for which they determine the copula in a unique way. (Wysocki, 2012) stated that the Archimedean copula is uniquely characterized by the diagonal section. The diagonal section of an Archimedean copula C is given by $\delta(t) = \varphi^{[-1]}(2\varphi(t))$. (Wysocki, 2012) introduced a family of $\mathbb{F}$ of functions called diagonal generators of Archimedean copula. The family of $\mathbb{F}$ has several properties given in the next definition.

Definition 2. The family $\mathbb{F}$, consisting all functions $f : I \rightarrow I$ have following properties:

(D21) f is strictly increasing bijection with $f(0) = 0$ and $f(1) = 1$;

(D22) $f(t) < t$, $t \in [0, 1]$;

(D23) f is convex;

(D24) f is continuously differentiable and $\lim_{t \rightarrow 1^-} f'(t) = 2$

Then, they showed that each diagonal generator f defined in $\mathbb{F}$, there exists an Archimedean copula with the asymptotic representation

$$C_k(u, v) = \lim_{k \rightarrow \infty} f^k[f^{-k}(u) + f^{-k}(v) - 1]$$

In this paper, we construct a convex Bézier polynomial type diagonal section which belongs to the family of $\mathbb{F}$. We can increase the modelling freedom by increasing the polynomial degree, because more parameters are imported to the copula by increasing the polynomial degree. The next section presents a new convex Bézier diagonal section and related diagonal copula.

3 Convex Bézier type diagonal section and its dependence properties

In this section, we explain how the diagonal section δ can be used in order to construct multi-parameter bivariate Archimedean copulas with the help of Bézier polynomial. Let $\delta_{\alpha, m}(t)$ be a convex Bézier polynomial type diagonal section

$$\delta_{\alpha, m}(t) = \sum_{j=0}^m \alpha_j P_{j, m}(t), \quad t \in [0, 1] \quad (3.1)$$

where $P_{j,m}(t) = \binom{m}{j} t^j (1-t)^{m-j}$ is the Binomial probability and α_j , $j = 0, \dots, m$ are the control points. The next proposition provides some properties of the proposed diagonal section.

Proposition 1. A Bezier type convex diagonal section $\delta_{\alpha,m}$ belongs to the family $\mathbb{F}$ if the following constraints are hold:

1. $\alpha_0 = 0$, $\alpha_m = 1$;
2. $\alpha_j < \frac{j}{m}$, $j = 1, \dots, m$;
3. $0 < \alpha_1 < \dots < \alpha_{m-1} < 1$;
4. $\alpha_{j+2} - 2\alpha_{j+1} + \alpha_j \geq 0$, $j = 0, \dots, m-2$;
5. $\alpha_{m-1} = \frac{m-2}{m}$.

Proof. From the endpoint property of Bézier function, $\delta_{\alpha,m}(1) = \alpha_m = 1$ and $\delta_{\alpha,m}(0) = \alpha_0 = 0$. See (Duncan, 2005). For the proof of Proposition 1.2, let $\alpha_j = \frac{j}{m} - r_j$, $j = 0, \dots, m$ then

$$\begin{aligned} \delta_{\alpha,m}(t) &= \sum_{j=0}^m \alpha_j \binom{m}{j} t^j (1-t)^{m-j} \\ &= \sum_{j=0}^m \left(\frac{j}{m} - r_j \right) \binom{m}{j} t^j (1-t)^{m-j} \\ &= \sum_{j=0}^m \left(\frac{j}{m} \right) \binom{m}{j} t^j (1-t)^{m-j} - \sum_{j=0}^m r_j \binom{m}{j} t^j (1-t)^{m-j} \\ &= t \sum_{j=1}^m \binom{m-1}{j-1} t^{j-1} (1-t)^{m-j} - \sum_{j=0}^m r_j \binom{m}{j} t^j (1-t)^{m-j} \\ &= t \sum_{l=0}^{m-1} t^l (1-t)^{m-l-1} \binom{m-1}{l} - \sum_{j=0}^m r_j \binom{m}{j} t^j (1-t)^{m-j} \\ &= t - \sum_{j=0}^m r_j \binom{m}{j} t^j (1-t)^{m-j} < t. \end{aligned}$$

thus if the inequality $\alpha_j < \frac{j}{m}$, $j = 0, \dots, m$ is satisfied then $\delta_{\alpha,m}(t) < t$. First derivative of the $\delta_{\alpha,m}$ can be obtained as

$$\delta'_{\alpha,m}(t) = m \sum_{j=0}^{m-1} (\alpha_{j+1} - \alpha_j) t^j (1-t)^{m-j-1} \binom{m-1}{j}$$

, see (Duncan, 2005). If the inequality $(\alpha_{j+1} - \alpha_j) > 0$, $j = 1, \dots, m-1$ is satisfied then $\delta_{\alpha,m}$ is increasing, thus proof of the Proposition 1.3 is completed.

Also, the second derivative of $\delta_{\alpha,m}(t)$ can be obtained as

$$\delta''_{\alpha,m}(t) = m(m-1) \sum_{j=0}^{m-2} (\alpha_{j+2} - 2\alpha_{j+1} + \alpha_j) t^j (1-t)^{m-2-j} \binom{m-2}{j}$$

If the inequality $\alpha_{j+2} - 2\alpha_{j+1} + \alpha_j \geq 0$, $j = 0, \dots, m-2$ is satisfied then $\delta_{\alpha,m}$ is convex.

If $\delta_{\alpha,m}$ is continuously differentiable and $\lim_{t \rightarrow 1-1} \delta'_{\alpha,m}(t) = 2$ then $\delta_{\alpha,m}$ is 2-Lipschitz. We should prove that $\delta_{\alpha,m}$ is 2-Lipschitz. Let $t_2 \geq t_1$ be any points of $[0, 1]$. We show that

$$|\delta_{\alpha,m}(t_2) - \delta_{\alpha,m}(t_1)| \leq L|t_2 - t_1|$$

where L is a Lipschitz constant. Then

$$\begin{aligned} \delta_{\alpha,m}(t_2) &= \sum_{j=0}^m \alpha_j \binom{m}{j} (1-t_2)^{m-j} (t_1 + (t_2 - t_1))^j \\ &= \sum_{j=0}^m \alpha_j \binom{m}{j} (1-t_2)^{m-j} \left(\sum_{k=0}^j \binom{j}{k} t_1^k (t_2 - t_1)^{j-k} \right) \\ &= \sum_{j=0}^m \sum_{k=0}^j \alpha_j \frac{m! t_1^k (t_2 - t_1)^{j-k} (1-t_2)^{m-j}}{k!(j-k)!(m-j)!} \end{aligned}$$

We can invert the order of the summation and write $k + l = j$, then

$$\delta_{\alpha,m}(t_2) = \sum_{k=0}^m \sum_{l=0}^{m-k} \alpha_{k+l} \frac{m!}{k!l!(m-k-l)!} t_1^k (t_2 - t_1)^l (1-t_2)^{m-k-l}$$

In the similar way, we can construct $\delta_{\alpha,m}(t_1)$,

$$\begin{aligned} \delta_{\alpha,m}(t_1) &= \sum_{k=0}^m \alpha_k \binom{m}{k} t_1^k ((t_2 - t_1) + (1-t_2))^{m-k} \\ &= \sum_{k=0}^m \alpha_k \binom{m}{k} t_1^k \left(\sum_{l=0}^{m-k} (t_2 - t_1)^l (1-t_2)^{m-k-l} \right) \\ &= \sum_{k=0}^m \sum_{l=0}^{m-k} \alpha_k \frac{m!}{k!l!(m-k-l)!} t_1^k (t_2 - t_1)^l (1-t_2)^{m-k-l}. \end{aligned}$$

Then,

$$\begin{aligned} &|\delta_{\alpha,m}(t_2) - \delta_{\alpha,m}(t_1)| \\ &= \left| \sum_{k=0}^m \sum_{l=0}^{m-k} \frac{m!}{k!l!(m-k-l)!} t_1^k (t_2 - t_1)^l (1-t_2)^{m-k-l} (\alpha_{k+l} - \alpha_k) \right| \\ &\leq L \left| \sum_{k=0}^m \sum_{l=0}^{m-k} \frac{m!}{k!l!(m-k-l)!} t_1^k (t_2 - t_1)^l (1-t_2)^{m-k-l} \right| \\ &= L \left| \sum_{l=0}^m \sum_{k=0}^{m-l} \frac{(t_2 - t_1)^l m!}{l!(m-l)!} \frac{l}{m} \left(\sum_{k=0}^{m-l} \binom{m-l}{k} t_1^k (1-t_2)^{m-l-k} \right) \right| \\ &= L \left| \sum_{l=0}^m \binom{m}{l} (t_2 - t_1)^l \frac{l}{m} (t_1 + 1 - t_2)^{m-l} \right| \\ &= L |P_{l,m}(t_2 - t_1)| \\ &\leq L|t_2 - t_1|. \end{aligned}$$

It is obvious that Lipschitz constant L depends on the control points α . Also, Lipschitz constant L can be written as $L = \sup_{t \in [0,1]} |\delta'_{\alpha,m}(t)|$. Finally, from the end point rule of Bézier polynomial

$$\lim_{t \rightarrow 1-1} \delta'_{\alpha,m}(t) = m(\alpha_m - \alpha_{m-1}) = 2, \text{ thus } \alpha_{m-1} = \frac{m-2}{m}. \quad \square$$

$\delta_{\alpha,m}$ which belongs to the family $\mathbb{F}$ is called as diagonal generator of the Archimedean copulas. To every diagonal generator defined in $\mathbb{F}$, there is a corresponding unique Archimedean copula C with strict additive generator. For every Bézier type convex diagonal section $\delta_{\alpha,m}$ belongs to the family $\mathbb{F}$, an Archimedean copula $C(u, v)$ may have an asymptotic representation as

$$C(u, v) = \lim_{k \rightarrow \infty} C_k(u, v) = \lim_{k \rightarrow \infty} \delta_{\alpha,m}^k \left\{ \delta_{\alpha,m}^{-k}(u) + \delta_{\alpha,m}^{-k}(v) - 1 \right\}.$$

From (Wysocki, 2012) in Theorem 2, the convergence of $C_k(u, v)$ may be obtained as

$$\lim_{k \rightarrow \infty} C_k(u, v) = h\{g(u) + g(v)\},$$

where $\lim_{k \rightarrow \infty} g_k(t) = g(t)$ and $\lim_{k \rightarrow \infty} h_k(t) = h(t)$. Also, g_k and h_k are defined as

$$g_k(t) = 2^k \left(1 - \delta_{\alpha,m}^{-k}(t) \right), \quad t \in [0, 1],$$

$$h_k(x) = \delta_{\alpha,m}^k \left(1 - \frac{x}{2^k} \right), \quad x \in [0, 2^k].$$

We note that, derivation of g and h may be analytically impossible due to the complexity of Bézier polynomial especially for the highest polynomial degree m . The next proposition defines the tail dependence properties of Archimedean copula $C(u, v)$ whose diagonal section is $\delta_{\alpha,m}$.

Proposition 2. Archimedean copulas with diagonal section $\delta_{\alpha,m}$ have the following tail dependence coefficients

$$\lambda_U = 0$$

$$\lambda_L = m\alpha_1$$

Proof. The tail dependence coefficients based on Bézier type diagonal section can be defined as

$$\lambda_L = \lim_{t \rightarrow 0^+} \frac{\delta_{\alpha,m}(t)}{t} = \delta'_{\alpha,m}(0^+)$$

$$\lambda_U = \lim_{t \rightarrow 1^-} \frac{1 - 2t + \delta_{\alpha,m}(t)}{1 - t} = 2 - \delta'_{\alpha,m}(1^-).$$

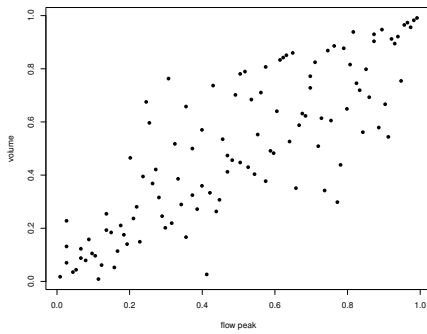
First order derivative of Bézier polynomial is derived by

$$\delta'_{\alpha,m}(t) = m \sum_{i=0}^{m-1} (\alpha_{i+1} - \alpha_i) P_{i,m-1}(t)$$

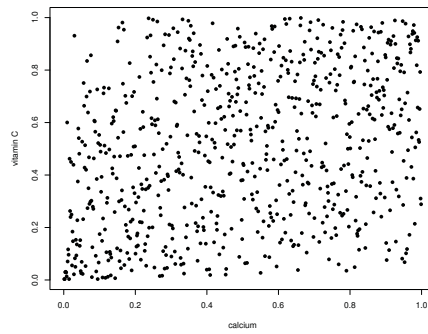
From the end-point rule of the Bézier polynomial, $\lim_{t \rightarrow 0^+} \delta'_{\alpha,m}(t)$ and $\lim_{t \rightarrow 1^-} \delta'_{\alpha,m}(t)$ are equal to $m(\alpha_1 - \alpha_0)$ and $m(\alpha_m - \alpha_{m-1})$ respectively (see (Duncan, 2005)). Because of $\alpha_0 = 0, \alpha_m = 1$ and $\alpha_{m-1} = \frac{m-2}{m}$ then desired results are obtained. $\square$

4 Application

In this section, we fit real data sets to the proposed diagonal sections using the NMDS method. The NMDS method gives us information to distinguish different parametric model on the basis of the diagonal section behaviour and provides a graphical tool to assess which Archimedean



(a) Scatter Plots of Loss-Alea data



(b) Scatter Plots of Women daily nutrient intake of calcium and vitamin C

Figure 1: Scatter plots of real data sets

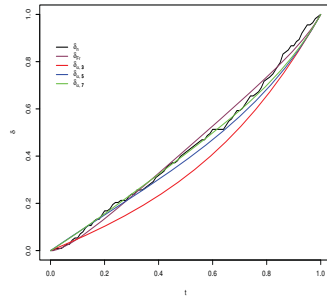
Table 1: The parameter estimation of Archimedean copulas

Data	δ	CvM	$\hat{\theta}$	$\hat{\alpha}_0$	$\hat{\alpha}_1$	$\hat{\alpha}_2$	$\hat{\alpha}_3$	$\hat{\alpha}_4$	$\hat{\alpha}_5$	$\hat{\alpha}_6$	$\hat{\alpha}_7$
variable-width	δ_{Fr}	0.0004	9.405	-	-	-	-	-	-	-	-
	δ_G	0.0005	2.849	-	-	-	-	-	-	-	-
	δ_C	0.0003	3.698	-	-	-	-	-	-	-	-
	$\delta_{\alpha,3}$	0.0047	-	0	0.166	0.333	1	-	-	-	-
	$\delta_{\alpha,5}$	0.0007	-	0	0.149	0.299	0.449	0.600	1	-	-
	$\delta_{\alpha,7}$	0.0003	-	0	0.102	0.224	0.346	0.469	0.591	0.714	1
calc. vit-C	δ_{Fr}	7.8×10^{-5}	1.965	-	-	-	-	-	-	-	-
	δ_G	0.0001	1.266	-	-	-	-	-	-	-	-
	δ_C	3.2×10^{-5}	0.533	-	-	-	-	-	-	-	-
	$\delta_{\alpha,3}$	5.3×10^{-5}	-	0	0.118	0.333	1	-	-	-	-
	$\delta_{\alpha,5}$	7.0×10^{-6}	-	0	0.0537	0.154	0.377	0.600	1	-	-
	$\delta_{\alpha,7}$	6.1×10^{-6}	-	0	0.042	0.094	0.206	0.335	0.524	0.714	1

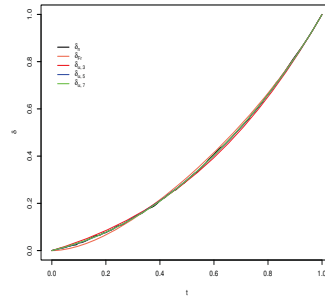
copulas with diagonal sections δ more closely capture the specific features of the data.

Let (X_i, Y_i) , $i = 1, \dots, n$ be a random sample from the Archimedean copulas and (U_i, V_i) , $i = 1, \dots, n$ be associated with the pseudo-observations. Consider a set of m diagonal function of Archimedean copula $\delta_1, \dots, \delta_m$ which belong to the different Archimedean copula. Dissimilarity between the empirical estimate of the diagonal function δ_n and the parametric diagonal functions δ_i for $i = 1, \dots, m$ can be defined by

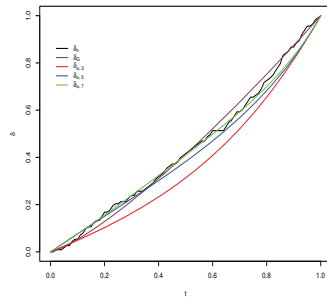
$$d(\delta_n, \delta_i) = \int_0^1 |\delta_n(t) - \delta_i(t)|^2 dt, \quad i = 1, \dots, m \quad (4.1)$$



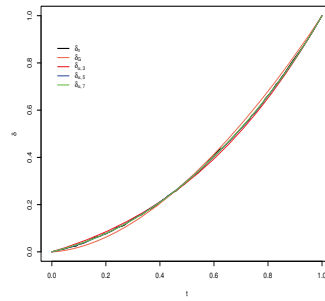
(a) Parametric $(\delta_{Fr}, \delta_{\alpha,m})$ and Non-parametric estimations of diagonal sections for Loss-Alea data



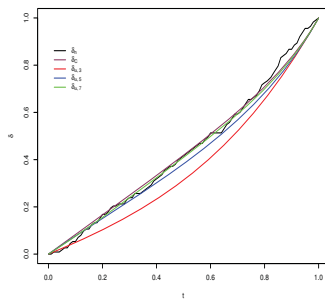
(b) Parametric $(\delta_{Fr}, \delta_{\alpha,m})$ and Non-parametric estimations of diagonal sections for nutrient data



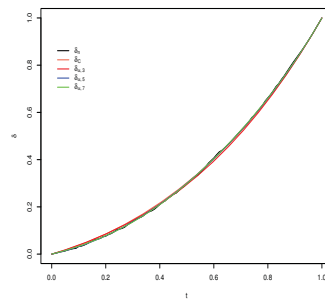
(c) Parametric $(\delta_G, \delta_{\alpha,m})$ and Non-parametric estimations of diagonal sections for Loss-Alea data



(d) Parametric $(\delta_G, \delta_{\alpha,m})$ and Non-parametric estimations of diagonal sections for nutrient data



(e) Parametric $(\delta_C, \delta_{\alpha,m})$ and Non-parametric estimations of diagonal sections for Loss-Alea data



(f) Parametric $(\delta_C, \delta_{\alpha,m})$ and Non-parametric estimations of diagonal sections for nutrient data

Figure 2: Parametric and Non-parametric estimations of diagonal sections for real data sets

Similarly, the dissimilarity between the i -th and the j -th diagonal function is computed as

$$d(\delta_i, \delta_j) = \int_0^1 |\delta_i(t) - \delta_j(t)|^2 dt, \quad 1 < i \neq j < m. \quad (4.2)$$

Let us give a procedure of the graphical tool for the selection of appropriate Archimedean copula in a given data set. The procedure can be given by the following:

- For $i = 1, \dots, m$ estimate dependence parameter(s) θ of a diagonal function δ_i from the family of the Archimedean copula.
- For $i = 1, \dots, m$ compute the dissimilarity between δ_i and the empirical estimate $\Delta_{(emp,i)} = d(\delta_n, \delta_i)$ by using equation 1.
- For the m Archimedean copulas diagonal function $\delta_1, \dots, \delta_m$ compute mutual dissimilarities between $\Delta_{(i,j)} = d(\delta_i, \delta_j)$ by using equation 2.
- Symmetric square matrix of the dimension $m + 1$, $D = \sigma_{(i,j)}$ can be defined as the following:

$$\begin{aligned} \sigma_{(1,j)} &= \Delta_{(emp,j+1)}, \quad j = 2, \dots, m + 1 \\ \sigma_{(i,j)} &= \Delta_{(i-1,j-1)}, \quad i, j = 2, \dots, m + 1, \quad i < j \\ \sigma_{(i,i)} &= 0, \quad i = 1, \dots, m + 1 \end{aligned}$$

- Using the dissimilarity matrix D , a non-metric multi-dimensional scaling (NMDS) technique can be performed. For more details about NMDS, see (Durante et al., 2015).

The problem of estimating parameters of Bézier type convex diagonal section can be solved by minimizing the squared distance between empirical diagonal section and parametric diagonal section as follows:

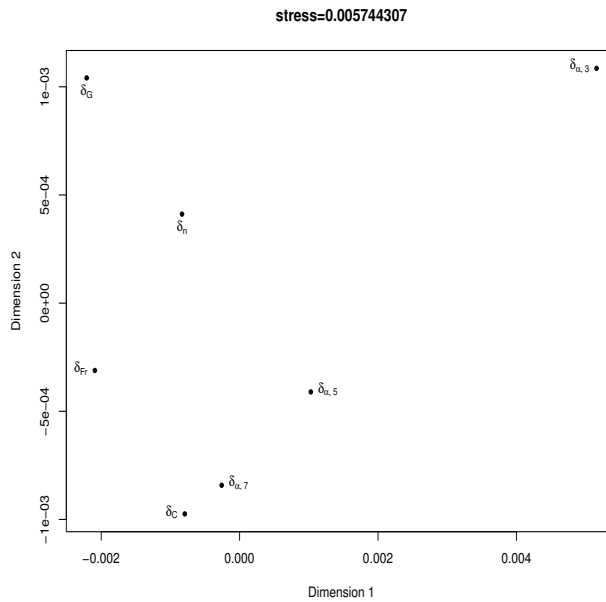
$$\min_{\alpha} \left\{ \delta_n - \delta_{\alpha,m} \right\}^2 \quad (4.3)$$

subject to the constraints

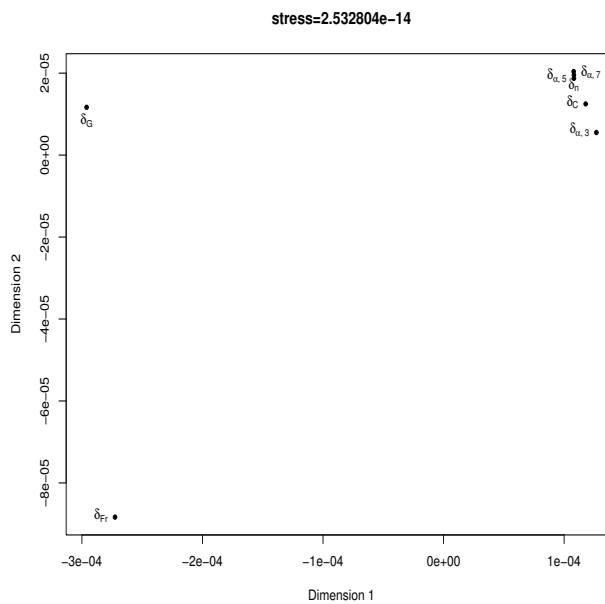
1. $\alpha_0 = 0, \alpha_m = 1$;
2. $\alpha_i < \frac{i}{m}, \quad i = 1, \dots, m$;
3. $0 < \alpha_1 < \dots < \alpha_{m-1} < 1$;
4. $\alpha_{i+2} - 2\alpha_{i+1} + \alpha_i \geq 0, \quad i = 0, \dots, m - 2$;
5. $\alpha_{m-1} = \frac{m-2}{m}$.

We note that (4.3) is equivalent to

$$\int_0^1 n \left(\delta_n(t) - \delta_{\alpha,m}(t) \right)^2 d\delta_n(t). \quad (4.4)$$



(a) Loss-Alea data set



(b) Nutrient data set

Figure 3: The two-dimensional representation of goodness-of-fit for two real data sets

Now, we fit two real data sets to Archimedean copulas with diagonal sections using NMDS method. As an illustration, we model the two bivariate data set examples from (Susam and Erdogan, In Press). The first data set is called Loss-Alea and the second one is called nutrient.

Table 1 reports the estimated control points of $\delta_{\alpha,m}$ and parameters of three Archimedean copulas for two real data sets. Also, Cramér-von-Mises distances between empirical diagonal section and parametric diagonal sections are shown in Table 1. It is concluded from Table 1 that the closest fit to the empirical diagonal function is attained for $\delta_{\alpha,m}$. Also, it is obvious that the Cramér-von-Mises distance between empirical diagonal section and $\delta_{\alpha,m}$ gets closure to zero when polynomial degree m increases. Also, the non-parametric and parametric estimates of diagonal section for real data sets are obtained for four Archimedean copulas as shown in Figure 2. From the Figure 2, $\delta_{\alpha,7}$ is a good choice for real data sets especially for nutrient data set. Because the nutrient data set has left-tail dependence structure. On the other hand, Loss-Alea data exhibits right-tail dependence, thus Gumbel copula is a good choice for modelling its dependence structure. It is concluded from Figure 2 that the convex Bézier type diagonal sections have a good fit to the empirical diagonal at the left-tail.

5 Conclusion

In this study, we propose a convex Bézier type diagonal section $\delta_{\alpha,m}$ which belongs to the family of $\mathbb{F}$. We derive its tail dependence coefficients with closed form. Bézier polynomial type diagonal section of a new parametric Archimedean family of copulas has left tail dependence structure. It is also shown that, when constructing diagonal section with Bézier polynomial, a multi-parameter Archimedean copula family can be constructed in a organized way. Thus, we can easily define the Bézier polynomial which satisfies the all properties of diagonal section defined in Definition 2. The proposed diagonal section $\delta_{\alpha,m}$ allows us to work with more parameterized Archimedean copula while increasing polynomial degree m . Thus, a better goodness-of-fit results can be achieved by modeling data with $\delta_{\alpha,m}$ rather than the traditional parametric models. Also, the new multi-parametric Archimedean family of copula has a satisfactory performance for the real data set which has a left tail dependence structure.

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