

Picard-Mann hybrid iteration process for nonexpansive semigroup in CAT(0) spaces

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ABSTRACT

In this paper, consider Picard-Mann hybrid iteration process for nonexpansive semigroups in CAT(0) spaces. Then, prove strong and Δ -convergence theorems for such iterative process in CAT(0) spaces. The results obtained in this paper extend and improve some recent known results.

Keywords: Fixed point, Nonexpansive semigroup, Δ -convergence, CAT(0) space.

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1 Introduction

Fixed point theory in CAT(0) spaces was first studied by Kirk (Kirk, 2003). Since then the fixed point theory for nonexpansive mappings in CAT(0) spaces has been rapidly developed and many of papers have appeared (Dhompongsa, Kirk and Panyanak, 2007; Dhompongsa and Panyanak, 2008; Kirk, 2004; Kirk and Panyanak, 2008). It is worth mentioning that fixed point theorems in CAT(0) spaces (specially in R-trees) can be applied to graph theory, biology, and computer science (Bartolini, Ciaccia and Patella, 2002; Bestvina, 2002; Bridson and Haefliger, 1999; Espínola and Kirk, 2006; Kirk, 2004; Park, 2010).

Let (X, d) be a metric space, D a closed convex subset of X and $S : D \rightarrow D$ a mapping. If $d(Sx, Sy) \leq d(x, y)$, for all $x, y \in D$, then S is nonexpansive on D . We denote the set of all fixed points of S by $F(S)$, i.e., $F(S) = \{x \in X : Sx = x\}$.

A family $\mathcal{S} := \{S(s) : 0 \leq s < \infty\}$ of mappings on a closed convex subset D of a CAT(0) space X is called a nonexpansive semigroup if it satisfies the following conditions:

- (i) for each $t \geq 0$, $S(t)$ is a nonexpansive mapping on D ;
- (ii) $S(0)x = x$ for all $x \in D$;
- (iii) $S(s + t) = S(s) \circ S(t)$ for all $s, t \geq 0$;

(iv) for all $x \in D$, $s \rightarrow S(s)x$ is continuous.

We use $F(\mathcal{S})$ to denote the common fixed point set of the semigroup $\mathcal{S}$, i.e., $F(\mathcal{S}) = \{x \in D : S(s)x = x, \forall s \geq 0\}$.

The Mann iterative process (Mann, 1953) is defined in a CAT(0) space by

$$x_{n+1} = \alpha_n x_n \oplus (1 - \alpha_n) Sx_n, \quad \forall n \geq 0$$

where $\{\alpha_n\}$ is a sequence in $[0, 1]$. Li et al. (Li and Yip, 2012) proved the convergence results of implicit Mann iteration processes with bounded perturbations for approximating a common fixed point of nonexpansive semigroup in CAT(0) spaces.

The Ishikawa iterative sequence (Ishikawa, 1974) is defined by

$$\begin{aligned} y_n &= \alpha_n x_n \oplus (1 - \alpha_n) Sx_n \\ x_{n+1} &= \beta_n x_n \oplus (1 - \beta_n) Sy_n, \end{aligned}$$

for all $n \geq 0$ where $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences in $[0, 1]$. In 2013 Liu et al. (Liu, Chen, Li and Xiao, 2013) proved the convergence result of the implicit Ishikawa iteration processes for approximating a common fixed point of nonexpansive semigroup in CAT(0) spaces.

In 2003, Suzuki (Suzuki, 2003) introduced an iterative process $\{x_n\}$ for nonexpansive semigroup on D , where D is a compact and convex subset of a Banach space X defined by $x_1 \in D$ and

$$x_{n+1} = \lambda S(t_n)x_n + (1 - \lambda)x_n$$

where $\lambda \in (0, 1)$ and $\{t_n\} \subset [0, \infty)$. Then the author proved that $\{x_n\}$ converges strongly to a common fixed point of $\{S(t) : t \geq 0\}$. Recently Cho et al. (Cho, Ćirić and Wang, 2011) generalized the result of Suzuki for CAT(0) spaces.

In 2013, Khan (Khan, 2013) introduced the following Picard-Mann hybrid iterative process for a nonexpansive mappings in Banach space and claimed that this process is independent of Picard and Mann iterative process, the convergence process is faster than Picard and Mann iteration process. For any initial point $x_1 \in D$,

$$\begin{cases} y_n = (1 - \alpha_n)x_n + \alpha_n Sx_n, \\ x_{n+1} = Sy_n, \end{cases} \quad (1.1)$$

where $\{\alpha_n\}$ is a real sequence in $(0, 1)$.

In this paper, we introduce Picard-Mann hybrid iteration process for nonexpansive semigroups in CAT(0) spaces, and then prove strong and Δ - convergence theorems for such iterative process in CAT(0) spaces. Moreover, we present a convergence theorem for a sequence of nonexpansive mappings in CAT(0) spaces. Our results generalized some recent known results.

2 Preliminaries

Throughout in this paper, we denote by $\mathbb{N}$ the set of positive integers and by $\mathbb{R}$ the set of real numbers. Refer to (Bridson and Haefliger, 1999) for the some definitions. suppose (X, d) be a metric space, $x, y \in X$ and $[0, l] \subset \mathbb{R}$. A map $c : [0, l] \rightarrow X$ is said to be a geodesic path joining

the point x to y such that $c(0) = x$, $c(l) = y$, with $d(c(t), c(t')) = |t - t'|$ for all $t, t' \in [0, l]$. In short, we use a geodesic from x to y instead of a geodesic path joining x to y . Notice that if c is an isometry, then $d(x, y) = l$. The image of c is called a geodesic (or metric) segment joining x and y . If it is unique, this geodesic segment is denoted by $[x, y]$. A metric space (X, d) is said to be a geodesic space if every two points of X are joined by a geodesic. Moreover X is said to be uniquely geodesic if there is exactly one geodesic joining x and y for each $x, y \in X$. A subset $Y \subseteq X$ is said to be convex if Y includes every geodesic segment joining any two of its points.

In a geodesic metric space (X, d) , geodesic triangle $\triangle(x_1, x_2, x_3)$ consists of three points x_1, x_2, x_3 in X (the vertices of $\triangle$) and a geodesic segment between each pair of vertices (the edges of $\triangle$). A triangle $\overline{\triangle}(x_1, x_2, x_3) := \triangle(\bar{x}_1, \bar{x}_2, \bar{x}_3)$ in the Euclidean plane $\mathbb{E}^2$, is said to be a comparison triangle for the geodesic triangle $\triangle(x_1, x_2, x_3)$ in (X, d) such that $d_{\mathbb{E}^2}(\bar{x}_i, \bar{x}_j) = d(x_i, x_j)$ for $i, j \in \{1, 2, 3\}$.

Comparison Axiom (Bridson and Haefliger, 1999): Let $\triangle$ be a geodesic triangle in X and let $\overline{\triangle}$ be a comparison triangle for $\triangle$ in a geodesic metric space (X, d) . Then we say that $\triangle$ satisfy the $CAT(0)$ inequality if for all $x, y \in \triangle$ and all comparison points $\bar{x}, \bar{y} \in \overline{\triangle}$,

$$d(x, y) \leq d_{\mathbb{E}^2}(\bar{x}, \bar{y}).$$

A geodesic metric space is called a $CAT(0)$ space (Bridson and Haefliger, 1999) if all geodesic triangles of appropriate size satisfy the comparison axiom.

We refer following results from Dhompongsa and Panyanak (Dhompongsa and Panyanak, 2008).

Definition 2.1. (Dhompongsa and Panyanak, 2008) For any $x, y \in X$ and $t \in [0, 1]$, there exists a unique point $z \in [x, y]$ such that

$$d(x, z) = td(x, y), \quad d(y, z) = (1 - t)d(x, y). \quad (2.1)$$

Notation $(1 - t)x \oplus ty$ is used for the unique point z satisfying (2.1)

We also denote by $[x, y]$ the geodesic segment joining from x to y , that is, $[x, y] = \{(1 - t)x \oplus ty : t \in [0, 1]\}$ (Bridson and Haefliger, 1999).

A subset D of $CAT(0)$ space X is said to be convex if $[x, y] \subset D$ for all $x, y \in D$.

Definition 2.2. (Dhompongsa, Kirk and Sims, 2006) Let $\{x_n\}$ be a bounded sequence in a $CAT(0)$ space X . For $x \in X$, we set

$$r(x, \{x_n\}) = \limsup_{n \rightarrow \infty} d(x, x_n).$$

The asymptotic radius $r(\{x_n\})$ of $\{x_n\}$ is given by

$$r(\{x_n\}) = \inf\{r(x, \{x_n\}) : x \in X\}.$$

The asymptotic center $A(\{x_n\})$ of $\{x_n\}$ is the set

$$A(\{x_n\}) = \{x \in X : r(x, \{x_n\}) = r(\{x_n\})\}.$$

In a $CAT(0)$ space, $A(\{x_n\})$ consists of exactly one point (Dhompongsa et al., 2006, Proposition 7).

Definition 2.3. (Kirk and Panyanak, 2008) A sequence $\{x_n\}$ in X is said to Δ -converge to $p \in X$ if p is the unique asymptotic center of $\{u_n\}$ for every subsequence $\{u_n\}$ of $\{x_n\}$.

In this case, we write $\Delta\text{-}\lim x_n = p$ and call p the Δ -limit of $\{x_n\}$.

The following lemmas plays an important role in our paper

Lemma 2.4. (Kirk and Panyanak, 2008) Every bounded sequence in a complete $CAT(0)$ space has a Δ -convergent subsequence.

Lemma 2.5. (Dhompongsa et al., 2007) If D is a closed convex subset of a complete $CAT(0)$ space and if $\{x_n\}$ is a bounded sequence in D , then the asymptotic center of $\{x_n\}$ is in D .

Lemma 2.6. (Dhompongsa and Panyanak, 2008) If $\{x_n\}$ is a bounded sequence in complete $CAT(0)$ space X with $A(\{x_n\}) = \{x\}$ and $\{u_n\}$ is a subsequence of $\{x_n\}$ with $A(\{u_n\}) = \{u\}$ and the sequence $\{d(x_n; u)\}$ converges, then $x = u$.

Lemma 2.7. (Dhompongsa and Panyanak, 2008) Let X be a $CAT(0)$ space. Then for all $x, y, z \in X$ and all $t \in [0, 1]$ we have

- (i) $d((1-t)x \oplus ty, z) \leq (1-t)d(x, z) + td(y, z),$
- (ii) $d((1-t)x \oplus ty, z)^2 \leq (1-t)d(x, z)^2 + td(y, z)^2 - t(1-t)d(x, y)^2.$

Lemma 2.8. (Dhompongsa and Panyanak, 2008) Let D be a nonempty closed convex subset of a complete $CAT(0)$ space X , and $S : D \rightarrow D$ be a nonexpansive mapping. If $\{x_n\}$ is a sequence in D such that $\lim_{n \rightarrow \infty} d(x_n, Sx_n) = 0$ and $\Delta\text{-}\lim_{n \rightarrow \infty} x_n = v$, then $v = Sv$.

3 Main Result

In this section we present some strong and Δ -convergent theorems of Picard-Mann hybrid iteration process for nonexpansive semigroups in a $CAT(0)$ space.

Theorem 3.1. Let D be a nonempty closed convex subset of a complete $CAT(0)$ space X . Let $\mathcal{S} := \{S(t) : t \geq 0\}$ be nonexpansive semigroups. Assume that $\mathcal{F} = F(\mathcal{S})$ Let $\{x_n\}$ be sequence generated by (1.1), where $\{\alpha_n\}$ and $\{t_n\}$ satisfy the following conditions:

- (i) $\{\alpha_n\} \in [a, b] \subset (0, 1)$
- (ii) $t_n > 0, \liminf_{n \rightarrow \infty} t_n = 0, \limsup_{n \rightarrow \infty} t_n > 0, \lim_{n \rightarrow \infty} (t_{n+1} - t_n) = 0.$

Then for $t > 0$ we have $\lim_{n \rightarrow \infty} d(x_n, S(t)x_n) = 0$ and $\lim_{n \rightarrow \infty} d(x_n, p)$ exists for all $p \in \mathcal{F}$.

Proof. Let $p \in \mathcal{F}$, then by using Lemma 2.7(i) we have,

$$\begin{aligned} d(y_n, p) &= d((1 - \alpha_n)x_n \oplus \alpha_n S(t_n)x_n, p) \\ &\leq (1 - \alpha_n)d(x_n, p) + \alpha_n d(S(t_n)x_n, p) \\ &\leq (1 - \alpha_n)d(x_n, p) + \alpha_n d(x_n, p) \\ &= d(x_n, p) \end{aligned} \tag{3.1}$$

and

$$\begin{aligned} d(x_{n+1}, p) &= d(S(t_n)y_n, p) \\ &\leq d(y_n, p) \\ &\leq d(x_n, p) \end{aligned} \quad (3.2)$$

Hence $\lim_{n \rightarrow \infty} d(x_n, p)$ exists, and therefore $\{x_n\}$ is bounded. Also by Lemma 2.7(ii) we have,

$$\begin{aligned} d(y_n, p)^2 &= d((1 - \alpha_n)x_n \oplus \alpha_n S(t_n)x_n, p)^2 \\ &\leq (1 - \alpha_n)d(x_n, p)^2 + \alpha_n d(S(t_n)x_n, p)^2 - \alpha_n(1 - \alpha_n)d(x_n, S(t_n)x_n)^2 \\ &\leq (1 - \alpha_n)d(x_n, p)^2 + \alpha_n d(x_n, p)^2 - \alpha_n(1 - \alpha_n)d(x_n, S(t_n)x_n)^2 \\ &= d(x_n, p)^2 - \alpha_n(1 - \alpha_n)d(x_n, S(t_n)x_n)^2. \end{aligned} \quad (3.3)$$

$$\begin{aligned} d(x_{n+1}, p)^2 &= d(S(t_n)y_n, p)^2 \\ &\leq d(y_n, p)^2 \\ &\leq d(x_n, p)^2 - \alpha_n(1 - \alpha_n)d(x_n, S(t_n)x_n)^2. \end{aligned} \quad (3.4)$$

Thus we have,

$$a(1 - b)d(x_n - S(t_n)x_n)^2 \leq \alpha_n(1 - \alpha_n)d(x_n, S(t_n)x_n)^2 \leq d(x_n, p)^2 - d(x_{n+1}, p)^2,$$

since $\lim_{n \rightarrow \infty} d(x_n, p)$ exists, by taking Limit in above inequality we obtain that

$$\lim_{n \rightarrow \infty} d(x_n, S(t_n)x_n) = 0.$$

Now we show that for a fixed $t > 0$

$$\lim_{n \rightarrow \infty} d(x_n, S(t)x_n) = 0.$$

With the same proof, we only show that $\lim_{n \rightarrow \infty} d(x_n, S(t)x_n) = 0$, without loss of generality, as in (Saejung, 2008) we can assume that

$$\lim_{n \rightarrow \infty} t_n = \lim_{n \rightarrow \infty} \frac{d(x_n, S(t_n)x_n)}{t_n} = 0.$$

$$\begin{aligned} d(x_n; S(t)x_n) &\leq \sum_{k=0}^{\lfloor \frac{t}{t_n} \rfloor - 1} d(S((k+1)t_n)x_n, S(kt_n)x_n) + d(S(\lfloor \frac{t}{t_n} \rfloor t_n)x_n, S(t)x_n) \\ &\leq \lfloor \frac{t}{t_n} \rfloor d(S(t_n)x_n, x_n) + d(S(t - \lfloor \frac{t}{t_n} \rfloor t_n)x_n, x_n) \\ &\leq \frac{t}{t_n} d(S(t_n)x_n, x_n) + \max\{d(S(s)x_n; x_n) : 0 \leq s \leq t_n\}, \end{aligned}$$

Now by continuity of the mapping $t \rightarrow S(t)x, x \in D$ and $\lim_{n \rightarrow \infty} d(x_n, S(t_n)x_n) = 0$, we obtain that $\lim_{n \rightarrow \infty} d(x_n, S(t)x_n) = 0$.

□

Theorem 3.2. Let D be a nonempty closed convex subset of a complete CAT(0) space X . Let $S := \{S(t) : t \geq 0\}$ be nonexpansive semigroups. Assume that $\mathcal{F} = F(S) \neq \emptyset$. Let $\{x_n\}$ be sequence generated by (1.1), where $\{\alpha_n\}$ and $\{t_n\}$ satisfy the following conditions:

- (i) $\{\alpha_n\} \in [a, b] \subset (0, 1)$
- (ii) $t_n > 0, \liminf_{n \rightarrow \infty} t_n = 0, \limsup_{n \rightarrow \infty} t_n > 0, \lim_{n \rightarrow \infty} (t_{n+1} - t_n) = 0$.

Then the sequence $\{x_n\}$, Δ -converges to an element of $\mathcal{F}$.

Proof. It follows from Theorem 3.1 that

$$\lim_{n \rightarrow \infty} d(S(t)x_n, x_n) = 0$$

for each $t > 0$. Next steps of the proof as same as (Eslamian and Dhompangsa, 2013).

Now we let $W_w(x_n) := \cup A(\{u_n\})$ where the union is taken over all subsequences $\{u_n\}$ of $\{x_n\}$. We claim that $W_w(x_n) \subset \mathcal{F}$. Let $u \in W_w(x_n)$, then there exists a subsequence $\{u_n\}$ of $\{x_n\}$ such that $A(\{u_n\}) = \{u\}$. By Lemmas 2.4 and Lemmas 2.5 there exists a subsequence $\{v_n\}$ of $\{u_n\}$ such that $\Delta - \lim_{n \rightarrow \infty} v_n = v \in D$. We show that $v \in \mathcal{F}$, indeed

$$\begin{aligned} d(v_n, S(t)v) &\leq d(v_n, S(t)v_n) + d(S(t)v_n, S(t)v) \\ &\leq d(v_n, S(t)v_n) + d(v_n, v), \end{aligned}$$

hence

$$\limsup_{n \rightarrow \infty} d(v_n, S(t)v) = \limsup_{n \rightarrow \infty} d(v_n, v).$$

By uniqueness of the asymptotic center we obtain that $S(t)v = v$ for all $t > 0$ and hence $v \in \mathcal{F}(S)$ and hence $v \in \mathcal{F}$. By Theorem 3.1 the limit $\lim_{n \rightarrow \infty} d(x_n, v)$ exists. Hence by Lemma 2.6, $u = v \in \mathcal{F}$. This shows that $W_w(x_n) \subset \mathcal{F}$. Next we show that $W_w(x_n)$ consists of exactly one point. Let $\{u_n\}$ be a subsequence of $\{x_n\}$ with $A(\{u_n\}) = \{u\}$ and let $A(\{x_n\}) = \{x\}$. Since $u \in W_w(x_n) \subset \mathcal{F}$ and $d(x_n, v)$ converges, by Lemma 2.6 we have $x = u$. □

Theorem 3.3. Let D be a nonempty compact convex subset of a complete CAT(0) space X . Let $S := \{S(t) : t \geq 0\}$ be nonexpansive semigroups. Assume that $\mathcal{F} = F(S) \neq \emptyset$. Let $\{x_n\}$ be sequence generated by (1.1), where $\{\alpha_n\}$ and $\{t_n\}$ satisfy the following conditions:

- (i) $\{\alpha_n\} \in [a, b] \subset (0, 1)$
- (ii) $t_n > 0, \liminf_{n \rightarrow \infty} t_n = 0, \limsup_{n \rightarrow \infty} t_n > 0, \lim_{n \rightarrow \infty} (t_{n+1} - t_n) = 0$.

Then the sequence $\{x_n\}$, converges strongly to an element of $\mathcal{F}$.

Proof. It follows from Theorem 3.1 that

$$\lim_{n \rightarrow \infty} d(S(t)x_n, x_n) = 0$$

for each $t > 0$. By compactness of C , there exists a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ such that $x_{n_i} \rightarrow w$. We shall show that $w \in \mathcal{F}$, indeed for all $t > 0$ we have

$$\begin{aligned} d(w, S(t)w) &\leq d(w, x_{n_i}) + d(x_{n_i}, S(t)x_{n_i}) + d(S(t)x_{n_i}, S(t)w) \\ &\leq 2d(w, x_{n_i}) + d(x_{n_i}, S(t)x_{n_i}) \rightarrow 0 \quad \text{as } n \rightarrow \infty \end{aligned}$$

hence we have $w = S(t)w$, i.e., $w \in \mathcal{F}(S)$ and hence $w \in \mathcal{F}$. Since $\lim_{n \rightarrow \infty} d(x_n, w)$ exists, we obtain the result. $\square$

Definition 3.4. (Schu, 1991) Let D be a nonempty closed convex subset of a CAT(0) space X and $S_n : D \rightarrow D$, where $n \in \mathbb{N}$. Then the family $\{S_n\}$ is called uniformly asymptotically regular on S , if for all $i \in \mathbb{N}$ and any bounded subset K of D we have

$$\lim_{n \rightarrow \infty} \sup_{x \in K} d(S_i(S_n x), S_n x) = 0$$

Theorem 3.5. Let S be a nonempty closed convex subset of a complete CAT(0) space X . Let $S_n : D \rightarrow D$ be uniformly asymptotically regular and nonexpansive mappings such that $\mathcal{F} = \bigcap_{n=1}^{\infty} F(S_n) \neq \emptyset$. Let $\{x_n\}$ be sequence generated by (1.1), where $\{\alpha_n\} \in [a, b] \subset (0, 1)$. Then the sequence $\{x_n\}$, Δ -converges to an element of $\mathcal{F}$.

Proof. Let $p \in \mathcal{F}$. Then by Lemma 2.7(i) we have

$$\begin{aligned} d(y_n, p) &= d((1 - \alpha_n)x_n \oplus \alpha_n S_n x_n, p) \\ &\leq (1 - \alpha_n)d(x_n, p) + \alpha_n d(S_n x_n, p) \\ &\leq (1 - \alpha_n)d(x_n, p) + \alpha_n d(x_n, p) \\ &= d(x_n, p) \end{aligned} \tag{3.5}$$

and

$$\begin{aligned} d(x_{n+1}, p) &= d(S_n y_n, p) \\ &\leq d(y_n, p) \\ &\leq d(x_n, p) \end{aligned} \tag{3.6}$$

Hence we have $d(x_{n+1}, p) \leq d(x_n, p)$, this gives that the limit $\lim_{n \rightarrow \infty} d(x_n, p)$ exists. Applying Lemma 2.7(ii) we have

$$\begin{aligned} d(y_n, p)^2 &= d((1 - \alpha_n)x_n \oplus \alpha_n S_n x_n, p)^2 \\ &\leq (1 - \alpha_n)d(x_n, p)^2 + \alpha_n d(S_n x_n, p)^2 - \alpha_n(1 - \alpha_n)d(x_n, S_n x_n)^2 \\ &\leq (1 - \alpha_n)d(x_n, p)^2 + \alpha_n d(x_n, p)^2 - \alpha_n(1 - \alpha_n)d(x_n, S_n x_n)^2 \\ &= d(x_n, p)^2 - \alpha_n(1 - \alpha_n)d(x_n, S_n x_n)^2. \end{aligned} \tag{3.7}$$

$$\begin{aligned}
 d(x_{n+1}, p)^2 &= d(S_n y_n, p)^2 \\
 &\leq d(y_n, p)^2 \\
 &\leq d(x_n, p)^2 - \alpha_n(1 - \alpha_n)d(x_n, S_n x_n)^2.
 \end{aligned}
 \tag{3.8}$$

So,

$$\begin{aligned}
 \sum_{n=1}^{\infty} a(1-b)d(x_n - S(t_n)x_n)^2 &\leq \sum_{n=1}^{\infty} \alpha_n(1 - \alpha_n)d(x_n, S(t_n)x_n)^2 \\
 &\leq \sum_{n=1}^{\infty} d(x_n, p)^2 - d(x_{n+1}, p)^2, < d(x_1, p)^2 < \infty,
 \end{aligned}$$

this follows that

$$\lim_{n \rightarrow \infty} d(x_n, S_n x_n) = 0.$$

Also we have

$$\begin{aligned}
 d(x_{n+1}, x_n) &\leq d(x_{n+1}, S_n x_n) + d(S_n x_n, x_n) \\
 &= d(S_n y_n, S_n x_n) + d(S_n x_n, x_n) \\
 &\leq d(y_n, x_n) + d(S_n x_n, x_n) \\
 &= d((1 - \alpha_n)x_n \oplus \alpha_n S_n x_n, x_n) + d(S_n x_n, x_n) \\
 &\leq (1 - \alpha_n)d(x_n, x_n) + \alpha_n d(S_n x_n, x_n) + d(S_n x_n, x_n) \\
 &\rightarrow 0 \quad \text{as } n \rightarrow \infty,
 \end{aligned}
 \tag{3.9}$$

and therefore

$$d(x_{n+1}, S_n x_n) \leq d(x_{n+1}, x_n) + d(x_n, S_n x_n) \rightarrow 0.$$

Now, by our assumption, for each $i \in \mathbb{N}$ we have

$$\begin{aligned}
 d(x_{n+1}, S_i x_{n+1}) &\leq d(x_{n+1}, S_n x_n) + d(S_n x_n, S_i(S_n x_n)) + d(S_i(S_n x_n), S_i x_{n+1}) \\
 &\leq 2d(x_{n+1}, S_n x_n) + \sup_{u \in \{x_n\}} d(S_n u, S_i(S_n u)) \rightarrow 0 \text{ as } n \rightarrow \infty.
 \end{aligned}$$

Next steps of the proof as same as (Eslamian and Dhompangsa, 2013).

We let $W_w(x_n) := \bigcup A(\{u_n\})$ where the union is taken over all subsequences $\{u_n\}$ of $\{x_n\}$. We claim that $W_w(x_n) \subset \mathcal{F}$. Let $u \in W_w(x_n)$, then there exists a subsequence $\{u_n\}$ of $\{x_n\}$ such that $A(\{u_n\}) = \{u\}$. By Lemmas 2.4 and Lemmas 2.5 there exists a subsequence $\{v_n\}$ of $\{u_n\}$ such that $\Delta - \lim_n v_n = v \in C$. By Lemma 2.8 we have $v \in \mathcal{F}$, and hence the limit $\lim_{n \rightarrow \infty} d(x_n, v)$ exists. Hence by Lemma 2.6, $u = v \in \mathcal{F}$. This shows that $W_w(x_n) \subset \mathcal{F}$. Next we show that $W_w(x_n)$ consists of exactly one point. Let $\{u_n\}$ be a subsequence of $\{x_n\}$ with $A(\{u_n\}) = \{u\}$ and let $A(\{x_n\}) = \{x\}$. Since $u \in W_w(x_n) \subset \mathcal{F}$ and $d(x_n, v)$ converges, by Lemma 2.6 we have $x = u$. $\square$

References

- Bartolini, I., Ciaccia, P. and Patella, M. 2002. String matching with metric trees using an approximate distance, *International Symposium on String Processing and Information Retrieval*, Springer, pp. 271–283.
- Bestvina, M. 2002. $\mathbb{R}$ -trees in topology, geometry, and group theory, *Handbook of geometric topology*, North-Holland, Amsterdam, pp. 55–91.
- Bridson, M. R. and Haefliger, A. 1999. *Metric spaces of non-positive curvature*, Vol. 319 of *Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]*, Springer-Verlag, Berlin.
URL: <https://doi.org/10.1007/978-3-662-12494-9>
- Cho, Y. J., Ćirić, L. and Wang, S.-h. 2011. Convergence theorems for nonexpansive semigroups in $\text{CAT}(0)$ spaces, *Nonlinear Anal.* **74**(17): 6050–6059.
URL: <https://doi.org/10.1016/j.na.2011.05.082>
- Dhompongsa, S., Kirk, W. A. and Panyanak, B. 2007. Nonexpansive set-valued mappings in metric and Banach spaces, *J. Nonlinear Convex Anal.* **8**(1): 35–45.
- Dhompongsa, S., Kirk, W. A. and Sims, B. 2006. Fixed points of uniformly Lipschitzian mappings, *Nonlinear Anal.* **65**(4): 762–772.
URL: <https://doi.org/10.1016/j.na.2005.09.044>
- Dhompongsa, S. and Panyanak, B. 2008. On Δ -convergence theorems in $\text{CAT}(0)$ spaces, *Comput. Math. Appl.* **56**(10): 2572–2579.
URL: <https://doi.org/10.1016/j.camwa.2008.05.036>
- Eslamian, M. and Dhompongsa, S. 2013. Modified Ishikawa iteration process for two nonexpansive semigroups in $\text{CAT}(0)$ spaces, *Politehn. Univ. Bucharest Sci. Bull. Ser. A Appl. Math. Phys.* **75**(1): 99–106.
- Espínola, R. and Kirk, W. A. 2006. Fixed point theorems in $\mathbb{R}$ -trees with applications to graph theory, *Topology Appl.* **153**(7): 1046–1055.
URL: <https://doi.org/10.1016/j.topol.2005.03.001>
- Ishikawa, S. 1974. Fixed points by a new iteration method, *Proc. Amer. Math. Soc.* **44**: 147–150.
URL: <https://doi.org/10.2307/2039245>
- Khan, S. H. 2013. A Picard-Mann hybrid iterative process, *Fixed Point Theory Appl.* pp. 2013:69, 10.
URL: <https://doi.org/10.1186/1687-1812-2013-69>
- Kirk, W. A. 2003. Geodesic geometry and fixed point theory, *Seminar of Mathematical Analysis (Malaga/Seville, 2002/2003)*, Vol. 64 of *Colecc. Abierta*, Univ. Sevilla Secr. Publ., Seville, pp. 195–225.

- Kirk, W. A. 2004. Fixed point theorems in $CAT(0)$ spaces and $\mathbb{R}$ -trees, *Fixed Point Theory Appl.* (4): 309–316.
URL: <https://doi.org/10.1155/S1687182004406081>
- Kirk, W. A. and Panyanak, B. 2008. A concept of convergence in geodesic spaces, *Nonlinear Anal.* **68**(12): 3689–3696.
URL: <https://doi.org/10.1016/j.na.2007.04.011>
- Li, X.-s. and Yip, T.-l. 2012. Implicit Mann approximation with perturbations for nonexpansive semigroups in $CAT(0)$ spaces, *Fixed Point Theory Appl.* pp. 2012:145, 13.
URL: <https://doi.org/10.1186/1687-1812-2012-145>
- Liu, Z.-b., Chen, Y.-s., Li, X.-s. and Xiao, Y.-b. 2013. Implicit Ishikawa approximation methods for nonexpansive semigroups in $CAT(0)$ spaces, *Abstr. Appl. Anal.* pp. Art. ID 503198, 8.
URL: <https://doi.org/10.1155/2013/503198>
- Mann, W. R. 1953. Mean value methods in iteration, *Proc. Amer. Math. Soc.* **4**: 506–510.
URL: <https://doi.org/10.2307/2032162>
- Park, S. 2010. The KKM principle in abstract convex spaces: equivalent formulations and applications, *Nonlinear Anal.* **73**(4): 1028–1042.
URL: <https://doi.org/10.1016/j.na.2010.04.029>
- Saejung, S. 2008. Strong convergence theorems for nonexpansive semigroups without Bochner integrals, *Fixed Point Theory Appl.* pp. Art. ID 745010, 7.
URL: <https://doi.org/10.1155/2010/806837>
- Schu, J. 1991. Approximation of fixed points of asymptotically nonexpansive mappings, *Proc. Amer. Math. Soc.* **112**(1): 143–151.
URL: <https://doi.org/10.2307/2048491>
- Suzuki, T. 2003. On strong convergence to common fixed points of nonexpansive semigroups in Hilbert spaces, *Proc. Amer. Math. Soc.* **131**(7): 2133–2136.
URL: <https://doi.org/10.1090/S0002-9939-02-06844-2>