

Singularity and Weighted sharing of product of difference-differential polynomials

Rajeshwari S.,¹ Bhuvaneshwari T² and Veena B. N³

^{1,2}Department of Mathematics

Presidency University

Bangalore, India

¹rajeshwaripreetham@gmail.com

²anjali.bhuvaneshwari@gmail.com

³Department of Mathematics

MSRIT

Bangalore, India

mvymath@gmail.com

ABSTRACT

In this study, we concern with singularity and weighted dividing of difference-differential polynomials of mero-morphic functions. Our investigation also revise IM analogues of the theorem for mero-morphic functions dividing the value $\gamma(t)$ with weight ι .

Keywords: Nevanlinna Theory, singularity, Value sharing, Mero-morphic Functions.

2000 Mathematics Subject Classification: Primary 30D35.

1 Introduction and Proceeding results

In this literature survey, we indicate $\mathbb{C}$ the numbers which is complex and by a mero-morphic function $\phi(t)$ we always mean a non-sustained mero-morphic function defined in the whole complex plane. We use some basic definitions, symbols and estimations distinctly throughout this paper for instance (Hayman, 1994 and Yang, 1993). For mero-morphic functions in the complex plane $S(s, \phi) = o\{\tau(s, \phi)\}$, outer a set of restricted length i.e., the inaccuracy expression which is tiny in collation with the characteristic for “almost” values of r .

To prove our results we use following list of definitions.

Definition 1.1. Let ‘ α ’ be a restricted complex number and κ be a positive integers. We indicate by $\Theta_{(\kappa)}(s, \alpha, \phi)$ the counting for zeros of $\phi - \alpha$ with multiplicities atleast κ and by $\overline{\Theta}_{(\kappa)}(s, \alpha, \phi)$ the one for which multiplicity is not counted. Similarly, we indicate by $\Theta_{(\kappa)}(s, \alpha, \phi)$ the counting function for zeros of $\phi - \alpha$ with multiplicities at most κ and $\overline{\Theta}_{(\kappa)}(s, \alpha, \phi)$ the one for which multiplicity is not counted. Then

$$\Theta_{\kappa}(s, \alpha, \phi) = \overline{\Theta}_{(1)}(s, \alpha, \phi) + \overline{\Theta}_{(2)}(s, \alpha, \phi) + \cdots + \overline{\Theta}_{(\kappa)}(s, \alpha, \phi).$$

Definition 1.2. Let $\phi(t)$ and $\psi(t)$ be two mero-morphic functions in the complex plane $\mathbb{C}$. If $\phi(t) - \alpha$ and $\psi(t) - \alpha$ presume the same zeros with the same multiplicities, then we say that $\phi(t)$ and $\psi(t)$ divides the value α CM, and if we do not consider the multiplicity, then we say that $\phi(t)$ and $\psi(t)$ divides the value α IM, where α is a complex number.

Definition 1.3. (Lahiri, 2001) Let κ be a non negative integers or infinity. For $\alpha \in \mathbb{C} \cup \{\infty\}$. We indicate by $E_\kappa(\alpha; \phi)$ the set of all α -points of ϕ where an α -point of multiplicity m is counted m times if $m \leq \kappa$ and $\kappa + 1$ times if $m > \kappa$. If $E_\kappa(\alpha; \phi) = E_\kappa(\alpha; \psi)$, we say that ϕ, ψ divides the value α with weight κ .

Definition 1.4. (Lahiri, 2001) Let ϕ and ψ be two non sustained mero-morphic functions thus ϕ and ψ divides the value 1 IM. Let t_0 be a 1- point of ϕ with multiplicity p and also a 1- point of ψ with multiplicity q . We indicate by $\Theta_L(s, 1; \phi)$ the counting function of those 1-point of ϕ and ψ , where $p > q$, by $\Theta_E^{(\kappa)}(s, 1; \phi)$ ($\kappa \geq 2$ is an integer) the counting function of those 1- points of ϕ and ψ , where $p = q \geq \kappa$, where each point in these counting functions is counted only once. In the same manner we can define $\Theta_L(s, 1; \psi)$ and $\Theta_E^{(\kappa)}(s, 1; \psi)$.

Definition 1.5. (Lahiri, 2001) Let ϕ and ψ be two mero-morphic functions thus ϕ and ψ divides the value α IM. We indicate by $\Theta_*(s, \alpha; \phi, \psi)$ the reduced counting function of those α -points of ϕ whose multiplicities differ from the multiplicities of the corresponding α - points of ψ . Clearly $\Theta_*(s, \alpha; \phi, \psi) = \Theta_*(s, \alpha; \psi, \phi)$ and $\Theta_*(s, \alpha; \phi, \psi) = \Theta_L(s, \alpha; \phi) + \Theta_L(s, \alpha; \psi)$.

In 2018, P. Sahoo and G. Biswas, found the possible answer to questions posed by K. Zhang and H. X. Yi, 2014 and obtained below results.

Theorem 1.1. Let ϕ and ψ be two transcendental entire functions of restricted order, $\gamma(t) (\neq 0)$ be a common tiny function concerning ϕ and ψ with restrictedly many zeros $c_j (j = 1, 2, \dots, d)$ be distinct restricted complex numbers, and $\rho, \varrho, \iota, d, \kappa$ and $v_j (j = 1, 2, \dots, d)$ be non negative integers. If the differential-difference polynomials $(\phi^\rho(\phi^\varrho - 1) \prod_{j=1}^d \phi(t + c_j)^{v_j})^{(\kappa)}$ and $(\psi^\rho(\psi^\varrho - 1) \prod_{j=1}^d \psi(t + c_j)^{v_j})^{(\kappa)}$ divides (γ, ι) , and one of

- (a) $\iota \geq 2$ and $\rho \geq \max\{\varrho + 5\sigma, 2\kappa + \varrho + \sigma + 5\}$;
- (b) $\iota = 1$ and $\rho \geq \max\{\varrho + 5\sigma, 5\kappa/2 + 3\varrho/2 + 3\sigma/2 + 5\}$;
- (c) $\iota = 0$ and $\rho \geq \max\{\varrho + 5\sigma, 5\kappa + 4\varrho + 4\sigma + 8\}$, holds, then $\phi = b\psi$, where $b^\varrho = b^{\rho+\sigma} = 1$

Theorem 1.2. Let ϕ and ψ be two transcendental entire functions of restricted order, $\gamma(t) (\neq 0)$ be a common tiny function concerning ϕ and ψ with restrictedly many zeros, $c_j (j = 1, 2, \dots, d)$ be distinct restricted complex numbers, and $\rho, \varrho, \iota, d, \kappa$ and $v_j (j = 1, 2, \dots, d)$ be non negative integers. Suppose that the differential-difference polynomials $(\phi^\rho(\phi - 1)^\varrho \prod_{j=1}^d \phi(t + c_j)^{v_j})^{(\kappa)}$ and $(\psi^\rho(\psi - 1)^\varrho \prod_{j=1}^d \psi(t + c_j)^{v_j})^{(\kappa)}$ divides (γ, ι) . If $\varrho \leq \kappa + 1$ and one of

- (a) $\iota \geq 2$ and $\rho \geq 2\kappa + \varrho + \sigma + 5$;
- (b) $\iota = 1$ and $\rho \geq 5\kappa/2 + 3\varrho/2 + 3\sigma/2 + 5$;
- (c) $\iota = 0$ and $\rho \geq 5\kappa + 4\varrho + 4\sigma + 8$, holds or if $\varrho > \kappa + 1$ and one of
- (a) $\iota \geq 2$ and $\rho \geq 4\kappa - \varrho + \sigma + 9$;
- (b) $\iota = 1$ and $\rho \geq 5\kappa - \varrho + 3\sigma/2 + 10$;
- (c) $\iota = 0$ and $\rho \geq 10\kappa - \varrho + 4\sigma + 15$,

holds, then either $\phi = \psi$ or ϕ and ψ satisfy the algebraic equation $R(\phi, \psi) = 0$ where $R(\phi, \psi)$

is given by

$$R(\varpi_1, \varpi_2) = \varpi_1^\rho (\varpi_1 - 1)^e \prod_{j=1}^d \varpi_1(t + c_j)^{\varpi_j} - \varpi_2^\rho (\varpi_2 - 1)^e \prod_{j=1}^d \varpi_2(t + c_j)^{v_j}.$$

In this paper we investigate on the above theorems by adding "divides ∞ IM" and obtain the following results.

Theorem 1.3. Let ϕ and ψ be two transcendental entire functions of restricted order, $\gamma(t) (\neq 0)$ be a common tiny function concerning ϕ and ψ with restrictedly many zeros $c_j (j = 1, 2, \dots, d)$ be distinct restricted complex numbers, and $\rho, \varrho, \iota, d, \kappa$ and $v_j (j = 1, 2, \dots, d)$ be non negative integers. Suppose that the differential-difference polynomials $(\phi^\rho (\phi^e - 1) \prod_{j=1}^d \phi(t + c_j)^{v_j})^{(\kappa)}$ and $(\psi^\rho (\psi^e - 1) \prod_{j=1}^d \psi(t + c_j)^{v_j})^{(\kappa)}$ divides (γ, ι) and ϕ, ψ divides ∞ IM then one of

- (a) $\iota \geq 2$ and $\rho \geq 2\kappa + \varrho + \lambda + 7$;
- (b) $\iota = 1$ and $\rho \geq \frac{5\kappa}{2} + \frac{3\varrho}{2} + \frac{3\lambda}{2} + 12$;
- (c) $\iota = 0$ and $\rho \geq 5\kappa + 4\varrho + 4\lambda + 14$, then $\phi = b\psi$, where $b^e = b^{\rho+\lambda} = 1$

Theorem 1.4. Let ϕ and ψ be two transcendental entire functions of restricted order, $\alpha(t) (\neq 0)$ be a common tiny function concerning ϕ and ψ with restrictedly many zeros, $c_j (j = 1, 2, \dots, d)$ be distinct restricted complex numbers, and $\rho, \varrho, \iota, d, \kappa$ and $v_j (j = 1, 2, \dots, d)$ be non-negative integer. Suppose that the differential-difference polynomials $(\phi^\rho (\phi - 1)^e \prod_{j=1}^d \phi(t + c_j)^{v_j})^{(\kappa)}$ and $(\psi^\rho (\psi - 1)^e \prod_{j=1}^d \psi(t + c_j)^{v_j})^{(\kappa)}$ divides (γ, ι) , ϕ and ψ divides ∞ IM. If $\varrho \leq \kappa + 1$ and one of

- (a) $l \geq 2$ and $\rho \geq 2\kappa + \varrho + \lambda + 7$;
- (b) $\iota = 1$ and $\rho \geq \frac{5\kappa}{2} + \frac{3\varrho}{2} + \frac{3\lambda}{2} + 12$,
- (c) $\iota = 0$ and $\rho \geq 5\kappa + 4\varrho + 4\lambda + 14$; holds or if $\varrho > \kappa + 1$ and one of
 - (a) $\iota \geq 2$ and $\rho \geq 4\kappa - \varrho + \lambda + 11$;
 - (b) $\iota = 1$ and $\rho \geq 5\kappa - \varrho + \frac{3\lambda}{2} + 16$;
 - (c) $\iota = 0$ and $\rho \geq 10\kappa - \varrho + 4\lambda + 21$,

holds, then either $\phi = \psi$ or ϕ and ψ satisfy the algebraic equation $R(\phi, \psi) = 0$ where $R(\phi, \psi)$ is given by $R(\varpi_1, \varpi_2) = \varpi_1^\rho (\varpi_1 - 1)^e \prod_{j=1}^d \varpi_1(t + c_j)^{v_j} - \varpi_2^\rho (\varpi_2 - 1)^e \prod_{j=1}^d \varpi_2(t + c_j)^{v_j}$.

2 Lemmas

Lemma 2.1. (Chen and Chen 2012) Let ϕ be an entire function of restricted order and $\Phi = \phi^\rho (\phi^e - 1) \prod_{j=1}^d \phi(t + c_j)^{v_j}$ Then

$$\tau(s, \Phi) = (\rho + \varrho + \lambda)\tau(s, \phi) + S(s, \phi).$$

Lemma 2.2. (Banerjee 2008) Let Φ, Ψ be two non sustained mero-morphic functions dividing $(1, \iota), (\infty, 0)$ where $2 \leq \iota < \infty$ and $H \neq 0$. Then

$$\begin{aligned} \tau(s, \Phi) &\leq \Theta_2(s, 0; \Phi) + \Theta_2(s, 0; \Psi) + \overline{\Theta}(s, \infty; \Phi) + \overline{\Theta}(s, \infty; \Psi) + \\ &\quad \overline{\Theta}_*(s, \infty; \Phi, \Psi) - \varrho(s, 1; \Psi) - \Theta_E^{(3)}(s, 1; \Phi) - \overline{\Theta}_L(s, 1; \Psi) + S(s, \Phi) + S(s, \Psi). \\ \tau(s, \Psi) &\leq \Theta_2(s, 0; \Phi) + \Theta_2(s, 0; \Psi) + \overline{\Theta}(s, \infty; \Psi) + \overline{\Theta}(s, \infty; \Phi) + \\ &\quad \overline{\Theta}_*(s, \infty; \Phi, \Psi) - \varrho(s, 1; \Phi) - \Theta_E^{(3)}(s, 1; \Psi) - \overline{\Theta}_L(s, 1; \Phi) + S(s, \Phi) + S(s, \Psi). \end{aligned}$$

Lemma 2.3. (Sahoo, 2010) Let Φ, Ψ be two non sustained mero-morphic functions dividing $(1, 1), (\infty, 0)$ and $H \neq 0$. Then

$$\tau(s, \Phi) \leq \Theta_2(s, 0; \Phi) + \Theta_2(s, 0; \Psi) + \frac{3}{2}\overline{\Theta}(s, \infty; \Phi) + \overline{\Theta}(s, \infty; \Psi) + \overline{\Theta}_*(s, \infty; \Phi, \Psi) + \frac{1}{2}\overline{\Theta}(s, 0; \Phi) + S(s, \Phi) + S(s, \Psi).$$

$$\tau(s, \Psi) \leq \Theta_2(s, 0; \Phi) + \Theta_2(s, 0; \Psi) + \overline{\Theta}(s, \infty; \Psi) + \overline{\Theta}(s, \infty; \Phi) + \overline{\Theta}_*(s, \infty; \Phi, \Psi) + \frac{1}{2}\overline{\Theta}(s, 0; \Psi) + S(s, \Phi) + S(s, \Psi).$$

Lemma 2.4. (Sahoo, 2010) Let Φ, Ψ be two non sustained mero-morphic functions dividing $(1, 0), (\infty, 0)$ and $H \neq 0$. Then

$$\tau(s, \Phi) \leq \theta_2(s, 0; \Phi) + \theta_2(s, 0; \Psi) + 3\overline{\theta}(s, \infty; \Phi) + 2\overline{\theta}(s, \infty; \Psi) + \overline{\theta}_*(s, \infty; \Phi, \Psi) + 2\overline{\theta}(s, 0; \Phi) + \overline{\theta}(s, 0; \Psi) + S(s, \Phi) + S(s, \Psi).$$

$$\tau(s, \Psi) \leq \theta_2(s, 0; \Phi) + \theta_2(s, 0; \Psi) + 3\overline{\theta}(s, \infty; \Psi) + 2\overline{\theta}(s, \infty; \Phi) + \overline{\theta}_*(s, \infty; \Phi, \Psi) + 2\overline{\theta}(s, 0; \Psi) + \overline{\theta}(s, 0; \Phi) + S(s, \Phi) + S(s, \Psi).$$

Lemma 2.5. (Yang and Yi, 1995) Let $\phi(t)$ be a non-sustained mero-morphic function, and let $\alpha_0(t), \alpha_1(t), \dots, \alpha_\rho(t) (\neq 0)$ be a tiny functions concerning ϕ . Then

$$\tau(s, \alpha_\rho \phi^\rho + \alpha_{\rho-1} \phi^{\rho-1} + \dots + \alpha_1 \phi + \alpha_0) = \rho \tau(s, \phi) + S(s, \phi).$$

Lemma 2.6. (Liu, Liu and Cao, 2011) Let $\phi(t)$ be a non-sustained mero-morphic function, and p, κ be a positive integers. Then

$$\begin{aligned} \tau(s, \phi^{(\kappa)}) &\leq \tau(s, \phi) + \kappa \overline{\theta}(s, \phi) + S(s, \phi), \\ \theta_p\left(s, \frac{1}{\phi^{(\kappa)}}\right) &\leq \tau(s, \phi^{(\kappa)}) - \tau(s, \phi) + \theta_{p+\kappa}\left(s, \frac{1}{\phi}\right) + S(s, \phi), \\ \theta_P\left(s, \frac{1}{\phi^{(\kappa)}}\right) &\leq \theta_{p+\kappa}\left(s, \frac{1}{\phi}\right) + \kappa \overline{\theta}(s, \phi) + S(s, \phi). \end{aligned}$$

Lemma 2.7. (Sahoo and Biswas, 2018) Let ϕ and ψ be entire functions, ρ, ϱ, κ be a positive integers, and let

$$\Phi = (\phi^\rho(\phi^\varrho - 1) \prod_{j=1}^d \phi(t + c_j)^{v_j})^{(\kappa)}, \quad \Psi = (\psi^\rho(\psi^\varrho - 1) \prod_{j=1}^d \psi(t + c_j)^{v_j})^{(\kappa)}$$

suppose non zero sustained α_1 and α_2 thus $\overline{\theta}(s, \alpha_1; \Phi) = \overline{\theta}(s, 0; \Psi)$ and $\overline{\theta}(s, \alpha_2; \Psi) = \overline{\theta}(s, 0; \Phi)$ then $\rho \leq 2\kappa + \varrho + \lambda + 2$.

Lemma 2.8. (Sahoo and Biswas, 2018) Let ϕ and ψ be entire functions, ρ, ϱ, κ be a positive integers, and let

$$\Phi = (\phi^\rho(\phi - 1) \prod_{j=1}^d \phi(t + c_j)^{v_j})^{(\kappa)}, \quad \Psi = (\psi^\rho(\psi - 1) \prod_{j=1}^d \psi(t + c_j)^{v_j})^{(\kappa)}$$

suppose non zero sustained c_1 and c_2 thus $\overline{\theta}(s, c_1; \Phi) = \overline{\theta}(s, 0; \Psi)$ and $\overline{\theta}(s, c_2; \Psi) = \overline{\theta}(s, 0; \Phi)$ then $\rho \leq 2\kappa + \varrho + \lambda + 2$ for $\varrho \leq \kappa + 1$ and $\rho \leq 4\kappa - \varrho + \lambda + 4$ for $\varrho > \kappa + 1$.

Lemma 2.9. (Chen and Chen, 2012) Suppose ϕ and ψ be two transcendental entire functions of restricted order, $c_j (j = 1, 2, \dots, d)$ be distinct restricted complex numbers, and $\rho, \varrho, d, v_j (j = 1, 2, \dots, d)$ are integers. If

$$\phi^\rho(\phi^\varrho - 1) \prod_{j=1}^d \phi(t + c_j)^{v_j} = \psi^\rho(\psi^\varrho - 1) \prod_{j=1}^d \psi(t + c_j)^{v_j}$$

then $\phi = b\psi$, where $b^\varrho = b^{\rho+\lambda} = 1$.

3 Main Proof

Proof of Theorem 1.3. Let

$$\Phi_1 = \phi^\rho(\phi^\varrho - 1) \prod_{j=1}^d \phi(t + c_j)^{v_j} \quad \Psi_1 = \psi^\rho(\psi^\varrho - 1) \prod_{j=1}^d \psi(t + c_j)^{v_j}$$

$\Phi = \frac{(\Phi_1)^{(\kappa)}}{\gamma(t)}$ and $\Psi = \frac{(\Psi_1)^{(\kappa)}}{\gamma(t)}$. Then Φ and Ψ are transcendental mero-morphic functions that divides $(1, \iota)$ and ϕ, ψ divides ∞IM . Since ϕ and ψ are transcendental, $\gamma(t)$ is a tiny function concerning both Φ_1 and Ψ_1 . Let us consider following cases separately.

Case 1. presume $\Delta \neq 0$. Now we examine the subsequent three sub cases,

Sub case 1.1. Suppose that $2 \leq \iota < \infty$. Then using lemma 2.2, we obtain

$$\begin{aligned} (\rho + \varrho + \lambda)\tau(s, \phi) &\leq \tau(s, \Phi) + S(s, \Phi) \\ &\leq \theta_2(s, 0; \Phi) + \theta_2(s, 0; \Psi) + \bar{\theta}(s, \infty; \Phi) + \bar{\theta}(s, \infty; \Psi) \\ &\quad + \bar{\theta}_*(s, \infty; \Phi, \Psi) - \varrho(s, 1; \Psi) - \theta_\Phi^{(3)}(s, 1; \Phi) - \bar{\theta}_2(s, \infty; \Psi) \\ &\quad + S(s, \Phi) + S(s, \Psi) \\ &\leq \theta_2(s, 0; \Phi) + \theta_2(s, 0; \Psi) + \bar{\theta}_*(s, \infty; \Phi, \Psi) \\ &\quad + \bar{\theta}(s, \infty; \Phi) + \bar{\theta}(s, \infty; \Psi) + S(s, \Phi) + S(s, \Psi). \end{aligned} \quad (3.1)$$

Noting that

$$\begin{aligned} \bar{\theta}_*(s, \infty; \Phi, \Psi) &= \bar{\theta}_L(s, \infty; \Phi) + \bar{\theta}_L(s, \infty; \Psi) \\ &\leq \bar{\theta}(s, \infty; \Phi) + \bar{\theta}(s, \infty; \Psi). \end{aligned} \quad (3.2)$$

Substitute (3.2) in (3.1), we get

$$\begin{aligned} (\rho + \varrho + \lambda)\tau(s, \phi) &\leq \theta_2(s, 0; \Phi) + \theta_2(s, 0; \Psi) + 2\bar{\theta}(s, \infty; \Phi) + \bar{\theta}(s, \infty; \Psi) \\ &\quad + S(s, \Phi) + S(s, \Psi) \\ &\leq \theta_2(s, 0; (\Phi_1)^{(\kappa)}) + \theta_2(s, 0; (\Psi_1)^{(\kappa)}) + \\ &\quad + 2\bar{\theta}(s, \infty; (\Phi_1)^{(\kappa)}) + \bar{\theta}(s, \infty; (\Psi_1)^{(\kappa)}) + S(s, \phi) + S(s, \psi) \\ &\leq \theta_{\kappa+2}(s, 0; \Phi_1) + \theta_{\kappa+2}(s, 0; \Psi_1) + 2\bar{\theta}(s, \infty; \Phi_1) \\ &\quad + \bar{\theta}(s, \infty; \Psi_1) + S(s, \phi) + S(s, \psi) \\ &\leq (\kappa + \varrho + \lambda + 4)\tau(s, \phi) + (\kappa + \varrho + \lambda + 3)\tau(s, \psi) \\ &\quad + S(s, \phi) + S(s, \psi). \end{aligned} \quad (3.3)$$

processing same manner we get

$$(\rho + \varrho + \lambda)\tau(s, \psi) \leq (\kappa + \varrho + \lambda + 4)\tau(s, \psi) + (\kappa + \varrho + \lambda + 3)\tau(s, \phi) + S(s, \phi) + S(s, \psi). \quad (3.4)$$

Adding (3.3) and (3.4), we get

$$(\rho - 2\kappa - \varrho - \lambda - 7)\{\tau(s, \phi) + \tau(s, \psi)\} \leq S(s, \phi) + S(s, \psi),$$

which is opposition to $\rho \geq 2\kappa + \varrho + \lambda + 7$.

Sub case 1.2.

when $\iota = 1$. Then using (3.2) and lemma 2.3, we reduce

$$\begin{aligned} (\rho + \varrho + \lambda)\tau(s, \phi) &= \tau(s, \Phi) + S(s, \Phi_1) \\ &\leq \theta_2(s, 0; \Phi) + \theta_2(s, 0; \Psi) + \frac{3}{2}\bar{\theta}(s, \infty; \Phi) + \bar{\theta}(s, \infty; \Psi) \\ &\quad + \bar{\theta}_*(s, \infty; \Phi, \Psi) + \frac{1}{2}\bar{\theta}(s, 0; \Phi) + S(s, \Phi) + S(s, \Psi) \\ &\leq \theta_2(s, 0; \Phi) + \theta_2(s, 0; \Psi) + \frac{5}{2}\bar{\theta}(s, \infty; \Phi) + \bar{\theta}(s, \infty; \Psi) \\ &\quad + \frac{1}{2}\bar{\theta}(s, 0; \Phi) + S(s, \Phi) + S(s, \Psi) \\ &\leq \theta_{\kappa+2}(s, 0; \Phi_1) + \theta_{\kappa+2}(s, 0; \Psi_1) + \frac{5}{2}\bar{\theta}(s, \infty; \Phi_1) + \bar{\theta}(s, \infty; \Psi_1) \\ &\quad + \frac{1}{2}\bar{\theta}(s, 0; \Phi_1) + S(s, \phi) + S(s, \psi) \\ &\leq \frac{3\kappa + 3\varrho + 3\lambda + 10}{2}\tau(s, \phi) + (\kappa + \varrho + \lambda + 3)\tau(s, \psi) + S(s, \phi) + S(s, \psi). \end{aligned} \quad (3.5)$$

Similarly we have,

$$(\rho + \varrho + \lambda)\tau(s, \psi) \leq \frac{3\kappa + 3\varrho + 3\lambda + 10}{2}\tau(s, \psi) + (\kappa + \varrho + \lambda + 3)\tau(s, \phi) + S(s, \phi) + S(s, \psi). \quad (3.6)$$

Adding (3.5) and (3.6), we get

$$\left(\rho - \frac{5\kappa}{2} - \frac{3\varrho}{2} - \frac{3\lambda}{2} + 11\right)\{\tau(s, \phi) + \tau(s, \psi)\} \leq S(s, \phi) + S(s, \psi),$$

which contradicts to $\rho \geq \frac{5\kappa}{2} + \frac{3\varrho}{2} + \frac{3\lambda}{2} + 12$.

Sub Case 1.3. when $\iota = 0$. As in sub case 1.1 and using (3.2) and lemma 2.4, we simplify

$$\begin{aligned} (\rho + \varrho + \lambda)\tau(s, \phi) &= \tau(s, \Phi) + S(s, \Phi) \\ &\leq \theta_2(s, 0; \Phi) + \theta_2(s, 0; \Psi) + 3\bar{\theta}(s, \infty; \Phi) + 2\bar{\theta}(s, \infty; \Psi) \\ &\quad + \bar{\theta}_*(s, \infty; \Phi, \Psi) + 2\bar{\theta}(s, 0; \Phi) + \bar{\theta}(s, 0; \Psi) + S(s, \Phi) + S(s, \Psi) \\ &\leq \theta_{\kappa+2}(s, 0; \Phi_1) + \theta_{\kappa+2}(s, 0; \Psi_1) + 4\bar{\theta}(s, \infty; \Phi_1) + 2\bar{\theta}(s, \infty; \Psi_1) \\ &\quad + 2\bar{\theta}(s, 0; \Phi_1) + \bar{\theta}(s, 0; \Psi_1) + S(s, \phi) + S(s, \psi) \\ &\leq (3\kappa + 3\varrho + 3\lambda + 8)\tau(s, \phi) + (2\kappa + 2\varrho + 2\lambda + 5)\tau(s, \psi) \\ &\quad + S(s, \phi) + S(s, \psi). \end{aligned} \quad (3.7)$$

Similarly we have,

$$(\rho + \varrho + \lambda)\tau(s, \psi) \leq (3\kappa + 3\varrho + 3\lambda + 8)\tau(s, \psi) + (2\kappa + 2\varrho + 2\lambda + 5)\tau(s, \phi) + S(s, \phi) + S(s, \psi). \quad (3.8)$$

Adding (3.7) and (3.8), we get

$$(\rho - 5\kappa - 4\varrho - 4\gamma - 13)\{\tau(s, \phi) + \tau(s, \psi)\} \leq S(s, \phi) + S(s, \psi),$$

which contradicts to $\rho \geq 5\kappa + 4\varrho + 4\gamma + 14$.

Case 2. We now presume that $\Delta \equiv 0$. That is

$$\left(\frac{\Phi_1''}{\Phi_1''} - \frac{2\Phi_1''}{\Phi_1' - 1}\right) - \left(\frac{\Psi_1''}{\Psi_1''} - \frac{2\Psi_1''}{\Psi_1' - 1}\right) = 0.$$

Integrating twice, we get

$$\frac{1}{\Phi_1 - 1} = \frac{A_0}{\Psi_1 - 1} + B_0. \quad (3.9)$$

Where $A_0 (\neq 0)$ and B_0 are sustained. From (3.9), we have the following sub cases.

Sub case 2.1. Let $B_0 \neq 0$ and $A_0 = B_0$. Then from (3.9), we get

$$\frac{1}{\Phi_1 - 1} = \frac{B_0\Psi_1}{\Psi_1 - 1}. \quad (3.10)$$

If $B_0 = -1$, then from (3.10), we have, $\Phi_1\Psi_1 = 1$.

i.e.,

$$(\phi^\rho(\phi^\varrho - 1) \prod_{j=1}^d \phi(t + c_j)^{v_j})^{(\kappa)} \cdot (\psi^\rho(\psi^\varrho - 1) \prod_{j=1}^d \psi(t + c_j)^{v_j})^{(\kappa)} = \gamma.$$

Since the number of zeros of $\gamma(t)$ is restricted, it follows that ϕ and ψ has restrictedly many zeros. we put $\phi(t) = h(t)e^{\beta(t)}$, where $h(t)$ is a non zero polynomial and $\beta(t)$ is a non sustained polynomials. Now replacing $\sum_{j=1}^d v_j \beta(t + c_j)$ by $\lambda^*(t)$ and $\prod_{j=1}^d h(t + c_j)^{v_j}$ by $\varrho u^*(t)$ we reduce that

$$\begin{aligned} (\phi^\rho(\phi^\varrho - 1) \prod_{j=1}^d \phi(t + c_j)^{v_j})^{(\kappa)} &= (h^\rho(t)e^{\rho\beta(t)}(h^\varrho(t)e^{\varrho\beta(t)} - 1) \prod_{j=1}^d h(t + c_j)^{v_j} e^{v_j\beta(t+c_j)})^{(\kappa)} \\ &= (h^\rho(t)\varrho u^*(t)e^{\rho\beta(t)+\lambda^*(t)}(h^\varrho(t)e^{\varrho\beta(t)} - 1))^{(\kappa)} \\ &= (h^{\rho+\varrho}(t)\varrho u^*(t)e^{(\rho+\varrho)\beta(t)+\lambda^*(t)} - (h^\rho(t)\varrho u^*(t)e^{\rho\beta(t)+\lambda^*(t)}))^{(\kappa)} \\ &= e^{(\rho+\varrho)\beta(t)+\lambda^*(t)}P_1(\beta(t), \lambda^*(t), h(t), \varrho u^*(t), \dots, \beta^{(\kappa)}(t), \lambda^{*(\kappa)}(t), h^{(\kappa)}(t), \varrho u^{*(\kappa)}(t)) \\ &\quad - e^{\rho\beta(t)+\lambda^*(t)}P_2(\beta(t), \lambda^*(t), h(t), \varrho u^*(t), \dots, \beta^{(\kappa)}(t), \lambda^{*(\kappa)}(t), h^{(\kappa)}(t), \varrho u^{*(\kappa)}(t)) \\ &= e^{\rho\beta(t)+\lambda^*(t)}(P_1e^{\varrho\beta(t)} - P_2). \end{aligned}$$

Obviously $P_1e^{\varrho\beta(t)} - P_2$ has infinite number of zeros which contradicts with the fact that ψ is an entire function. Now we presume that $\Phi_1 \equiv \Psi_1$. Then

$$(\phi^\rho(\phi^\varrho - 1) \prod_{j=1}^d \phi(t + c_j)^{v_j})^{(\kappa)} = (\psi^\rho(\psi^\varrho - 1) \prod_{j=1}^d \psi(t + c_j)^{v_j})^{(\kappa)}.$$

Integrating we get,

$$(\phi^\rho(\phi^\varrho - 1) \prod_{j=1}^d \phi(t + c_j)^{v_j})^{(\kappa-1)} = (\psi^\rho(\psi^\varrho - 1) \prod_{j=1}^d \psi(t + c_j)^{v_j})^{(\kappa-1)} + C_{\kappa-1},$$

where $C_{\kappa-1}$ is a sustained. If $C_{\kappa-1} \neq 0$, from lemma 2.7, we obtain $\rho \leq 2\kappa$, a opposition. Consequently $C_{\kappa-1} = 0$. Repeat the process κ -times, we obtain

$$\phi^\rho(\phi^\varrho - 1) \prod_{j=1}^d \phi(t + c_j)^{v_j} = \psi^\rho(\psi^\varrho - 1) \prod_{j=1}^d \psi(t + c_j)^{v_j},$$

which by lemma 2.9 gives $\phi = b\psi$, where $b^\varrho = b^{\rho+\lambda} = 1$.

Sub case 2.2. Let $B_0 \neq 0$ and $A_0 \neq B_0$. Then by (3.10), we get

$$\Phi_1 = \frac{(B_0 + 1)\Psi_1 - (B_0 - A_0 + 1)}{B_0\Psi_1 + (A_0 - B_0)},$$

and so,

$$\bar{\theta}\left(s, \frac{B_0 - A_0 + 1}{B_0 + 1}, \Psi_1\right) = \bar{\theta}(s, 0; \Phi_1).$$

moving forward as in Sub case 2.1, we obtain a opposition.

Sub case 2.3. Let $B_0 = 0$ and $A_0 \neq 0$. Then by (3.10), we get

$$\Phi_1 = \frac{\Psi_1 + A_0 - 1}{A_0} \quad \text{and} \quad \Psi_1 = A_0\Phi_1 - (A_0 - 1).$$

If $A_0 \neq 1$, we have

$$\bar{\theta}\left(s, \frac{A_0 - 1}{A_0}, \Phi_1\right) = \bar{\theta}(s, 0; \Psi_1) \quad \text{and} \quad \bar{\theta}(s, A_0 - 1, \Psi_1) = \bar{\theta}(s, 0; \Phi_1),$$

a opposition. Then $A_0 = 1$, which implies $\Phi_1 \equiv \Psi_1$ and therefore

$$(\phi^\rho(\phi^\varrho - 1) \prod_{j=1}^d \phi(t + c_j)^{v_j})^{(\kappa)} = (\psi^\rho(\psi^\varrho - 1) \prod_{j=1}^d \psi(t + c_j)^{v_j})^{(\kappa)}.$$

Proof of Theorem 1.4. Let $\Phi_2 = \phi^\rho(\phi - 1)^\varrho \prod_{j=1}^d \phi(t + c_j)^{v_j}$, $\Psi_2 = \psi^\rho(\psi - 1)^\varrho \prod_{j=1}^d \psi(t + c_j)^{v_j}$, $\Phi = \frac{(\Phi_2)^{(\kappa)}}{\gamma(t)}$ and $\Psi = \frac{(\Psi_2)^{(\kappa)}}{\gamma(t)}$. Then Φ and Ψ are transcendental mero-morphic function that divides $(1, \iota)$ and ϕ and ψ share ∞ IM. Putting in a similar way at Theorem 1.8 we get either $\Phi_2\Psi_2 = 1$ or $\Phi_2 = \Psi_2$. If $\Phi_2 = \Psi_2$, then putting the same of theorem 1.8 and by using lemma 2.8 we get

$$(\phi^\rho(\phi - 1)^\varrho \prod_{j=1}^d \phi(t + c_j)^{v_j})^{(\kappa)} = (\psi^\rho(\psi - 1)^\varrho \prod_{j=1}^d \psi(t + c_j)^{v_j})^{(\kappa)} \quad (3.11)$$

Set $h = \frac{\phi}{\psi}$. If h is a sustained, then by substituting $\phi = \psi h$ in (3.11) we reduce that

$$\psi^\varrho(h^{\rho+\varrho+\lambda} - 1) - {}^\varrho C_1 \psi^{\varrho-1}(h^{\rho+\varrho+\lambda-1} - 1) + \dots + (-1)^\varrho(h^{\rho+\lambda} - 1) = 0,$$

which implies $h = 1$ and consequently $\phi = \psi$. If h is not a sustained, then we know by (3.11) that ϕ and ψ satisfy algebraic equation $R(\phi, \psi) = 0$ where $R(\phi, \psi)$ is given by

$$R(\varpi_1, \varpi_2) = \varpi_1^\rho(\varpi_1 - 1)^\varrho \prod_{j=1}^d \varpi_1(t + c_j)^{v_j} - \varpi_2^\rho(\varpi_2 - 1)^\varrho \prod_{j=1}^d \varpi_2(t + c_j)^{v_j}.$$

If $\Phi_2\Psi_2 = 1$, using the same method as in Theorem 1.8 we arrive at a opposition which ends the theorem.

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