

The exponential variation property for the Weibull distribution

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ABSTRACT

In this paper, our aim is to characterize the Weibull distribution widely used to model lifetimes in reliability analysis. Namely, for the exponential variation index defined as the squared of the coefficient of variation, we show that the two-parameter Weibull distribution is over-, equi- and under-varied if and only if its shape parameter is less, equal and greater than one, respectively. In reliability, similar behaviours on the shape parameter are known for the decay, constant and growth of the Weibull failure rate curve. To this end, we have stated the main result which is analytically proved by means of sharp inequalities involving the polygamma functions.

Keywords: Coefficient of variation; Digamma function; Equi-variation; Gamma function; Over-variation; Trigamma function; Under-variation..

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1 Introduction and result

Abid et al. (2019, 2020) have recently introduced the exponential variation index, so-called Jørgensen variation index, for continuous positive random variable X on $(0, \infty)$ as the ratio of variance to the squared expectation. More precisely, this positive quantity is written by

$$VI(X) := \frac{\text{Var} X}{(\mathbb{E} X)^2} \begin{matrix} \geq \\ \leq \end{matrix} 1, \quad (1.1)$$

i.e., X is over- (equi- and under-varied) compared to the exponential random variable with the same expectation $\mathbb{E}X$ if $\text{Var}X > (\mathbb{E}X)^2$ ($\text{Var}X = (\mathbb{E}X)^2$ and $\text{Var}X < (\mathbb{E}X)^2$, respectively). It can be viewed as the squared of the standard coefficient of variation; see also Kokonendji et al. (2020) for general multivariate extensions.

In the framework of reliability, an increasing/decreasing failure rate on the average (IFRA/DFRA) distribution which is also the corresponding IFR/DFR one implies $\text{VI} \leq (\geq) 1$, that is under- (over-) variation in the large sense (i.e., including equi-variation); see, e.g., Barlow and Proschan (1981) in the sense of the coefficient of variation. Note that IFR distributions are more frequent than DFR, but over-varied data are not the most frequent situation in reliability.

Among the distributions widely used to model lifetimes in reliability theory, the Weibull (1951) distribution plays a central role as one of the most important generalizations of the exponential distribution; see, e.g., Smeitink and Dekker (1990) and Yannaros (1994). It has been considered for many types of manufactured items, such as vacuum tubes, ball bearings and electrical insulation. It is also used in biomedical applications, for example, as the distribution for times to occurrence (diagnosis) of tumors in human populations or in laboratory animals. Tanaka and Ichikawa (1983) have proposed an approximate formula for the standard coefficient of variation $\sqrt{\text{VI}(X)}$ for the Weibull distribution for the purposes of mental arithmetic. Herein, our objective is to characterize the Weibull distribution in the sense of under- and over-variation mentioned above.

The probability density function of the well-known two-parameter Weibull variable $X \sim \mathcal{W}(\alpha, \beta)$ with scale parameter $\alpha > 0$, shape parameter $\beta > 0$ is expressed as

$$f_{\mathcal{W}}(x) = \frac{\beta}{\alpha} \left(\frac{x}{\alpha} \right)^{\beta-1} \exp \left[- \left(\frac{x}{\alpha} \right)^{\beta} \right] \mathbb{1}_{[0, \infty)}(x),$$

where $\mathbb{1}_A$ denotes the indicator function of any given event A (e.g., ter Berg, 2009; Weibull, 1951). For $\beta = 1$, $\mathcal{W}(\alpha, 1)$ is reduced to the exponential distribution. Since the expectation and variance of $\mathcal{W}(\alpha, \beta)$ are $\mathbb{E}X = \alpha \Gamma(1 + 1/\beta)$ and $\text{Var}X = \alpha^2 [\Gamma(1 + 2/\beta) - \Gamma^2(1 + 1/\beta)]$, respectively, where $\Gamma(\cdot)$ is the classical gamma function, its exponential variation index satisfied the following property in this theorem:

Theorem 1.1. *Let $X \sim \mathcal{W}(\alpha, \beta)$ with $\alpha, \beta > 0$. Then:*

$$0 < \text{VI}(X) \leq 1 \Leftrightarrow 0 < \beta \leq 1, \forall \alpha > 0.$$

This result has been numerically illustrated in Kokonendji et al. (2020, Table 1) without proof. In reliability this result is often used, in the same way of β with respect to 1, for the classical failure rate (or bathtub) curve from $\lambda_{\mathcal{W}}(t) = (\beta/\alpha)(t/\alpha)^{\beta-1} \mathbb{1}_{(0, \infty)}(t)$; i.e., $\lambda_{\mathcal{W}}(\cdot)$ is decreasing/constant/increasing for $\beta \leq 1$ (or $\beta \neq 1$), respectively. Accordingly, the Weibull model can be also used for over-, equi- and under-varied data. We

show Theorem 1.1 in the following last section with concluding remarks. Equivalent results of Theorem 1.1 are trivial, for instance, for the lognormal variable $Y \sim \mathcal{LN}(\mu, \sigma^2)$ with $\mu \in \mathbb{R}$, $\sigma^2 > 0$, such that $0 < \text{VI}(Y) \leq 1 \Leftrightarrow 0 < \sigma^2 \leq \log 2$, $\forall \mu \in \mathbb{R}$, and for the gamma variable $Y \sim \mathcal{G}(\alpha, \beta)$ with $\alpha, \beta > 0$ such that $0 < \text{VI}(Y) \leq 1 \Leftrightarrow 0 < \beta \leq 1$, $\forall \alpha > 0$.

2 Proof of Theorem 1.1 and concluding remarks

2.1 Proof of Theorem 1.1

Apply (1.1) to $X \sim \mathcal{W}(\alpha, \beta)$, it reveals

$$\text{VI}(X) = \frac{\alpha^2 \left[\Gamma(1 + 2/\beta) - \Gamma^2(1 + 1/\beta) \right]}{\alpha^2 \Gamma^2(1 + 1/\beta)} = \frac{\Gamma(1 + 2/\beta)}{\Gamma^2(1 + 1/\beta)} - 1$$

which does not depend on $\alpha > 0$. Using the relation $\Gamma(x+1) = x\Gamma(x)$, $\forall x > 0$, it follows that

$$\text{VI}(X) = 2\beta \frac{\Gamma(2/\beta)}{\Gamma^2(1/\beta)} - 1, \quad \forall \alpha, \beta > 0;$$

and, therefore,

$$0 < \text{VI}(X) \leq 1 \Leftrightarrow 0 < \beta \Gamma(2/\beta) / \Gamma^2(1/\beta) \leq 1, \quad \forall \alpha, \beta > 0. \quad (2.1)$$

For fixed $\alpha > 0$, the right inequalities of (2.1) are equivalent to

$$\Gamma(2/\beta) - \Gamma^2(1/\beta) / \beta \geq 0, \quad \forall \beta > 0. \quad (2.2)$$

Setting $x = 1/\beta$ and using Legendre's doubling formula (e.g., Abramowitz and Stegun, 1972; Chap. 6) with $\Gamma(x) > 0$ for all $x > 0$, Inequalities (2.2) becomes successively

$$\begin{aligned} \forall x > 0, \Gamma(2x) - x\Gamma^2(x) \geq 0 &\Leftrightarrow \Gamma(x) \left[\frac{2^{(2x-1)}}{\pi^{1/2}} \Gamma\left(x + \frac{1}{2}\right) - x\Gamma(x) \right] \geq 0, \quad \forall x > 0, \\ &\Leftrightarrow \frac{2^{(2x-1)}}{\pi^{1/2}} \Gamma\left(x + \frac{1}{2}\right) - x\Gamma(x) \geq 0, \quad \forall x > 0, \\ &\Leftrightarrow \frac{4^x}{2} \Gamma\left(x + \frac{1}{2}\right) - \pi^{1/2} x \Gamma(x) \geq 0, \quad \forall x > 0. \end{aligned}$$

Now, let us introduce the function $\mathcal{C}(\cdot)$ defined on $(0, \infty)$ by

$$\mathcal{C}(x) := \frac{4^x}{2} \Gamma\left(x + \frac{1}{2}\right) - \pi^{1/2} x \Gamma(x), \quad \forall x > 0. \quad (2.3)$$

One has the following pointwise and asymptotic behaviours of $\mathcal{C}(\cdot)$ in the first lemma.

Lemma 2.1. *The real-valued function $\mathcal{C}(\cdot)$ defined on $(0, \infty)$ given by (2.3) satisfies: (i) $\mathcal{C}(\cdot)$ is continuous on $(0, \infty)$, (ii) $\mathcal{C}(1) = 0$, (iii) $\lim_{x \searrow 0} \mathcal{C}(x) = -\sqrt{\pi}/2$, (iv) $\lim_{x \rightarrow \infty} \mathcal{C}(x) = \infty$ and, (v) $\lim_{x \rightarrow \infty} \mathcal{C}(x)/x = \infty$ which means that the $0y$ -axis is a parabolic branch for the curve of $\mathcal{C}(\cdot)$.*

Proof. Parts (i), (ii) and (iii) are trivial. For Parts (iv) and (v), we first have the expansion

$$\frac{\Gamma(x+1/2)}{\Gamma(x)} = \sqrt{x} \left(1 - \frac{1}{8x} + \frac{1}{128x^2} + \frac{5}{1024x^3} + o\left(\frac{1}{x^4}\right) \right)$$

(e.g., Graham et al., 1994). Then, derived from the Euler-Maclaurin formula it follows that

$$\frac{\Gamma(x+1/2)}{\Gamma(x)} \approx \sqrt{x}, \text{ as } x \rightarrow \infty.$$

Therefore, we obtain

$$\forall x > 0, \mathcal{C}(x) = x\Gamma(x) \left[\frac{4^x}{2} \frac{\Gamma(x+1/2)}{x\Gamma(x)} - \pi^{1/2} \right] \approx \frac{4^x}{2} \sqrt{x} \Gamma(x), \text{ as } x \rightarrow \infty, \quad (2.4)$$

and also

$$\frac{\mathcal{C}(x)}{x} \approx \frac{4^x}{2\sqrt{x}} \Gamma(x), \text{ as } x \rightarrow \infty. \quad (2.5)$$

Since $\lim_{x \rightarrow \infty} \Gamma(x) = \infty$, the desired results are easily deduced from (2.4) and (2.5), respectively. $\square$

In what follows, we show that the function $\mathcal{C}(\cdot)$ defined on $(0, \infty)$ is strictly increasing on $(0, \infty)$ through the lemma below.

Lemma 2.2. *The continuous function $\mathcal{C}(\cdot)$ defined on $(0, \infty)$ into $(-\sqrt{\pi}/2, \infty)$ given by (2.3) is differentiable on $(0, \infty)$. Its first derivative $\mathcal{C}'(\cdot)$ is positive on $(0, \infty)$ with*

$$\mathcal{C}'(x) = \frac{4^x}{2} \left[\log(4) \Gamma\left(x + \frac{1}{2}\right) + \Gamma'\left(x + \frac{1}{2}\right) \right] - \pi^{1/2} \left[\Gamma(x) + x\Gamma'(x) \right]$$

and $\Gamma'(x) = \int_0^\infty t^{x-1} \log(t) \exp(-t) dt$ for all $x > 0$.

Proof. By introducing the digamma function defined by $\psi(x) = \Gamma'(x)/\Gamma(x)$, $\forall x > 0$, and using the formula $\psi(2x) = (1/2)\psi(x) + (1/2)\psi(x+1/2) + \log(2)$ in conjunction with $\Gamma(2x) = (2\pi)^{-1/2} 2^{2x-1/2} \Gamma(x)\Gamma(x+1/2)$, $\mathcal{C}'(x)$ can be rewritten successively, for all $x > 0$, as follows:

$$\begin{aligned} \mathcal{C}'(x) &= \frac{4^x}{2} \left[\log(4) \Gamma\left(x + \frac{1}{2}\right) + \Gamma\left(x + \frac{1}{2}\right) \psi\left(x + \frac{1}{2}\right) \right] - \pi^{1/2} \left(\Gamma(x) + x\Gamma(x)\psi(x) \right) \\ &= \frac{4^x}{2} \Gamma\left(x + \frac{1}{2}\right) \left[\log(4) + \psi\left(x + \frac{1}{2}\right) \right] - \pi^{1/2} \Gamma(x) \left(1 + x\psi(x) \right) \\ &= 2^{2x} \Gamma\left(x + \frac{1}{2}\right) \left[\psi(2x) - \frac{\psi(x)}{2} \right] - \pi^{1/2} \Gamma(x) \left(1 + x\psi(x) \right) \\ &= \pi^{1/2} \Gamma(x) \left\{ \frac{\Gamma(2x)}{\Gamma^2(x)} \left(2\psi(2x) - \psi(x) \right) - \left(1 + x\psi(x) \right) \right\}. \end{aligned}$$

Since $\psi(x+1) = \psi(x) + 1/x$, we finally get

$$\mathcal{C}'(x) = \pi^{1/2}\Gamma(x) \left\{ \frac{\Gamma(2x)}{\Gamma^2(x)} \left(2\psi(2x) - \psi(x) \right) - x\psi(x+1) \right\} =: \pi^{1/2}\Gamma(x)\mathcal{D}(x).$$

Consider the function $\mathcal{D}$ defined on $(0, \infty)$ as follows:

$$\forall x > 0, \mathcal{D}(x) = \frac{\Gamma(2x)}{\Gamma^2(x)} \left(2\psi(2x) - \psi(x) \right) - x\psi(x+1).$$

Our remaining objective is to prove that $\mathcal{D}(x) > 0, \forall x > 0$.

From a reformulation of Theorem 2.1.a of Batir (2005), one first gets

$$\log(x+1/2) < \psi(x+1) < \log\{x+\exp(-\gamma)\}, \forall x > 0,$$

where $\gamma = -\psi(1) = 0.577215\dots$ denotes the Euler-Mascheroni constant. Then, one deduces that

$$\frac{\Gamma(2x)}{\Gamma^2(x)} \left(2\psi(2x) - \psi(x) \right) - x \log\{x+\exp(-\gamma)\} < \mathcal{D}(x) < \frac{\Gamma(2x)}{\Gamma^2(x)} \left(2\psi(2x) - \psi(x) \right) - x \log\left(x+\frac{1}{2}\right),$$

for all $x > 0$. From now, it suffices to show that

$$\forall x > 0, 2\psi(2x) - \psi(x) \geq \frac{\Gamma^2(x)}{\Gamma(2x)} x \log\{x+\exp(-\gamma)\}. \quad (2.6)$$

Indeed, for $x \geq 1$ the above inequality (2.6) holds since, using the Legendre doubling formula and the monotonic decreasing of the function $x \mapsto \Gamma(x)/\Gamma(x+1/2)$ on $[1, \infty)$, it comes that

$$x \frac{\Gamma^2(x)}{\Gamma(2x)} = \frac{x\Gamma(x)\sqrt{2\pi}2^{1/2-2x}}{\Gamma(x+1/2)} \leq x2^{2(1-x)} \leq 1$$

and

$$2\psi(2x) - \psi(x) = \psi\left(x+\frac{1}{2}\right) + \log 4 > \log(4x+4) - \frac{2}{2x+1} \geq \log(x+e^{-\gamma}).$$

The first inequality derives from $\psi(x) > \log(x+1/2) - 1/x, \forall x > 0$, and the second large one is directly obtained from a simple function study.

As for $x \in (0, 1)$, one can make use of the following inequalities due to Alzer (1999, Theorem 2):

$$x^{\alpha(x-1)-\gamma} < \Gamma(x) < x^{\beta(x-1)-\gamma},$$

with $\alpha = 1 - \gamma$ and $\beta = (\pi^2/6 - \gamma)/2$, which gives

$$x \frac{\Gamma^2(x)}{\Gamma(2x)} \log(x+e^{-\gamma}) < 2^{x(2\gamma-1)+1} x^{(x-1)(\gamma-2+\pi^2/6)} \log(x+e^{-\gamma}) < \log(4x+4) - \frac{2}{2x+1}.$$

Finally, one gets (2.6) on $(0, 1)$ because we already have

$$2\psi(2x) - \psi(x) > \log(4x+4) - \frac{2}{2x+1}, \forall x > 0.$$

This concludes the proof of the lemma. $\square$

From Lemma 2.2, the function $\mathcal{C}(\cdot)$ is a one-to-one mapping defined on $(0, \infty)$ into $(-\sqrt{\pi}/2, \infty)$ with $\mathcal{C}(1) = 0$. Recalling the setting $x = 1/\beta$ stated above, one gets Theorem 1.1, namely the Weibull distribution of X is over-varied (i.e., $\text{VI}(X) > 1$) for $\beta < 1$, equi-varied (i.e., $\text{VI}(X) = 1$) for $\beta = 1$, and under-varied (i.e., $\text{VI}(X) < 1$) for $\beta > 1$.

2.2 Concluding remarks

In this paper, the Weibull distribution has been characterized in the sense of under-, equi- and over-variation distribution. This characterization has been formulated in term of Theorem 1.1 that we proved analytically. For this purpose, we have introduced the function $\mathcal{C}(\cdot)$ onto $(0, \infty)$ given in (2.3) and studied its continuity and differentiability through Lemma 2.1 and Lemma 2.2.

Moreover, Figure 1 depicts the function $\mathcal{C}(\cdot)$, its first derivative $\mathcal{C}'$ and second derivative $\mathcal{C}''$ by using the R software (R Core Team, 2020). In the matter of fact, the function $\mathcal{C}(\cdot)$ may be conjectured to be convex or to have positive second derivative on $(0, \infty)$ with

$$\mathcal{C}''(x) = 2 \log^2(2) 4^x \Gamma\left(x + \frac{1}{2}\right) + 2 \log(2) 4^x \Gamma'\left(x + \frac{1}{2}\right) + \frac{1}{2} 4^x \Gamma''\left(x + \frac{1}{2}\right) - \pi^{1/2} \Gamma''(x + 1)$$

and $\Gamma''(x) = \int_0^\infty t^{x-1} \log^2(t) \exp(-t) dt$ for all $x > 0$. For that, it will be necessary to consider further improving certain inequalities with respect to the trigamma function.

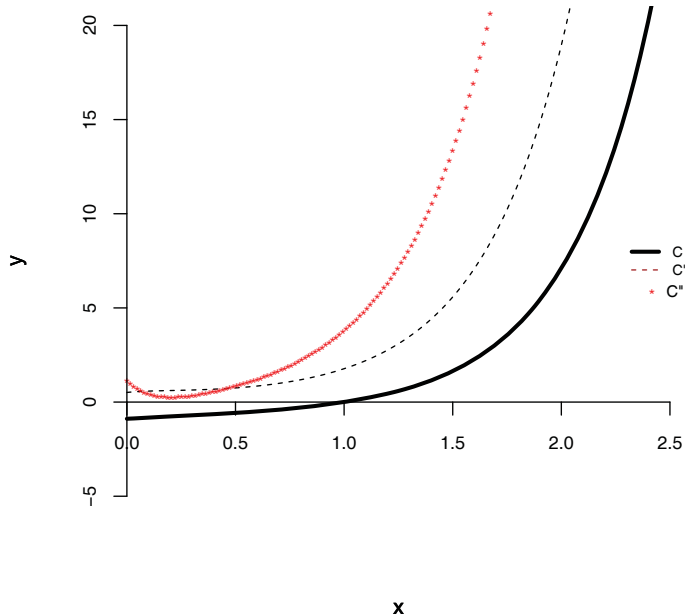


Figure 1: Graphs of the functions C , C' and C'' on $(0, \infty)$ with $C(0^+) = -\sqrt{\pi}/2$ and $C(1) = 0$.

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