

k —Almost Balancing Numbers

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ABSTRACT

In this work, we determine the general terms of k —almost balancing numbers and k —almost Lucas-balancing numbers of first and second type in terms of balancing and Lucas-balancing numbers for some integer $k \geq 1$.

Keywords: Balancing number, almost balancing number, cobalancing number, Pell equation, quadratic form.

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1 Introduction

A positive integer n is called a balancing number (see [2]) and is denoted by B_n if the Diophantine equation

$$1 + 2 + \cdots + (n - 1) = (n + 1) + (n + 2) + \cdots + (n + r) \quad (1.1)$$

verified for some positive integer r which is called balancer. Similarly a positive integer n is called a cobalancing number (see [11]) and is denoted by b_n if the Diophantine equation

$$1 + 2 + \cdots + n = (n + 1) + (n + 2) + \cdots + (n + r) \quad (1.2)$$

verified for some positive integer r which is called cobalancer (see also [9, 10, 13, 14]).

Recently many authors studied on several type of balancing numbers (see [5, 6, 7, 8, 15, 16, 18]). In [12], the authors defined that a positive integer n is called an almost balancing number if the Diophantine equation

$$|[(n + 1) + (n + 2) + \cdots + (n + r)] - (1 + 2 + \cdots + n - 1)| = 1 \quad (1.3)$$

verified for some positive integer r which is called almost balancer. If (1.3) equals to 1, then n is called an almost balancing number of first type and is denoted by B_n^* , and equals to -1 , then n is called an almost balancing number of second type and is denoted by B_n^{**} . Thus from (1.3), we get

$$C_n^* = \sqrt{8(B_n^*)^2 + 9} \quad \text{and} \quad C_n^{**} = \sqrt{8(B_n^{**})^2 - 7}$$

are integers which are called the almost Lucas-balancing number of first type and second type, respectively.

2 k -Almost Balancing Numbers.

Let $k \geq 1$ be an integer. Then by considering (1.2) and (1.3), a positive integer n is called a k -almost cobalancing number if the Diophantine equation

$$|[(n+1) + (n+2) + \cdots + (n+r)] - [1 + 2 + \cdots + n]| = k \quad (2.1)$$

holds for some positive integer r which is called k -almost cobalancer. If (2.1) equals to 1, then n is called a k -almost cobalancing number of first type and is denoted by b_n^{k*} , and equals to -1 , then n is called a k -almost cobalancing number second type and is denoted by b_n^{k**} . Hence

$$c_n^{k*} = \sqrt{8(b_n^{k*})^2 + 8b_n^{k*} + 8k + 1} \quad \text{and} \quad c_n^{k**} = \sqrt{8(b_n^{k**})^2 + 8b_n^{k**} - 8k + 1}$$

are integers which are called k -almost Lucas-cobalancing number of first type and second type, respectively. In [17], we determined the general terms of k -almost cobalancing numbers and k -almost Lucas-cobalancing numbers of first and second type.

Similarly in this section, we determine the general terms of k -almost balancing numbers and k -almost Lucas-balancing numbers of first and second type.

Again let $k \geq 1$ be an integer. Then by considering (1.1) and (1.3), a positive integer n is called a k -almost balancing number if the Diophantine equation

$$|[(n+1) + (n+2) + \cdots + (n+r)] - [1 + 2 + \cdots + (n-1)]| = k \quad (2.2)$$

holds for some positive integer r which is called k -almost balancer. If (2.2) equals to 1, then n is called a k -almost balancing number of first type and in this case

$$r = \frac{-2n - 1 + \sqrt{8n^2 + 8k + 1}}{2}. \quad (2.3)$$

If equals to -1 , then n is called a k -almost balancing number second type and in this case

$$r = \frac{-2n - 1 + \sqrt{8n^2 - 8k + 1}}{2}. \quad (2.4)$$

Let B_n^{k*} denote the k -almost balancing number of first type and let B_n^{k**} denote the k -almost balancing number second type. Then from (2.3) and (2.4), we get

$$C_n^{k*} = \sqrt{8(B_n^{k*})^2 + 8k + 1} \quad \text{and} \quad C_n^{k**} = \sqrt{8(B_n^{k**})^2 - 8k + 1} \quad (2.5)$$

are integers which are called k -almost Lucas-balancing number of first type and second type, respectively.

2.1 k -Almost Balancing and k -Almost Lucas-Balancing Numbers of First Type.

In this subsection, we will determine the general terms of k -almost balancing numbers and k -almost Lucas-balancing numbers of first type. We see in (2.3) that n is k -almost balancing number of first type if and only if $8n^2 + 8k + 1$ is a perfect square. So we set $8n^2 + 8k + 1 = y^2$ for some integer $y \geq 1$. Then $2(2n)^2 - y^2 = -8k - 1$ and taking $x = 2n$, we get the Pell equation (see [1, 4])

$$2x^2 - y^2 = -8k - 1. \quad (2.6)$$

For the set of all (positive) integer solutions of (2.6), we have $F = (2, 0, -1)$ of discriminant $\Delta = 8$, $\tau_8 = 3 + 2\sqrt{2}$ and

$$M = \begin{bmatrix} 3 & 4 \\ 2 & 3 \end{bmatrix} \quad (2.7)$$

(see [3]). Let Ω^* denote the set of all integer solutions of (2.6), that is, $\Omega^* = \{(x, y) : 2x^2 - y^2 = -8k - 1\}$. Then we have three cases:

1. $8k + 1$ is a perfect square and $\#Rep = 1$
2. $8k + 1$ is a perfect square and $\#Rep = 2q + 1$
3. $8k + 1$ is not a perfect square and $\#Rep = 2q$

for some integer $q \geq 1$.

Theorem 2.1. *If $8k + 1$ is a perfect square and $\#Rep = 1$, then $\Omega^* = \{(2hB_n, hC_n) : n \geq 1\}$, where $h = \sqrt{8k + 1}$.*

Proof. If $8k + 1$ is a perfect square and $\#Rep = 1$, then the set of representatives is $Rep = \{[0 \ h]\}$, where $h = \sqrt{8k + 1}$ and in this case $[0 \ h]M^n$ generates all integer solutions (x_n, y_n) of (2.6) for $n \geq 1$, where M is defined in (2.7). It can be easily seen that the n^{th} power of M is

$$M^n = \begin{bmatrix} C_n & 4B_n \\ 2B_n & C_n \end{bmatrix}$$

for $n \geq 1$, where $C_n = \sqrt{8B_n^2 + 1}$ is the Lucas-balancing number. So

$$[x_n \ y_n] = [0 \ h] \begin{bmatrix} C_n & 4B_n \\ 2B_n & C_n \end{bmatrix} = [2hB_n \ hC_n].$$

Therefore $\Omega^* = \{(2hB_n, hC_n) : n \geq 1\}$ as we wanted. □

From Theorem 2.1, we can give the following theorem

Theorem 2.2. *If $8k + 1$ is a perfect square and $\#Rep = 1$, then the general terms of k -almost balancing numbers and k -almost Lucas-balancing numbers of first are*

$$B_n^{k*} = hB_n \quad \text{and} \quad C_n^{k*} = hC_n$$

for $n \geq 0$, where $h = \sqrt{8k + 1}$.

Proof. We proved in Theorem 2.1 that $\Omega^* = \{(2hB_n, hC_n) : n \geq 1\}$. Since $x = 2n$, we get

$$B_n^{k*} = hB_n.$$

Thus from (2.5), we deduce that

$$C_n^{k*} = \sqrt{8(B_n^{k*})^2 + 8k + 1} = \sqrt{8(hB_n)^2 + 8k + 1} = \sqrt{h^2(8B_n^2 + 1)} = hC_n$$

as we claimed. □

Theorem 2.3. *If $8k+1$ is a perfect square and $\#Rep = 2q+1$, then $\Omega^* = \{(x_{(2q+1)n+i}, y_{(2q+1)n+i}) : n \geq 0\} \cup \{(x_{(2q+1)n}, y_{(2q+1)n}), (x_{(2q+1)n-i}, y_{(2q+1)n-i}) : n \geq 1\}$, where*

$$\begin{aligned}(x_{(2q+1)n+i}, y_{(2q+1)n+i}) &= (2k_{2i}B_n + k_{2i-1}C_n, 4k_{2i-1}B_n + k_{2i}C_n) \\(x_{(2q+1)n}, y_{(2q+1)n}) &= (2hB_n, hC_n) \\(x_{(2q+1)n-i}, y_{(2q+1)n-i}) &= (2k_{2i}B_n - k_{2i-1}C_n, -4k_{2i-1}B_n + k_{2i}C_n)\end{aligned}$$

and k_{2i-1}, k_{2i} are positive integers such that $2k_{2i-1}^2 - k_{2i}^2 = -8k - 1$ for $1 \leq i \leq q, 0 < k_1 < k_3 < \dots < k_{2q-1}, h < k_2 < k_4 < \dots < k_{2q}$ and $h = \sqrt{8k+1}$.

Proof. If $8k+1$ is a perfect square and $\#Rep = 2q+1$, then the set of representatives is

$$Rep = \{[0 \ h], [\pm k_{2i-1} \ k_{2i}]\},$$

where k_{2i-1} and k_{2i} are positive integers such that $2k_{2i-1}^2 - k_{2i}^2 = -8k - 1$ for $1 \leq i \leq q, 0 < k_1 < k_3 < \dots < k_{2q-1}, h < k_2 < k_4 < \dots < k_{2q}$ and $h = \sqrt{8k+1}$. In this case

1. $[0 \ h]M^n$ generates all integer solutions $(x_{(2q+1)n}, y_{(2q+1)n})$ for $n \geq 1$,
2. $[k_{2i-1} \ k_{2i}]M^n$ generates all integer solutions $(x_{(2q+1)n+i}, y_{(2q+1)n+i})$ for $n \geq 0$,
3. $[-k_{2i-1} \ k_{2i}]M^n$ generates all integer solutions $(x_{(2q+1)n-i}, y_{(2q+1)n-i})$ for $n \geq 1$.

Thus $\Omega^* = \{(2k_{2i}B_n + k_{2i-1}C_n, 4k_{2i-1}B_n + k_{2i}C_n) : n \geq 0\} \cup \{(2hB_n, hC_n), (2k_{2i}B_n - k_{2i-1}C_n, -4k_{2i-1}B_n + k_{2i}C_n) : n \geq 1\}$. $\square$

From Theorem 2.3, we deduce that

Theorem 2.4. *If $8k+1$ is a perfect square and $\#Rep = 2q+1$, then the general terms of k -almost balancing numbers and k -almost Lucas-balancing numbers of first type are*

$$\begin{aligned}B_{(2q+1)n}^{k*} &= hB_n \\C_{(2q+1)n}^{k*} &= hC_n \\B_{(2q+1)n+i}^{k*} &= \frac{2k_{2i}B_n + k_{2i-1}C_n}{2} \\C_{(2q+1)n+i}^{k*} &= 4k_{2i-1}B_n + k_{2i}C_n\end{aligned}$$

for $n \geq 0$ and

$$\begin{aligned}B_{(2q+1)n-i}^{k*} &= \frac{2k_{2i}B_n - k_{2i-1}C_n}{2} \\C_{(2q+1)n-i}^{k*} &= -4k_{2i-1}B_n + k_{2i}C_n\end{aligned}$$

for $n \geq 1$, where k_{2i-1}, k_{2i} are positive integers such that $2k_{2i-1}^2 - k_{2i}^2 = -8k - 1$ for $1 \leq i \leq q, 0 < k_1 < k_3 < \dots < k_{2q-1}, h < k_2 < k_4 < \dots < k_{2q}$ and $h = \sqrt{8k+1}$.

Proof. It can be derived from Theorem 2.3. $\square$

Remark 2.1. Note that balancing numbers and balancers should be positive integers from definition. But Behera and Panda ([2]) accepted 0 and 1 to be balancing numbers since $8(0)^2 + 1 = 1^2$ and $8(1)^2 + 1 = 3^2$ are perfect squares. In this case from (1.1), we get $r = -1$ and $r = 0$, respectively which are not positive integers. Nevertheless, Behera and Panda accepted 0 and 1 to be balancing numbers. Here we have same condition. Of course k -almost balancing numbers must be positive from definition. But when $n = 0$, for some values of k from (2.3), $8n^2 + 8k + 1 = 8k + 1$ is a perfect square. For instance, $8k + 1$ is a perfect square for $k = 1, 3, 6, 10, \dots$. Therefore for these values of k , we assume that 0 is a k -almost balancing number of first type, that is, $B_0^{k*} = 0$. So in Theorems 2.2 and 2.4, we accept 0 to be a k -almost balancing number of first type.

Theorem 2.5. If $8k + 1$ is not perfect square and $\#Rep = 2q$, then $\Omega^* = \{(x_{2qn+i}, y_{2qn+i}) : n \geq 0\} \cup \{(x_{2qn-i+1}, y_{2qn-i+1}) : n \geq 1\}$, where

$$\begin{aligned}(x_{2qn+i}, y_{2qn+i}) &= (2k_{2i}B_n + k_{2i-1}C_n, 4k_{2i-1}B_n + k_{2i}C_n) \\ (x_{2qn-i+1}, y_{2qn-i+1}) &= (2k_{2i}B_n - k_{2i-1}C_n, -4k_{2i-1}B_n + k_{2i}C_n)\end{aligned}$$

and k_{2i-1}, k_{2i} are positive integers such that $2k_{2i-1}^2 - k_{2i}^2 = -8k - 1$ for $1 \leq i \leq q, k_1 < k_3 < \dots < k_{2q-1}$ and $k_2 < k_4 < \dots < k_{2q}$.

Proof. If $8k + 1$ is not perfect square and $\#Rep = 2q$, then the set of representatives is

$$Rep = \{[\pm k_{2i-1} \quad k_{2i}]\},$$

where k_{2i-1}, k_{2i} are positive integers such that $2k_{2i-1}^2 - k_{2i}^2 = -8k - 1$ for $1 \leq i \leq q, k_1 < k_3 < \dots < k_{2q-1}$ and $k_2 < k_4 < \dots < k_{2q}$. In this case

1. $[k_{2i-1} \quad k_{2i}]M^n$ generates all integer solutions (x_{2qn+i}, y_{2qn+i}) for $n \geq 0$,
2. $[-k_{2i-1} \quad k_{2i}]M^n$ generates all integer solutions $(x_{2qn-i+1}, y_{2qn-i+1})$ for $n \geq 1$.

Thus $\Omega^* = \{(2k_{2i}B_n + k_{2i-1}C_n, 4k_{2i-1}B_n + k_{2i}C_n) : n \geq 0\} \cup \{(2k_{2i}B_n - k_{2i-1}C_n, -4k_{2i-1}B_n + k_{2i}C_n) : n \geq 1\}$. $\square$

From Theorem 2.5, we deduce that

Theorem 2.6. If $8k + 1$ is not perfect square and $\#Rep = 2q$, then the general terms of k -almost balancing numbers and k -almost Lucas-balancing numbers of first type are

$$\begin{aligned}B_{2qn+i}^{k*} &= \frac{2k_{2i}B_n + k_{2i-1}C_n}{2} \\ C_{2qn+i}^{k*} &= 4k_{2i-1}B_n + k_{2i}C_n\end{aligned}$$

for $n \geq 0$ and

$$\begin{aligned}B_{2qn-i+1}^{k*} &= \frac{2k_{2i}B_n - k_{2i-1}C_n}{2} \\ C_{2qn-i+1}^{k*} &= -4k_{2i-1}B_n + k_{2i}C_n\end{aligned}$$

for $n \geq 1$, where k_{2i-1}, k_{2i} are positive integers such that $2k_{2i-1}^2 - k_{2i}^2 = -8k - 1$ for $1 \leq i \leq q, k_1 < k_3 < \dots < k_{2q-1}$ and $k_2 < k_4 < \dots < k_{2q}$.

Proof. It can be easily derived from Theorem 2.5. $\square$

2.2 k -Almost Balancing and k -Almost Lucas-Balancing Numbers of Second Type.

In this subsection, we will determine the general terms of k -almost balancing numbers and k -almost Lucas-balancing numbers of second type. We see in (2.4) that n is a k -almost balancing number of second type if and only if $8n^2 - 8k + 1$ is a perfect square. So we set $8n^2 - 8k + 1 = y^2$ for some integer $y \geq 1$. Then $2(2n)^2 - y^2 = 8k - 1$ and taking $x = 2n$, we get the Pell equation

$$2x^2 - y^2 = 8k - 1. \quad (2.8)$$

Let Ω^{**} denote the set of all integer solutions of (2.8), that is, $\Omega^{**} = \{(x, y) : 2x^2 - y^2 = 8k - 1\}$. Then we have only one case: $\#Rep = 2q$ for some integer $q \geq 1$ since $8k - 1$ is not a perfect square, indeed,

Theorem 2.7. $8k - 1$ is not a perfect square for every $k \geq 1$.

Proof. Conversely, let us assume that $8k - 1$ is a perfect square, then $8k - 1 = t^2$ for some positive integer t . Then $k = \frac{t^2+1}{8}$. If t is even, say $t = 2u$ for some integer $u \neq 0$, then $k = \frac{4u^2+1}{8}$. If u is even, say $u = 2j$ for some integer $j \neq 0$, then $k = \frac{4(2j)^2+1}{8} = 2j^2 + \frac{1}{8} \notin \mathbb{Z}$ and if u is odd, say $u = 2j + 1$, then $k = \frac{4(2j+1)^2+1}{8} = 2j^2 + 2j + \frac{5}{8} \notin \mathbb{Z}$. If t is odd, say $t = 2u + 1$ for some integer $u \neq 0$, then $k = \frac{(2u+1)^2+1}{8} = \frac{2u^2+2u+1}{4}$. If u is even, say $u = 2j$, then $k = \frac{2(2j)^2+2(2j)+1}{4} = 2j^2 + j + \frac{1}{4} \notin \mathbb{Z}$ and if u is odd, say $u = 2j + 1$, then $k = \frac{2(2j+1)^2+2(2j+1)+1}{4} = 2j^2 + 3j + \frac{5}{4} \notin \mathbb{Z}$. Therefore $8k - 1$ is not a perfect square for every $k \geq 1$. $\square$

For the set of all integer solutions of (2.8), we can give the following theorem.

Theorem 2.8. $\Omega^{**} = \{(x_{2qn+i}, y_{2qn+i}) : n \geq 0\} \cup \{(x_{2qn-i+1}, y_{2qn-i+1}) : n \geq 1\}$ for every $k \geq 1$, where

$$\begin{aligned} (x_{2qn+i}, y_{2qn+i}) &= (2k_{2i}B_n + k_{2i-1}C_n, 4k_{2i-1}B_n + k_{2i}C_n) \\ (x_{2qn-i+1}, y_{2qn-i+1}) &= (-2k_{2i}B_n + k_{2i-1}C_n, 4k_{2i-1}B_n - k_{2i}C_n) \end{aligned}$$

and k_{2i-1}, k_{2i} are positive integers such that $2k_{2i-1}^2 - k_{2i}^2 = 8k - 1$ for $1 \leq i \leq q, k_1 < k_3 < \dots < k_{2q-1}$ and $k_2 < k_4 < \dots < k_{2q}$.

Proof. For the Pell equation $2x^2 - y^2 = 8k - 1$, the set of representatives is

$$Rep = \{[\pm k_{2i-1} \quad k_{2i}]\},$$

where k_{2i-1}, k_{2i} are positive integers such that $2k_{2i-1}^2 - k_{2i}^2 = 8k - 1$ for $1 \leq i \leq q, k_1 < k_3 < \dots < k_{2q-1}$ and $k_2 < k_4 < \dots < k_{2q}$. In this case

1. $[k_{2i-1} \quad k_{2i}]M^n$ generates all integer solutions (x_{2qn+i}, y_{2qn+i}) for $n \geq 0$,
2. $[k_{2i-1} \quad -k_{2i}]M^n$ generates all integer solutions $(x_{2qn-i+1}, y_{2qn-i+1})$ for $n \geq 1$.

Thus $\Omega^{**} = \{(2k_{2i}B_n + k_{2i-1}C_n, 4k_{2i-1}B_n + k_{2i}C_n) : n \geq 0\} \cup \{(-2k_{2i}B_n + k_{2i-1}C_n, 4k_{2i-1}B_n - k_{2i}C_n) : n \geq 1\}$. $\square$

Finally from Theorem 2.8,

Theorem 2.9. For every $k \geq 1$, the general terms of k -almost balancing numbers and k -almost Lucas-balancing numbers of second type are

$$B_{2qn+i}^{k**} = \frac{2k_{2i}B_n + k_{2i-1}C_n}{2}$$

$$C_{2qn+i}^{k**} = 4k_{2i-1}B_n + k_{2i}C_n$$

for $n \geq 0$ and

$$B_{2qn-i+1}^{k**} = \frac{-2k_{2i}B_n + k_{2i-1}C_n}{2}$$

$$C_{2qn-i+1}^{k**} = 4k_{2i-1}B_n - k_{2i}C_n$$

for $n \geq 1$, where k_{2i-1}, k_{2i} are positive integers such that $2k_{2i-1}^2 - k_{2i}^2 = 8k - 1$ for $1 \leq i \leq q$, $k_1 < k_3 < \dots < k_{2q-1}$ and $k_2 < k_4 < \dots < k_{2q}$.

Proof. It can be easily seen from Theorem 2.8. □

Remark 2.2. For some values of k , from (2.4), $8n^2 - 8k + 1$ is a perfect square for some positive integer n , but in this case r can be negative or zero. For example, for $k = 10$, $8n^2 - 79$ is a perfect square for $n = 4$ and $n = 5$. But in this case $r = -1$ and $r = 0$, respectively. But we accept 4 and 5 to be 10-almost balancing numbers of second type. As long as n is positive, we accept it to be a k -almost balancing numbers of second type not considering r is negative or zero.

References

- [1] Barbeau, E. J., 2003, *Pell's Equation*, Springer-Verlag, New York, Inc.
- [2] Behera, A. and Panda, G.K., 1999, *On the Square Roots of Triangular Numbers*, The Fibonacci Quart. 37(2), 98–105.
- [3] Flath, D. E., 1989, *Introduction to Number Theory*, Wiley.
- [4] Jacobson, M. Williams, H.C., 2009, *Solving the Pell Equation*, CMS Books in Mathematics, Springer Science, Business Media, LLC.
- [5] Kovacs, T., Liptai, K. and Olajos, P., 2010, *On (a, b) -Balancing Numbers*, Publ. Math. Deb. 77(3-4), 485–498.
- [6] Liptai, K., Luca, F., Pinter, A. and Szalay, L., 2009, *Generalized Balancing Numbers*, Indagationes Math. N.S. 20(1), 87–100.
- [7] Liptai, K., 2004, *Fibonacci Balancing Numbers*, The Fibonacci Quarterly 42(4), 330–340.
- [8] Liptai, K., 2006, *Lucas Balancing Numbers*, Acta Math. Univ. Ostrav. 14, 43–47.
- [9] Olajos, P., 2010, *Properties of Balancing, Cobalancing and Generalized Balancing Numbers*, Annales Mathematicae et Informaticae 37, 125–138.

- [10] Panda, G.K. and Ray, P.K., 2011, *Some Links of Balancing and Cobalancing Numbers with Pell and Associated Pell Numbers*, Bul. of Inst. of Math. Acad. Sinica 6(1), 41–72.
- [11] Panda, G.K. and Ray, P.K., 2005, *Cobalancing Numbers and Cobalancers*, Int. J. Math. Math. Sci. 8, 1189–1200.
- [12] Panda, G.K. and Panda, A.K., 2015, *Almost Balancing Numbers*, Jour. of the Indian Math. Soc. 82(3-4), 147–156.
- [13] Panda, A.K., 2017, *Some Variants of the Balancing Sequences*, Ph.D. dissertation, National Institute of Technology Rourkela, India.
- [14] Ray, P.K., 2009, *Balancing and Cobalancing Numbers*, Ph.D. dissertation, Department of Mathematics, National Institute of Technology Rourkela, India.
- [15] Ray, P.K., 2015, *Balancing and Lucas-balancing Sums by Matrix Methods*, Math. Reports 17(67), 225–233.
- [16] Szalay, L., 2007, *On the Resolution of Simultaneous Pell Equations*, Ann. Math. Inform. 34, 77–87.
- [17] Tekcan, A., 2020, *k –Almost Cobalancing Numbers*. International Journal of Applied Mathematics and Statistics 59(4), 88–94.
- [18] Tengely, S., 2013, *Balancing Numbers which are Products of Consecutive Integers*, Publ. Math. Deb. 83(1–2), 197–205.