

The asymmetric information in the financial market

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ABSTRACT

In the present work the authors study the asymmetric information in the financial markets. They examine a financial market model in which four agents trade. The model include n Brownian Motions. The first agent knows all information, the second agent knows minimum information, the third and the fourth agents know only partial information. The model analyzes the different informations of the four agents and compares their logarithmic utility functions. The filtration of the first agent is larger. It is initially specified. The filtration of the third and the fourth agents is smaller and the dynamics is non-Markovian. The model studies the logarithmic utility functions of the four agents. It results that, when $T \rightarrow +\infty$, the utility functions of the agents diverge.

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1 Introduction

Shiller (Shiller, 1980) and Summers (Summers, 1986) introduced the fads models and Guasoni (Guasoni, 2006) examines a continuous-time version of these models. He considers the informed agent who knows both fundamental and market values, instead partially informed agent knows only market values. In (Romano and Scolozzi, 2018) we extend the Guasoni model considering $n + 1$ Brownian Motion, and then compare the positions of the two agents. We assume $(\mathcal{F}_t^1)$ and $(\mathcal{F}_t^0)$ as the respective information flows of informed and partially informed agents. The first is the larger filtrations. We obtain it through the natural filtrations given by the $n + 1$ Brownian Motions $B_t^1, B_t^2, \dots, B_t^n, B_t^{n+1}$, the conditions of completeness and the condition of right continuity are satisfied. The filtration of the partially informed agents is generated by the process Y_t only, with $\mathcal{F}_t^0 \subset \mathcal{F}_t^1, \forall t$. Cao in (Cao and Shan, 2013) studies the asymmetric information in reference to the quality of human capital investments. In this paper, we study the issue of portfolio optimization in the presence of asymmetric information.

We examine a financial market model in which four agents trade. The model include n Brownian Motions. The first agent, he is called informed agent, knows all information. The second

is called uninformed agent and he knows minimum information, the third and the fourth agents know only partial information. They are called partially informed agents. Their information are not necessarily comparable. The model analyzes the different logarithmic utility functions of the agents, through Hitsuda representation (Hitsuda, 1968), and compares them. In this work, section 2 analyzes the model; section 3 studies the decomposition of the process Y_t under the filtration $\mathcal{F}^0$ for the uninformed agent and we examine the Markov property of the process Y_t . In section 4 the work considers the decomposition of the process A_t under the filtration $\mathcal{G}$ (for the first partially informed agent) and the decomposition of the process A_t^* under the filtration $\mathcal{G}^*$ (for the second partially informed agent). It is also studied the Markov property of A_t and of A_t^* . In section 5, the work examines the different level of market information of the four agents, and so it compares their utility functions. At last, in section 6, the work highlights the author's finding.

2 The model

Similarly to (Romano and Scolozzi, 2018) we suppose that in the financial market there is one risky asset S and one riskless asset. The model considers a probability space $(\Omega, \mathcal{F}, \mathcal{P})$. In refer to study the evolution of the risky asset S , we consider $n = h + k + l + m$ independent Brownian Motions defined on the probability space. Of course, it results $n \in \mathbb{N}$, $h \geq 1$, $k \geq 1$, $l \geq 1$, $m \geq 1$.

$$\begin{aligned} & (B_t^1)_{t \in [0, +\infty[}, (B_t^2)_{t \in [0, +\infty[}, \dots, (B_t^h)_{t \in [0, +\infty[}; \\ & (B_t^{h+1})_{t \in [0, +\infty[}, \dots, (B_t^{h+k})_{t \in [0, +\infty[}; \\ & (B_t^{h+k+1})_{t \in [0, +\infty[}, \dots, (B_t^{k+h+l})_{t \in [0, +\infty[}; \\ & (B_t^{h+k+l+1})_{t \in [0, +\infty[}, \dots, (B_t^{k+h+l+m})_{t \in [0, +\infty[}. \end{aligned}$$

Let the real parameter $\lambda > 0$, and let the Ornstein-Uhlenbeck process $(U_t)_{t \in [0, +\infty[}$ given by

$$U_t + \lambda \int_0^t U_s ds = \sum_{j=1}^m p_{h+k+l+j} B_t^{h+k+l+j}, \quad t \in [0, +\infty[$$

it results that:

$$U_t = \sum_{j=1}^m p_{h+k+l+j} \int_0^t e^{-\lambda(t-s)} dB_s^{h+k+l+j}.$$

Let the real numbers p_j , let $j = 1, 2, \dots, n$, and let the process $(Y_t)_{t \in [0, +\infty[}$ given by:

$$Y_t = \sum_{j=1}^h p_j B_t^j + \sum_{j=1}^k p_{h+j} B_t^{h+j} + \sum_{j=1}^l p_{h+k+j} B_t^{h+k+j} + U_t.$$

We consider the following functions:

$$\mu, \sigma : [0, +\infty[\longrightarrow [0, +\infty[$$

such that

$$\forall T > 0 \quad \mu \in L^1([0, T]), \quad \sigma \in L^2([0, T]).$$

The functions considered are deterministic Lebesgue measurable.

Now, considering the risky asset, we have that its price dynamics S evolves according to

$$dS_t = S_t [\mu_t dt + \sigma_t dY_t]$$

and its solution is:

$$S_t = S_0 \exp \left[\int_0^t \left(\mu_s - \frac{\sigma_s^2}{2} \right) ds + \int_0^t \sigma_s dY_s \right].$$

We suppose to have four agents: the first is the informed agent, the second is the uninformed agents, the third and the fourth are the partially informed agents. Their information is obtained from respectively from:

1. *all the n Brownian Motions;*
2. *only the process Y_t ;*
3. *only the $h + k$ Brownian Motions and from the following process*

$$A_t = \sum_{j=1}^l p_{h+k+j} B_t^{h+k+j} + U_t;$$

4. *all the $h + l$ Brownian Motions and from the following process*

$$A_t^* = \sum_{j=1}^k p_{h+j} B_t^{h+j} + U_t.$$

The first agent knows the filtration $\mathcal{F}^1 = (\mathcal{F}_t^1)_{t \in [0, +\infty[}$ given by the following n Brownian Motions $B_t^1, B_t^2, \dots, B_t^n$.

The second agent knows the filtration $\mathcal{F}^0 = (\mathcal{F}_t^0)_{t \in [0, +\infty[}$ given by the process Y_t only.

The third agent knows the filtration $\mathcal{G} = (\mathcal{G}_t)_{t \in [0, +\infty[}$ given by the following $h + k$ Brownian Motions $B_t^1, B_t^2, \dots, B_t^h, B_t^{h+1}, B_t^{h+2}, \dots, B_t^{h+k}$, and by the process A_t .

The fourth agent knows the filtration $\mathcal{G}^* = (\mathcal{G}_t^*)_{t \in [0, +\infty[}$ given by the following $h + l$ Brownian Motions

$B_t^1, B_t^2, \dots, B_t^h, B_t^{h+k+1}, B_t^{h+k+2}, \dots, B_t^{h+k+l}$, and by the process A_t^* .

Moreover $\mathcal{F}_t^0 \subset \mathcal{G}_t \subset \mathcal{F}_t^1$ and $\mathcal{F}_t^0 \subset \mathcal{G}_t^* \subset \mathcal{F}_t^1$, $\forall t > 0$.

In refer to the first agent, we have that the asset price dynamics is given by:

$$dS_t = S_t \left[(\mu_t - \lambda \sigma_t U_t) dt + \sigma_t \sum_{j=1}^n p_j dB_t^j \right]$$

and it refers to the following Brownian Motion

$$W_t = \left[\sum_{j=1}^n p_j^2 \right]^{-\frac{1}{2}} \sum_{j=1}^n p_j B_t^j.$$

3 Decomposition of Y_t under $\mathcal{F}^0$

Now, first we study the Markov property of Y_t and then we obtain the Brownian Motion B_t^0 , under $\mathcal{F}^0$.

Theorem 3.1. *Let Y_t be a Gaussian process. Let*

1. $E[Y_t] = 0 \quad \forall t \in [0, +\infty[.$
2. $cov(Y_s, Y_t) = \left(\sum_{j=1}^{h+k+l} p_j^2 \right) t \wedge s + \sum_{j=1}^m p_{h+k+l+j}^2 \frac{e^{-\lambda|t-s|} - e^{-\lambda(t+s)}}{2\lambda}.$

Considering $\lambda > 0$, $p_j = 0, \quad \forall j = 1, 2, \dots, h+k+l$ or $p_{h+k+l+j} = 0, \quad \forall j = 1, 2, \dots, m$, it results that Y_t is a Markov process.

It results that Y_t isn't a Markov process, in generality.

The process $(Z_t)_{t \in [0, +\infty[}$ given by

$$Z_t = \left(\sum_{j=1}^n p_j^2 \right)^{-\frac{1}{2}} \left[\sum_{j=1}^{h+k+l} p_j \left(B_t^j + \lambda \int_0^t B_u^j du \right) + \sum_{j=1}^m p_{h+k+l+j} B_t^{h+k+l+j} \right].$$

verifies the theorem 3.2:

Theorem 3.2. *Let Z_t be a Gaussian process. Let*

1. $E[Z_t] = 0 \quad \forall t \in [0, +\infty[.$
2. $cov(Z_t, Z_s) = t \wedge s + \left(\sum_{j=1}^n p_j^2 \right)^{-1} \left(\sum_{j=1}^{h+k+l} p_j^2 \right) \int_0^t \int_0^s (\lambda + \lambda^2 u \wedge v) dudv.$

Let

$$\tilde{f}(t, s) = -\alpha^2 (\lambda + \lambda^2 t \wedge s) \quad \forall t, s \geq 0$$

where

$$\alpha^2 = \left(\sum_{j=1}^n p_j^2 \right)^{-1} \sum_{j=1}^{h+k+l} p_j^2.$$

Theorem 3.3. Given the function $\tilde{f}(t, s)$, let the function

$$\tilde{g}(t, s) = \begin{cases} \lambda \eta(s) & \text{for } 0 \leq s \leq t \\ 0 & \text{otherwise} \end{cases}$$

satisfies the integral equation given by

$$\tilde{f}(t, s) = \tilde{g}(t, s) - \int_0^s \tilde{g}(t, u) \tilde{g}(s, u) du \quad \forall 0 \leq s \leq t \quad (1)$$

It results that $\eta(s)$ satisfies the Cauchy problem given by

$$\begin{cases} \eta'(s) = \lambda [\eta^2(s) - \alpha^2] \\ \eta(0) = -\alpha^2. \end{cases}$$

Now it is possible to consider the theorem 3.4:

Theorem 3.4. 1. Considering $(\mathcal{F}_t^0)_{t \in [0, +\infty[}$, let $(B_t^0)_{t \in [0, +\infty[}$ be a Brownian Motion and:

$$Z_t = B_t^0 - \int_0^t \left(\int_0^s \tilde{g}(s, u) dB_u^0 \right) ds.$$

2. Given the function $g : [0, T] \times [0, T] \rightarrow \mathbb{R}$ defined by:

$$g(t, s) = \begin{cases} -\lambda \eta(s) e^{\lambda \int_s^t \eta(u) du} & \text{for } 0 \leq s \leq t \\ 0 & \text{otherwise} \end{cases}$$

it results

$$B_t^0 = Z_t - \int_0^t \left(\int_0^s g(s, u) dZ_u \right) ds. \quad (2)$$

Proof. 1. Hitsuda (Hitsuda, 1968) proposition 2, and the function $\tilde{g}(t, s)$ which verifies the integral equation (1), imply the existence of the Brownian Motion B_t^0 .

2. We obtain (2) in a similar way already seen in (Romano and Scolozzi, 2018), theorem 6.

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□

Theorem 3.5. Let Y_t and Z_t verify the following relation:

$$Y_t + \lambda \int_0^t Y_u du = \left(\sum_{j=1}^n p_j^2 \right)^{\frac{1}{2}} Z_t$$

and therefore

$$Y_t = \left(\sum_{j=1}^n p_j^2 \right)^{\frac{1}{2}} \int_0^t e^{-\lambda(t-u)} dZ_u.$$

Proof.

$$\begin{aligned} Y_t + \lambda \int_0^t Y_u du &= \sum_{j=1}^{h+k+l} p_j B_t^j + U_t + \lambda \left[\int_0^t \left(\sum_{j=1}^{h+k+l} p_j B_u^j + U_u \right) du \right] = \\ &= \sum_{j=1}^{h+k+l} p_j B_t^j + U_t + \lambda \sum_{j=1}^{h+k+l} p_j \int_0^t B_u^j du + \lambda \int_0^t U_u du = \\ &= \sum_{j=1}^{h+k+l} p_j \left(B_t^j + \lambda \int_0^t B_u^j du \right) + U_t + \lambda \int_0^t U_u du = \\ &= \sum_{j=1}^{h+k+l} p_j \left(B_t^j + \lambda \int_0^t B_u^j du \right) + \sum_{j=1}^m p_{h+k+l+j} B_t^{h+k+l+j} = \left(\sum_{j=1}^n p_j^2 \right)^{\frac{1}{2}} Z_t. \diamond \quad \square \end{aligned}$$

It allows that we are able to consider the link between the Brownian Motion B_t^0 and the process Y_t .

Theorem 3.6. Let $Y_t = \sum_{j=1}^{h+k+l} p_j B_t^j + U_t$ and let $(\mathcal{F}_t^0)_{t \in [0, +\infty[}$ be its completed natural filtration.

It results:

1.

$$B_t^0 = \left(\sum_{j=1}^n p_j^2 \right)^{-\frac{1}{2}} \left[Y_t + \lambda \int_0^t \left(\int_0^s [1 + \eta(u)] e^{\lambda \int_u^s \eta(q) dq} dY_u \right) ds \right].$$

2.

$$Y_t = \left(\sum_{j=1}^n p_j^2 \right)^{\frac{1}{2}} \left[B_t^0 - \lambda \int_0^t \left(\int_0^s [1 + \eta(u)] e^{-\lambda(s-u)} dB_u^0 \right) ds \right].$$

In refer to the uninformed agent, using the previous relation 2, the asset price evolves according to:

$$dS_t = S_t \left[\left(\mu_t - \lambda \left(\sum_{j=1}^n p_j^2 \right)^{\frac{1}{2}} \sigma_t \int_0^t [1 + \eta(u)] e^{-\lambda(t-u)} dB_u^0 \right) dt + \sigma_t \left(\sum_{j=1}^n p_j^2 \right)^{\frac{1}{2}} dB_t^0 \right].$$

4 Decomposition of A_t under $\mathcal{G}$ and of A_t^* under $\mathcal{G}^*$

Now, first we study the Markov property of

$$A_t = \sum_{j=1}^l p_{h+k+j} B_t^{h+k+j} + U_t$$

and of

$$A_t^* = \sum_{j=1}^k p_{h+j} B_t^{h+j} + U_t$$

and then we obtain the Brownian Motions C_t^0 e C_t^{0*} , under $\mathcal{G}$ and $\mathcal{G}^*$, which represent them.

Theorem 4.1. *Let A_t e A_t^* be Gaussian processes and moreover:*

1. $E[A_t] = 0, \quad E[A_t^*] = 0 \quad \forall t \in [0, +\infty[.$
2. $\text{cov}(A_s, A_t) = \left(\sum_{j=1}^l p_{h+k+j}^2 \right) t \wedge s + \sum_{j=1}^m p_{h+k+l+j}^2 \frac{e^{-\lambda|t-s|} - e^{-\lambda(t+s)}}{2\lambda};$
 $\text{cov}(A_s^*, A_t^*) = \left(\sum_{j=1}^k p_{h+j}^2 \right) t \wedge s + \sum_{j=1}^m p_{h+k+l+j}^2 \frac{e^{-\lambda|t-s|} - e^{-\lambda(t+s)}}{2\lambda}.$

A_t and A_t^* aren't Markov processes, in generality.

The following processes $(D_t)_{t \in [0, +\infty[}$ and $(D_t^*)_{t \in [0, +\infty[}$ given by:

$$D_t = \left(\sum_{j=1}^{l+m} p_{h+k+j}^2 \right)^{-\frac{1}{2}} \left[\sum_{j=1}^l p_{h+k+j} \left(B_t^{h+k+j} + \lambda \int_0^t B_u^{h+k+j} du \right) + \sum_{j=1}^m p_{h+k+l+j} B_t^{h+k+l+j} \right];$$

$$D_t^* = \left(\sum_{j=1}^k p_{h+j}^2 + \sum_{j=1}^m p_{h+k+l+j}^2 \right)^{-\frac{1}{2}} \left[\sum_{j=1}^k p_{h+j} \left(B_t^{h+j} + \lambda \int_0^t B_u^{h+j} du \right) + \sum_{j=1}^m p_{h+k+l+j} B_t^{h+k+l+j} \right]$$

They satisfied the theorem 4.2:

Theorem 4.2. *Let D_t and D_t^* be Gaussian processes and moreover*

1. $E[D_t] = 0 \quad E[D_t^*] = 0 \quad \forall t \in [0, +\infty[.$

$$2. \operatorname{cov}(D_t, D_s) = t \wedge s +$$

$$+ \left(\sum_{j=1}^{l+m} p_{h+k+j}^2 \right)^{-1} \left(\sum_{j=1}^l p_{h+k+j}^2 \right) \int_0^t \int_0^s (\lambda + \lambda^2 u \wedge v) \, dudv.$$

$$\operatorname{cov}(D_t^*, D_s^*) = t \wedge s +$$

$$+ \left(\sum_{j=1}^k p_{h+j}^2 + \sum_{j=1}^m p_{h+k+l+j}^2 \right)^{-1} \left(\sum_{j=1}^k p_{h+j}^2 \right) \int_0^t \int_0^s (\lambda + \lambda^2 u \wedge v) \, dudv.$$

Let

$$\tilde{\varphi}(t, s) = -\beta^2 (\lambda + \lambda^2 t \wedge s) \quad \forall t, s \geq 0$$

$$\tilde{\varphi}^*(t, s) = -(\beta^*)^2 (\lambda + \lambda^2 t \wedge s) \quad \forall t, s \geq 0$$

where

$$\beta^2 = \left(\sum_{j=1}^{l+m} p_{h+k+j}^2 \right)^{-1} \sum_{j=1}^l p_{h+k+j}^2$$

and

$$(\beta^*)^2 = \left(\sum_{j=1}^k p_{h+j}^2 + \sum_{j=1}^m p_{h+k+l+j}^2 \right)^{-1} \sum_{j=1}^k p_{h+j}^2.$$

Now, similarly to Hitsuda (Hitsuda, 1968) and Guasoni (Guasoni, 2006), we enunciate the theorem 4.3:

Theorem 4.3. 1. Given the function $\tilde{\varphi}(t, s)$, let the function

$$\tilde{\psi}(t, s) = \begin{cases} \lambda \theta(s) & \text{for } 0 \leq s \leq t \\ 0 & \text{otherwise} \end{cases}$$

satisfies the integral equation given by

$$\tilde{\varphi}(t, s) = \tilde{\psi}(t, s) - \int_0^s \tilde{\psi}(t, u) \tilde{\psi}(s, u) \, du \quad \forall 0 \leq s \leq t$$

It results that $\theta(s)$ satisfies the Cauchy problem given by:

$$\begin{cases} \theta'(s) = \lambda [\theta^2(s) - \beta^2] \\ \theta(0) = -\beta^2. \end{cases}$$

2. Given the function $\tilde{\varphi}^*(t, s)$, let the function

$$\tilde{\psi}^*(t, s) = \begin{cases} \lambda \theta^*(s) & \text{for } 0 \leq s \leq t \\ 0 & \text{otherwise} \end{cases}$$

satisfies the integral equation given by

$$\tilde{\varphi}^*(t, s) = \tilde{\psi}^*(t, s) - \int_0^s \tilde{\psi}^*(t, u) \tilde{\psi}^*(s, u) du \quad \forall 0 \leq s \leq t$$

It results that $\theta^*(s)$ satisfies the Cauchy problem given by:

$$\begin{cases} \theta^{*'}(s) = \lambda [\theta^{*2}(s) - (\beta^*)^2] \\ \theta^*(0) = -(\beta^*)^2. \end{cases}$$

Theorem 4.4. 1. Considering $(\mathcal{G}_t)_{t \in [0, +\infty[}$, let $(C_t^0)_{t \in [0, +\infty[}$ be a Brownian Motion:

$$D_t = C_t^0 - \int_0^t \left(\int_0^s \tilde{\psi}(s, u) dC_u^0 \right) ds.$$

Given the function $\psi : [0, T] \times [0, T] \rightarrow \mathbb{R}$ defined by:

$$\psi(t, s) = \begin{cases} -\lambda \theta(s) e^{\lambda \int_s^t \theta(u) du} & \text{for } 0 \leq s \leq t \\ 0 & \text{otherwise} \end{cases}$$

it results:

$$C_t^0 = D_t - \int_0^t \left(\int_0^s \psi(s, u) dD_u \right) ds.$$

2. Considering

$(\mathcal{G}_t^*)_{t \in [0, +\infty[}$, let $(C_t^{0*})_{t \in [0, +\infty[}$ be a Brownian Motion:

$$D_t^* = C_t^{0*} - \int_0^t \left(\int_0^s \tilde{\psi}^*(s, u) dC_u^{0*} \right) ds.$$

Given the function $\psi^* : [0, T] \times [0, T] \rightarrow \mathbb{R}$ defined by:

$$\psi^*(t, s) = \begin{cases} -\lambda \theta^*(s) e^{\lambda \int_s^t \theta^*(u) du} & \text{for } 0 \leq s \leq t \\ 0 & \text{otherwise} \end{cases}$$

it results

$$C_t^{0*} = D_t^* - \int_0^t \left(\int_0^s \psi^*(s, u) dD_u^* \right) ds.$$

Theorem 4.5. 1. Let A_t and D_t verify the equation:

$$A_t + \lambda \int_0^t A_u du = \left(\sum_{j=1}^{l+m} p_{h+k+j}^2 \right)^{\frac{1}{2}} D_t$$

and therefore

$$A_t = \left(\sum_{j=1}^{l+m} p_{h+k+j}^2 \right)^{\frac{1}{2}} \int_0^t e^{-\lambda(t-u)} dD_u.$$

2. Let A_t^* and D_t^* verify the equation:

$$A_t^* + \lambda \int_0^t A_u^* du = \left(\sum_{j=1}^k p_{h+j}^2 + \sum_{j=1}^m p_{h+k+l+j}^2 \right)^{\frac{1}{2}} D_t^*$$

and therefore

$$A_t^* = \left(\sum_{j=1}^k p_{h+j}^2 + \sum_{j=1}^m p_{h+k+l+j}^2 \right)^{\frac{1}{2}} \int_0^t e^{-\lambda(t-u)} dD_u^*.$$

It allows that we are able to consider the link between the Brownian Motion C_t^0 and the process A_t and between the Brownian Motion C_t^{0*} and the process A_t^* .

Theorem 4.6. Let A_t and A_t^* be Gaussian processes and let $(\mathcal{G}_t)_{t \in [0, +\infty[}$ and $(\mathcal{G}_t^*)_{t \in [0, +\infty[}$ be their completed natural filtrations.

It results:

1.

$$C_t^0 = \left(\sum_{j=1}^{l+m} p_{h+k+j}^2 \right)^{-\frac{1}{2}} \left[A_t + \lambda \int_0^t \left(\int_0^s [1 + \theta(u)] e^{\lambda \int_u^s \theta(q) dq} dA_u \right) ds \right].$$

2.

$$A_t = \left(\sum_{j=1}^{l+m} p_{h+k+j}^2 \right)^{\frac{1}{2}} \left[C_t^0 - \lambda \int_0^t \left(\int_0^s [1 + \theta(u)] e^{-\lambda(s-u)} dC_u^0 \right) ds \right].$$

3.

$$C_t^{0*} = \left(\sum_{j=1}^k p_{h+j}^2 + \sum_{j=1}^m p_{h+k+l+j}^2 \right)^{-\frac{1}{2}} \left[A_t^* + \lambda \int_0^t \left(\int_0^s [1 + \theta^*(u)] e^{\lambda \int_u^s \theta^*(q) dq} dA_u^* \right) ds \right].$$

4.

$$A_t^* = \left(\sum_{j=1}^k p_{h+j}^2 + \sum_{j=1}^m p_{h+k+l+j}^2 \right)^{\frac{1}{2}} \left[C_t^{0*} - \lambda \int_0^t \left(\int_0^s [1 + \theta^*(u)] e^{-\lambda(s-u)} dC_u^{0*} \right) ds \right].$$

In refer to the first partially informed agent, through relation 2, the asset price evolves as follows:

$$dS_t = S_t \left[\left(\mu_t - \lambda \left(\sum_{j=1}^{l+m} p_{h+k+j}^2 \right)^{\frac{1}{2}} \sigma_t \int_0^t [1 + \theta(u)] e^{-\lambda(t-u)} dC_u^0 \right) dt + \right. \\ \left. + \sigma_t \left(\sum_{j=1}^{h+k} p_j dB_t^j + \left(\sum_{j=1}^{l+m} p_{h+k+j}^2 \right)^{\frac{1}{2}} dC_t^0 \right) \right]$$

which refers to Brownian Motion

$$W_t^1 = \left(\sum_{j=1}^n p_j^2 \right)^{-\frac{1}{2}} \left[\sum_{j=1}^{h+k} p_j B_t^j + \left(\sum_{j=1}^{l+m} p_{h+k+j}^2 \right)^{\frac{1}{2}} C_t^0 \right].$$

In refer to the second partially informed agent, through relation 4, the asset price evolves as follows:

$$dS_t = S_t \left[\left(\mu_t - \lambda \left(\sum_{j=1}^k p_{h+j}^2 + \sum_{j=1}^m p_{h+k+l+j}^2 \right)^{\frac{1}{2}} \sigma_t \int_0^t [1 + \theta^*(u)] e^{-\lambda(t-u)} dC_u^{0*} \right) dt + \right. \\ \left. + \sigma_t \left(\sum_{j=1}^h p_j B_t^j + \sum_{j=1}^l p_{h+k+j} B_t^{h+k+j} + \left(\sum_{j=1}^k p_{h+j}^2 + \sum_{j=1}^m p_{h+k+l+j}^2 \right)^{\frac{1}{2}} dC_t^{0*} \right) \right]$$

which refers to the Brownian Motion

$$W_t^{1*} = \left(\sum_{j=1}^n p_j^2 \right)^{-\frac{1}{2}} \left[\sum_{j=1}^h p_j B_t^j + \sum_{j=1}^l p_{h+k+j} B_t^{h+k+j} + \right. \\ \left. + \left(\sum_{j=1}^k p_{h+j}^2 + \sum_{j=1}^m p_{h+k+l+j}^2 \right)^{\frac{1}{2}} C_t^{0*} \right].$$

5 The utility functions of the four agents

The agents have an initial wealth $x > 0$. Every agent invests H_s units of $S_s, \forall s \in [0, t]$ with $t > 0$, and he obtains the wealth given by X_t .

Every agents have a different level of market information: in refer to the informed agent, his wealth evolves according to Brownian Motion W_t ; in refer to the uninformed agent, his wealth evolves according to Brownian Motion B_t^0 . Considering the two partially informed agent, in refer to the first his wealth evolves according to Brownian Motions W_t^1 ; in refer to the second his wealth evolves according to Brownian Motions W_t^{1*} .

Now, we study and we compare the different utility functions.

At time t , the value of the wealth X_t , is given by:

$$X_t = x + \int_0^t H_s dS_s.$$

For every agent, the process of the units H_t must be predictable with respect to his filtrations, integrable with respect to process S_t , and such that almost certainly it results $X_t > 0, \forall t \in [0, T]$.

Let $\mathcal{U}$ be the utility function. At time T every agent maximizes his expected utility from terminal wealth and it is possible to solve the problem given by:

$$\sup \left\{ E \left[\mathcal{U} \left(x + \int_0^T H_t dS_t \right) \right] : H_t \text{ admissible} \right\}.$$

At every time t , the following process π_t given by

$$H_t = \pi_t \frac{X_t}{S_t}$$

guarantees the positivity of the wealth produced.

Therefore we obtain

$$X_t = x + \int_0^t \pi_s \frac{X_s}{S_s} dS_s$$

from it we deduce

$$\frac{dX_t}{X_t} = \pi_t \frac{dS_t}{S_t}.$$

Now, in refer to the informed agent, it results:

$$\frac{dX_t}{X_t} = \pi_t (\mu_t - \lambda \sigma_t U_t) dt + \pi_t \sigma_t \sum_{j=1}^n p_j dB_t^j.$$

In refer to the uninformed agent, it results:

$$\frac{dX_t}{X_t} = \kappa_t \left[\mu_t - \left(\sum_{j=1}^n p_j^2 \right)^{\frac{1}{2}} \lambda \sigma_t \int_0^t [1 + \eta(u)] e^{-\lambda(t-u)} dB_u^0 \right] dt + \kappa_t \sigma_t \left(\sum_{j=1}^n p_j^2 \right)^{\frac{1}{2}} dB_t^0.$$

In refer to the first partially informed agents, it results:

$$\begin{aligned} \frac{dX_t}{X_t} = & \kappa_t \left[\mu_t - \lambda \left(\sum_{j=1}^{l+m} p_{h+k+j}^2 \right)^{\frac{1}{2}} \sigma_t \int_0^t [1 + \theta(u)] e^{-\lambda(t-u)} dC_u^0 \right] dt + \\ & + \kappa_t \sigma_t \left[\sum_{j=1}^{h+k} p_j dB_t^j + \left(\sum_{j=1}^{l+m} p_{h+k+j}^2 \right) dC_t^0 \right]; \end{aligned}$$

In refer to the second partially informed agents, it results:

$$\begin{aligned} \frac{dX_t}{X_t} = & \kappa_t \left[\mu_t - \lambda \left(\sum_{j=1}^k p_{h+j}^2 + \sum_{j=1}^m p_{h+k+l+j}^2 \right)^{\frac{1}{2}} \sigma_t \int_0^t [1 + \theta^*(u)] e^{-\lambda(t-u)} dC_u^{0*} \right] dt + \\ & + \kappa_t \sigma_t \left[\sum_{j=1}^h p_j dB_t^j + \sum_{j=1}^l p_{h+k+j} dB_t^{h+k+j} + \left(\sum_{j=1}^k p_{h+j}^2 + \sum_{j=1}^m p_{h+k+l+j}^2 \right) dC_t^{0*} \right]. \end{aligned}$$

5.1 The informed agent

At time T , in refer to the informed agent, his wealth is given by:

$$X_T = xe^{\int_0^T \left[\pi_s (\mu_s - \lambda \sigma_s U_s) - \frac{1}{2} \left(\sum_{j=1}^n p_j^2 \right) \pi_s^2 \sigma_s^2 \right] ds + \left[\sum_{j=1}^n p_j^2 \right]^{\frac{1}{2}} \int_0^T \pi_s \sigma_s dW_s}.$$

Theorem 5.1. Let $\mathcal{U}(y) = \log y$. The optimal investment share is given by the process

$$\pi_s = \frac{\mu_s - \lambda \sigma_s U_s}{\sum_{j=1}^n p_j^2 \sigma_s^2} \quad (3)$$

The utility function is the following:

$$u(x, T) = \log x + \frac{1}{2 \sum_{j=1}^n p_j^2} E \left[\int_0^T \left(\frac{\mu_s}{\sigma_s} - \lambda U_s \right)^2 ds \right]. \quad (4)$$

Proof. $\forall \pi_s$ admissible, we have that

$$\begin{aligned} \mathcal{U}(X_T) = & \log(X_T) = \log x + \\ & + \int_0^T \left[\pi_s (\mu_s - \lambda \sigma_s U_s) - \frac{1}{2} \left(\sum_{j=1}^n p_j^2 \right) \pi_s^2 \sigma_s^2 \right] ds + \left[\sum_{j=1}^n p_j^2 \right]^{\frac{1}{2}} \int_0^T \pi_s \sigma_s dW_s \end{aligned}$$

it allows that:

$$E[\log(X_T)] = \log x + E\left[\int_0^T \left[\pi_s(\mu_s - \lambda\sigma_s U_s) - \frac{1}{2} \left(\sum_{j=1}^n p_j^2\right) \pi_s^2 \sigma_s^2\right] ds\right]$$

from which it results the maximum value.

The optimal portfolio share is given by (3) and the maximum value is given by (4). $\diamond$ $\square$

Theorem 5.2. *Let*

$$u(x, T) = \log x + \frac{1}{2 \sum_{j=1}^n p_j^2} \int_0^T \frac{\mu_s^2}{\sigma_s^2} ds + \frac{\sum_{j=1}^m p_{h+k+l+j}^2}{\sum_{j=1}^n p_j^2} \left[\frac{T\lambda}{4} - \frac{1 - e^{-2T\lambda}}{8} \right].$$

5.2 The uninformed agent

At time T , in refer to the uninformed agent, his wealth is given by:

$$X_T = xe^{\int_0^T \left[\kappa_s \left(\mu_s - \left(\sum_{j=1}^n p_j^2 \right)^{\frac{1}{2}} \lambda \sigma_s \int_0^s [1 + \eta(u)] e^{-\lambda(s-u)} dB_u^0 \right) - \frac{1}{2} \left(\sum_{j=1}^n p_j^2 \right) \kappa_s^2 \sigma_s^2 \right] ds}$$

$$e^{\left(\sum_{j=1}^n p_j^2 \right)^{\frac{1}{2}} \int_0^T \kappa_s \sigma_s dB_s^0}$$

Theorem 5.3. *Let $\mathcal{V}(y) = \log y$. The optimal investment share is given by the process*

$$\kappa_s = \frac{\mu_s - \left(\sum_{j=1}^n p_j^2 \right)^{\frac{1}{2}} \lambda \sigma_s \int_0^s [1 + \eta(u)] e^{-\lambda(s-u)} dB_u^0}{\left(\sum_{j=1}^n p_j^2 \right) \sigma_s^2} \quad (5)$$

The utility function is the following:

$$v(x, T) = \log x + \frac{1}{2 \left(\sum_{j=1}^n p_j^2 \right)} E \left[\int_0^T \left(\frac{\mu_s}{\sigma_s} - \left(\sum_{j=1}^n p_j^2 \right)^{\frac{1}{2}} \lambda \int_0^s [1 + \eta(u)] e^{-\lambda(s-u)} dB_u^0 \right)^2 ds \right]. \quad (6)$$

Proof. $\forall \kappa_s$ admissible, it results that

$$\begin{aligned} \mathcal{V}(X_T) &= \log(X_T) = \log x + \\ &+ \int_0^T \left[\kappa_s \left(\mu_s - \left(\sum_{j=1}^n p_j^2 \right)^{\frac{1}{2}} \lambda \sigma_s \int_0^s [1 + \eta(u)] e^{-\lambda(s-u)} dB_u^0 \right) - \frac{1}{2} \left(\sum_{j=1}^n p_j^2 \right) \kappa_s^2 \sigma_s^2 \right] ds + \\ &+ \left[\sum_{j=1}^n p_j^2 \right]^{\frac{1}{2}} \int_0^T \kappa_s \sigma_s dB_0^s \end{aligned}$$

it follows that

$$\begin{aligned} E[\log(X_T)] &= \log x + \\ &+ E \left[\int_0^T \left[\kappa_s \left(\mu_s - \left(\sum_{j=1}^n p_j^2 \right)^{\frac{1}{2}} \lambda \sigma_s \int_0^s [1 + \eta(u)] e^{-\lambda(s-u)} dB_u^0 \right) - \frac{1}{2} \left(\sum_{j=1}^n p_j^2 \right) \kappa_s^2 \sigma_s^2 \right] ds \right]. \end{aligned}$$

from which it follows the maximum value.

The optimal portfolio share is given by (5) and the maximum value is given by (6). $\diamond$ $\square$

Theorem 5.4. *Let*

$$v(x, T) = \log x + \frac{1}{2 \sum_{j=1}^n p_j^2} \int_0^T \frac{\mu_s^2}{\sigma_s^2} ds + \frac{\lambda^2}{2} \int_0^T \left[\int_0^s [1 + \eta(u)]^2 e^{-2\lambda(s-u)} du \right] ds.$$

Trought the theorem 5.5 we compare the utility functions of the informed and uninformed agent:

Theorem 5.5. *Let*

$$\lim_{T \rightarrow +\infty} \frac{1}{T} \int_0^T \left[\int_0^s [1 + \eta(u)]^2 e^{-2\lambda(s-u)} du \right] ds = \frac{\lambda}{4} (1 - \alpha)^2$$

where

$$\alpha = \left(\sum_{j=1}^n p_j^2 \right)^{-\frac{1}{2}} \left(\sum_{j=1}^{h+k+l} p_j^2 \right)^{\frac{1}{2}}.$$

Proof. Considering:

$$\lim_{s \rightarrow +\infty} \int_0^s [1 + \eta(u)]^2 e^{-2\lambda(s-u)} du = \frac{(1 - \alpha)^2}{2\lambda}.$$

and changing variable it results that:

$$\int_0^s [1 + \eta(s - u)]^2 e^{-2\lambda u} du.$$

The indicator function of the interval $[0, s]$ is denoted with $I_{[0,s]}$; it results that:

$$\lim_{s \rightarrow +\infty} \int_0^{+\infty} [1 + \eta(s-u)]^2 e^{-2\lambda u} I_{[0,s]}(u) du.$$

As $\lim_{s \rightarrow +\infty} \eta(s-u) = -\alpha$, it results that the integrand function tends punctually to the function $u \in [0, +\infty[\mapsto (1-\alpha)^2 e^{-2\lambda u}$.

Now we observe that the integrand function is given by:

$$0 \leq [1 + \eta(s-u)]^2 e^{-2\lambda u} I_{[0,s]}(u) \leq (1+\alpha) e^{-2\lambda u} \quad \forall u \in [0, +\infty[$$

and the decrease and increase functions are integrable on $[0, +\infty[$.

Considering the theorem of the Lebesgue dominated convergence, it resultst:

$$\lim_{s \rightarrow +\infty} \int_0^{+\infty} [1 + \eta(s-u)]^2 e^{-2\lambda u} I_{[0,s]}(u) du = \int_0^{+\infty} (1-\alpha)^2 e^{-2\lambda u} du.$$

Integrating it results:

$$\lim_{s \rightarrow +\infty} \int_0^s [1 + \eta(u)]^2 e^{-2\lambda(s-u)} du = \frac{(1-\alpha)^2}{2\lambda}$$

and it allows that

$$\lim_{T \rightarrow +\infty} \int_0^T \left[\int_0^s [1 + \eta(u)]^2 e^{-2\lambda(s-u)} du \right] ds = +\infty.$$

Now, with the De L'Hopital's rule, follows the thesis. $\diamond$

□

5.3 The two partially informed agents

At time T , in refer to the first partially informed agents, his wealth is given by:

$$X_T = xe \left[\kappa_s \left(\mu_s - \lambda \left(\sum_{j=1}^{l+m} p_{h+k+j}^2 \right)^{\frac{1}{2}} \sigma_s \int_0^s [1 + \theta(u)] e^{-\lambda(s-u)} dC_u^0 \right) \right] ds$$

$$- \frac{1}{2} \sum_{j=1}^n p_j^2 \int_0^T \kappa_s^2 \sigma_s^2 ds \int_0^T \kappa_s \sigma_s \left[\sum_{j=1}^{h+k} p_j dB_s^j + \left(\sum_{j=1}^{l+m} p_{h+k+j}^2 \right) dC_s^0 \right];$$

At time T , in refer to the second partially informed agents, his wealth is given by:

$$X_T = xe \left[\kappa_s \left(\mu_s - \lambda \left(\sum_{j=1}^k p_{h+j}^2 + \sum_{j=1}^m p_{h+k+l+j}^2 \right)^{\frac{1}{2}} \sigma_s \int_0^s [1 + \theta^*(u)] e^{-\lambda(s-u)} dC_u^{0*} \right) \right] ds$$

$$e^{-\frac{1}{2} \sum_{j=1}^n p_j^2 \int_0^T \kappa_s^2 \sigma_s^2 ds} \int_0^T \kappa_s \sigma_s \left[\sum_{j=1}^h p_j dB_t^j + \sum_{j=1}^l p_{h+k+j} dB_t^{h+k+j} + \left(\sum_{j=1}^k p_{h+j}^2 + \sum_{j=1}^m p_{h+k+l+j}^2 \right) dC_s^{0*} \right].$$

Theorem 5.6. 1. Let $\mathcal{W}(y) = \log y$.

The optimal investment share is given by the process

$$\kappa_s = \frac{\mu_s - \lambda \left(\sum_{j=1}^{l+m} p_{h+k+j}^2 \right)^{\frac{1}{2}} \sigma_s \int_0^s [1 + \theta(u)] e^{-\lambda(s-u)} dC_u^0}{\left(\sum_{j=1}^n p_j^2 \right) \sigma_s^2}.$$

The utility function is the following:

$$w(x, T) = \log x + \frac{1}{2 \left(\sum_{j=1}^n p_j^2 \right)} E \left[\int_0^T \left(\frac{\mu_s}{\sigma_s} - \lambda \left(\sum_{j=1}^{l+m} p_{h+k+j}^2 \right)^{\frac{1}{2}} \int_0^s [1 + \theta(u)] e^{-\lambda(t-u)} dC_u^0 \right)^2 ds \right].$$

2. Let $\mathcal{W}^*(y) = \log y$. The optimal investment share is given by the process

$$\kappa_s = \frac{\mu_s - \lambda \left(\sum_{j=1}^k p_{h+j}^2 + \sum_{j=1}^m p_{h+k+l+j}^2 \right)^{\frac{1}{2}} \sigma_s \int_0^s [1 + \theta^*(u)] e^{-\lambda(s-u)} dC_u^{0*}}{\left(\sum_{j=1}^n p_j^2 \right) \sigma_s^2}.$$

The utility function is the following:

$$w^*(x, T) = \log x + \frac{1}{2 \left(\sum_{j=1}^n p_j^2 \right)} E \left[\int_0^T \left(\frac{\mu_s}{\sigma_s} - \lambda \left(\sum_{j=1}^k p_{h+j}^2 + \sum_{j=1}^m p_{h+k+l+j}^2 \right)^{\frac{1}{2}} \int_0^s [1 + \theta^*(u)] e^{-\lambda(t-u)} dC_u^{0*} \right)^2 ds \right]$$

Theorem 5.7. 1. Let

$$w(x, T) = \log x + \frac{1}{2 \sum_{j=1}^n p_j^2} \int_0^T \frac{\mu_s^2}{\sigma_s^2} ds + \frac{\lambda^2 \sum_{j=1}^{l+m} p_{h+k+j}^2}{2 \sum_{j=1}^n p_j^2} \int_0^T \left[\int_0^s [1 + \theta(u)]^2 e^{-2\lambda(s-u)} du \right] ds.$$

2. Let

$$w^*(x, T) = \log x + \frac{1}{2 \sum_{j=1}^n p_j^2} \int_0^T \frac{\mu_s^2}{\sigma_s^2} ds + \frac{\lambda^2 \sum_{j=1}^k p_{h+j}^2 + \sum_{j=1}^m p_{h+k+l+j}^2}{2 \sum_{j=1}^n p_j^2} \int_0^T \left[\int_0^s [1 + \theta^*(u)]^2 e^{-2\lambda(s-u)} du \right] ds.$$

Theorem 5.8. 1. Let

$$\lim_{T \rightarrow +\infty} \frac{1}{T} \int_0^T \left[\int_0^s [1 + \theta(u)]^2 e^{-2\lambda(s-u)} du \right] ds = \frac{1}{2\lambda} (1 - \beta)^2$$

where

$$\beta = \left(\sum_{j=1}^{l+m} p_{h+k+j}^2 \right)^{-\frac{1}{2}} \left(\sum_{j=1}^l p_{h+k+j}^2 \right)^{\frac{1}{2}}.$$

2. Let

$$\lim_{T \rightarrow +\infty} \left(\frac{1}{T} \int_0^T \left[\int_0^s [1 + \theta^*(u)]^2 e^{-2\lambda(s-u)} du \right] ds \right) = \frac{1}{2\lambda} (1 - \beta^*)^2$$

where

$$\beta^* = \left(\sum_{j=1}^k p_{h+j}^2 + \sum_{j=1}^m p_{h+k+l+j}^2 \right)^{-\frac{1}{2}} \left(\sum_{j=1}^k p_{h+j}^2 \right)^{\frac{1}{2}}.$$

5.4 A comparison among the utility functions of the agents

Considering the four agents, their utility functions are given by: $u(x, T)$ in refer to the first agent; $v(x, T)$ in refer to the second agent; $w(x, T)$ in refet to the third agent and $w^*(x, T)$ in refer to the fourth agent. Now, considering $T \rightarrow +\infty$, we study the divergence of the utility functions for the four agents. The different level of market information of the four agents implies the difference among the expected utility of the agents.

Theorem 5.9. *Let*

1.

$$\lim_{T \rightarrow +\infty} \frac{u(x, T) - w(x, T)}{T} =$$

$$= \frac{\lambda}{2} \frac{\left(\sum_{j=1}^l p_{h+k+j}^2 \right)^{\frac{1}{2}}}{\sum_{j=1}^n p_j^2} \left[\left(\sum_{j=1}^{l+m} p_{h+k+j}^2 \right)^{\frac{1}{2}} - \left(\sum_{j=1}^l p_{h+k+j}^2 \right)^{\frac{1}{2}} \right] > 0;$$

$$\lim_{T \rightarrow +\infty} \frac{u(x, T) - w^*(x, T)}{T} =$$

$$= \frac{\lambda}{2} \frac{\left(\sum_{j=1}^k p_{h+j}^2 \right)^{\frac{1}{2}}}{\sum_{j=1}^n p_j^2} \left[\left(\sum_{j=1}^k p_{h+j}^2 + \sum_{j=1}^m p_{h+k+l+j}^2 \right)^{\frac{1}{2}} - \left(\sum_{j=1}^k p_{h+j}^2 \right)^{\frac{1}{2}} \right] > 0.$$

2. *Considering*

$$\alpha = \frac{\left(\sum_{j=1}^{h+k+l} p_j^2 \right)^{\frac{1}{2}}}{\left(\sum_{j=1}^n p_j^2 \right)^{\frac{1}{2}}}, \quad \beta = \frac{\left(\sum_{j=1}^l p_{h+k+j}^2 \right)^{\frac{1}{2}}}{\left(\sum_{j=1}^{l+m} p_{h+k+j}^2 \right)^{\frac{1}{2}}},$$

$$\beta^* = \left(\sum_{j=1}^k p_{h+j}^2 + \sum_{j=1}^m p_{h+k+l+j}^2 \right)^{-\frac{1}{2}} \left(\sum_{j=1}^k p_{h+j}^2 \right)^{\frac{1}{2}},$$

it allows that

$$\lim_{T \rightarrow +\infty} \frac{w(x, T) - v(x, T)}{T} = \frac{\lambda}{4} \left[(1 - \beta)^2 - (1 - \alpha)^2 \right] > 0 \quad (I)$$

$$\lim_{T \rightarrow +\infty} \frac{w^*(x, T) - v(x, T)}{T} = \frac{\lambda}{4} \left[(1 - \beta^*)^2 - (1 - \alpha)^2 \right] > 0 \quad (II)$$

3.

$$\lim_{T \rightarrow +\infty} \frac{w(x, T) - w^*(x, T)}{T} = \frac{\lambda}{4} \left[(1 - \beta)^2 - (1 - \beta^*)^2 \right].$$

We can deduce that the asymptotic utility of the informed agent is greater than the other agents. Moreover the asymptotic utility of the uninformed agent is smaller than the other agents. At last, the utility functions of the two partially informed agents are easily comparable if β is different to β^* .

6 Main Result

In (Romano and Scolozzi, 2018) the authors study the logarithmic utility functions of two types of agents: informed and partially informed. Informed agent refers to all $n + 1$ Brownian Motions, the partially informed agent refers to only one opportune Brownian Motion. Considering $T \rightarrow +\infty$, it results that the utility functions of the two agents diverge, and also the difference between their expected utility diverges: it implies that the informed agent has a positive excess utility. In this work the authors study the logarithmic utility functions of the four types of agents: the first agent knows all information, the second agent knows minimum information, the third and the fourth agents know only partial information. Considering $T \rightarrow +\infty$, the utility functions diverge. It results that the asymptotic utility of the informed agent is greater than the other agents. The utility functions of the two partially informed agents are comparable if, in theorem 5.9, β is different to β^* . About the logarithmic utility maximization problem of the agents, the present work confirms Guasoni's findings.

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