

A Nonparametric Estimator for Interval System Availability

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ABSTRACT

In this paper, we study the interval availability of a repairable system through a system of integral equations. A new nonparametric estimator of interval system availability is proposed. This estimator is asymptotically consistent. Multiple numerical simulations are presented to demonstrate the efficient numerical performance of the new estimator.

Keywords: Interval system availability, Laplace-Stieltjes Transform, Kernel estimation, Integral equation, Block-by-Block method

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1 Introduction

Consider a repairable system that has two states, namely ‘up’ if it is properly functioning and ‘down’ if it is under repair. Let the binary random variable $X(t)$ denote the state of the system, that is $X(t) = 1$ means that the system is in ‘up’ state, and $X(t) = 0$ otherwise. Suppose that a new system with a random lifetime U_1 starts working at time $t = 0$ until it fails and it will be repaired immediately taking a random time D_1 . Upon completing the repair, the system resumes functioning with a random lifetime U_2 and a random repair time D_2 , and so on and so forth. The instant availability of the system is the probability that the system functions properly at time t , i.e., $A(t) \equiv \mathbb{P}(X(t) = 1)$. The instant availability and its limit have been studied extensively, for example see, (Barlow and Proschan, 1975), (Biswas and Sarkar, 2000), (Huang and Mi, 2015), (Mi, 2006).

Suppose that a certain mission will be implemented at time t and will take w time units. It is important to explore the probability that the system is able to complete the mission successfully. That is, we want to find the probability $\mathbb{P}(X(u) = 1, t \leq u \leq t + w)$ which will be denoted as $A_w(t)$. A repairable system can be repaired in two ways. One is

perfect repair which repairs a failed system as good as new. The other one is non-perfect repair that a failed system can not be repaired as good as new. Perfect repair means the system is restored back to a level equivalent to a new system. However, it is often difficult to be implemented due to technical limitation, or it may require very long time, and it is usually more costly. Therefore, non-perfect repair is practically more feasible.

In this paper we consider a repair policy that allows $k \geq 0$ imperfect repairs on successive failures before a perfect repair or replacement by a new unit is used. This policy has the advantage of saving time and expenses and was studied in (Biswas and Sarkar, 2000), (Huang and Mi, 2013).

Let the lifetimes of the system be $U_1, \dots, U_k$ and the imperfect repair times be $D_1, \dots, D_k$. Suppose that $U_i, D_i, i \geq 1$ are independent, U_i and D_i have distribution F_i and G_i for each $1 \leq i \leq k$. From the repair policy stated in the above obviously (U_1, D_1) and (U_{k+1}, D_{k+1}) are i.i.d.. With these notations, the evolution of a k -system defined in (Mi, 2020), can be described as follows. A k -system has lifetime U_1 at time 0 and the associated repair time D_1 . At the time $U_1 + D_1$ the system resumes functioning with lifetime U_2 and repair time D_2 , respectively, and so on until the time $\sum_{j=1}^{k-1} (U_j + D_j) + U_k$ then a perfect repair or replacement with a new unit is applied. We will use $\{(U_i, D_i), 1 \leq i \leq k\}$ to denote a k -system as described above.

In (Huang and Mi, 2013) the authors derived the equation which determines the interval availability $A_w(t)$ as shown below.

Theorem 1.1. *Suppose that $F_i(t)$ and $G_i(t)$, $1 \leq i \leq k$ are continuous CDF with support set $[0, \infty)$. The interval availability $A_w(t)$ is the unique bounded solution of the renewal equation*

$$A_w(t) = c(t) + \int_0^t A_w(t-s) dH(s) \quad t \geq 0 \quad (1.1)$$

where $c(t) = \sum_{i=0}^{k-1} \left[\left(\binom{i}{j=0}^* H_j \right)(t) - \left(\left(\binom{i}{j=0}^* H_j \right) * F_{i+1}(w + \cdot) \right)(t) \right]$, $H(s) = \binom{k}{i=1}^* H_i$ and $H_i = F_i * G_i$ is the convolution of F_i and G_i . The availability function $A_w(t)$ is of bounded variation $V_0^x(A) < +\infty$ on any finite interval $[0, x]$.

Let $1 \leq i \leq k$ be fixed. Suppose that in the first cycle of the k -system the lifetime and repair time are U_i and D_i respectively, and then the system evolves in the same manner as the original k -system until time $\sum_{j=i}^k (U_j + D_j)$. We may call the new system as the i th delayed k -system and denote its interval availability as $A_{iw}(t)$. Note that $A_w(t)$ mentioned in Theorem 1.1 is actually $A_{1w}(t)$. By introducing $A_{iw}(t)$ we are able to reformat the problem in the form of an integral system shown below and then solve

it numerically.

$$\begin{aligned} A_{iw}(t) &= \bar{F}_i(t+w) + \int_0^t A_{i+1,w}(t-s) dH_i(s), \quad t \geq 0, \quad 1 \leq i \leq k-1, \\ A_{k,w}(t) &= \bar{F}_k(t+w) + \int_0^t A_{1w}(t-s) dH_k(s), \quad t \geq 0, \end{aligned} \quad (1.2)$$

where $\bar{F}_i(t) = 1 - F_i(t)$.

If we denote $A_{k+1,w} = A_{1w}$, then the integral systems can be in the following form:

$$A_{iw}(t) = \bar{F}_i(t+w) + \int_0^t A_{i+1,w}(t-s) dH_i(s), \quad t \geq 0, \quad 1 \leq i \leq k \quad (1.3)$$

If $F_i(\cdot)$ and $G_i(\cdot)$ belong to a certain parametric family, one can first try to estimate the pertinent parameters for estimating $F_i(\cdot)$ and $G_i(\cdot)$, and then get an estimate of $A_w(t)$ by solving (1.1) or (1.3). However, in many situations either the parametric forms of $F_i(\cdot)$ and $G_i(\cdot)$ are unknown or the knowledge about them is rather limited. Thus, nonparametric methods become necessary.

2 The Kernel Estimator of Interval Availability And Its Properties

It is obvious that the availability $A_{iw}(t) = 0$ when $t < 0$. Hence, if we define

$$J_{iw}(t) = \begin{cases} \bar{F}_i(w+t), & t \geq 0, \\ 0, & t < 0. \end{cases}$$

then, for any $t \in (-\infty, +\infty)$, the integral system (1.3) can be rewritten as

$$A_{iw}(t) = J_{iw}(t) + \int_0^t A_{i+1,w}(t-s) dH_i(s), \quad 1 \leq i \leq k \quad (2.1)$$

In this section, we want to explore the steady interval availability of $A_{iw}(t)$, $1 \leq i \leq k$ when $t \rightarrow \infty$. Actually the expression of $\lim_{t \rightarrow \infty} A_{1w}(t)$ was derived in (Huang and Mi, 2013). In this article, we will derive $\lim_{t \rightarrow \infty} A_{iw}(t)$, $2 \leq i \leq k$.

It is known that in the case of $k = 1$ this limit exists and is given by

$$A_w(\infty) = \lim_{t \rightarrow \infty} A_w(t) = \frac{\int_w^\infty \bar{F}(x) dx}{\mu + \nu}$$

where $A_w(t) \equiv A_{1w}(t)$, $\mu = \mathbb{E}(U_1)$, $\nu = \mathbb{E}(D_1)$, and $U_1 \sim F$ provided $\mathbb{E}(U_1^2) < \infty$ (See, for instance, (Aven and Jensen, 2013)). If further $w = 0$, then $A_w(\infty)$ becomes the steady instant availability

$$A(\infty) = \frac{\int_0^\infty \bar{F}(x) dx}{\mu + \nu} = \frac{\mu}{\mu + \nu}$$

in which the denominator is the mean length of cycle $(0, U_1 + D_1)$, and the numerator is the mean functioning time in a cycle. With the same observation we see that $A_w(\infty)$ is actually the proportion of the contribution due to lifetime U to the mean length of a cycle.

2.1 Steady Interval Availability

In order to derive the steady interval availability $A_{iw}(\infty)$, $1 \leq i \leq k$, the Tauberian Theorem (see (Feller, 1991)) will be applied, and the unilateral (one-sided) Laplace-Stieltjes (LS) transform is needed. This transform is defined as follows.

Let $\xi(t)$, $-\infty < t < \infty$ be a function of bounded variation on any finite interval. Its unilateral LS transform $\hat{\xi}(s)$ is given by

$$\hat{\xi}(s) \equiv \lim_{\epsilon \rightarrow 0^+} \int_{-\epsilon}^{\infty} e^{-st} d\xi(t),$$

if there exists a constant $s_0 > 0$ and the integral converges for all $s > -s_0$. It can be shown that

$$\hat{\xi}(s) = -\xi(0^-) + s \int_0^{\infty} e^{-st} \xi(t) dt$$

and in particular if $\xi(0^-) = 0$ then

$$\hat{\xi}(s) = s \int_0^{\infty} e^{-st} \xi(t) dt. \quad (2.2)$$

We will use the unilateral LS transform only, so LS transform always means the unilateral LS transform whenever it is involved throughout the rest of this article.

If X is a random variable with cumulative distribution function F , then its Laplace-Stieltjes transform is given by the expectation:

$$\begin{aligned} \hat{F}(s) &= \mathbb{E}[e^{-sX}] \\ &= 1 - s\mathbb{E}(X) + \frac{s^2}{2!}\mathbb{E}(X^2) - \frac{s^3}{3!}\mathbb{E}(X^3) + \cdots + (-1)^n \frac{s^n}{n!}\mathbb{E}(X^n) + \cdots \end{aligned} \quad (2.3)$$

if $\hat{F}(s)$ is well-defined in a neighborhood of $s = 0$.

It was shown in (Mi, 2006) that $A_w(t)$ has at most countable discontinuity point. This ensures that its LS transform $\hat{A}_w(s)$ exists.

Let $K(t)$ be a CDF with $K(0) = 0$. For any non-negative function $g(t)$ of bounded variation with $g(t) = 0, \forall t < 0$, we define the convolution of K and g as:

$$(K * g)(t) = \int_0^t g(t-x) dK(x), \quad t \geq 0$$

It can be verified that the LS transform of the convolution $K * g(t)$ equals to the product of the LS transforms of the two functions $K(t)$ and $g(t)$, i.e., $\widehat{K * g}(s) = \hat{K}(s) \cdot \hat{g}(s)$.

Applying the LT transform to both sides of the integral equations (2.1) and then repeat-

ing substitution, we obtain

$$\begin{aligned}
 \hat{A}_i(s) &= \hat{J}_i(s) + \hat{H}_i \cdot \hat{A}_{i+1}(s) \\
 \hat{A}_i(s) &= \hat{J}_i(s) + \hat{H}_i \cdot (\hat{J}_{i+1}(s) + \hat{H}_{i+1} \cdot \hat{A}_{i+2}(s)) \\
 &= \dots \\
 &= \hat{J}_i(s) + \hat{H}_i \cdot \hat{J}_{i+1}(s) + \hat{H}_i \cdot \hat{H}_{i+1} \cdot \hat{J}_{i+2}(s) + \dots \\
 &\quad + \hat{H}_i \cdot \dots \cdot \hat{H}_{k-1} \cdot \hat{J}_k(s) + \hat{H}_i \cdot \dots \cdot \hat{H}_k \cdot \hat{J}_1(s) + \dots \\
 &\quad + \hat{H}_i \cdot \hat{H}_{i+1} \cdot \dots \cdot \hat{H}_{k-1} \cdot \hat{H}_k \cdot \hat{H}_1 \cdot \dots \cdot \hat{H}_{i-2} \cdot \hat{J}_{i-1}(s) \\
 &\quad + \hat{H}_i \cdot \hat{H}_{i+1} \cdot \dots \cdot \hat{H}_{k-1} \cdot \hat{H}_k \cdot \hat{H}_1 \cdot \dots \cdot \hat{H}_{i-2} \cdot \hat{H}_{i-1} \cdot \hat{A}_i(s), 1 \leq i \leq k,
 \end{aligned}$$

here the dependence of $\hat{A}_{iw}(s)$ and $\hat{J}_{iw}(s)$ on w is omitted for simplicity of notations.

With (2.2), we have:

$$\begin{aligned}
 \left(1 - \prod_{m=1}^k \hat{H}_m(s)\right) \hat{A}_i(s) &= s \int_0^{+\infty} e^{-st} J_i(t) dt + \hat{H}_i \cdot \left(s \int_0^{+\infty} e^{-st} J_{i+1}(t) dt \right) \\
 &\quad + \hat{H}_i \cdot \hat{H}_{i+1} \cdot \left(s \int_0^{+\infty} e^{-st} J_{i+2}(t) dt \right) + \dots \\
 &\quad + \prod_{\substack{m=1 \\ m \neq i-1}}^k \hat{H}_m(s) \cdot \left(s \int_0^{+\infty} e^{-st} J_{i-1}(t) dt \right)
 \end{aligned}$$

Next, with (2.3)

$$\begin{aligned}
 &s \sum_{m=1}^k (\mu_m + \nu_m) \hat{A}_i(s) + O(s^2) \\
 &= s \int_0^{+\infty} J_i(t) dt + s \int_0^{+\infty} J_{i+1}(t) dt + \dots + s \int_0^{+\infty} J_{i-1}(t) dt + O(s^2) \\
 &= s \sum_{m=1}^k \int_0^{+\infty} J_m(t) dt + O(s^2)
 \end{aligned}$$

then,

$$\hat{A}_i(s) = \frac{s \sum_{m=1}^k \int_0^{+\infty} J_m(t) dt + O(s^2)}{s \sum_{m=1}^k (\mu_m + \nu_m)}$$

Therefore, from Tauberian Theorem (see (Feller, 1991)) as $s \rightarrow 0^+$, we have

Theorem 2.1. For $1 \leq i \leq k$,

$$A_{iw}(+\infty) = \lim_{t \rightarrow +\infty} A_{iw}(t) = \lim_{s \rightarrow 0^+} \hat{A}_i(s) = \frac{\sum_{m=1}^k \int_w^{+\infty} \bar{F}_m(t) dt}{\sum_{m=1}^k (\mu_m + \nu_m)} \quad (2.4)$$

When $i = 1$, (2.4) was proved in (Huang and Mi, 2013). The result of Theorem 2.1 not only shows that the i -th ($2 \leq i \leq k$) delayed k -system has the same steady interval availability as the original k -system but also has important application in the numerical computations developed in the last section of this article.

2.2 Large Sample Property And Steady Behavior of Kernel Estimator

In general, the closed expression of $A_{iw}(t)$ for k -system is not available even in the simple case of $k = 1$. It is thus natural to seek for its estimator. Note that $A_{iw}(t)$ is a function of time t satisfying Eq. (1.3). This observation leads us to the following approach. First, the kernel method will be applied to estimate the CDFs $F_i(t)$ and $H_i(t)$ appearing in (2.1). Then we solve the system of integral equations obtained from substituting these CDFs in (2.1) by their kernel estimators for the non-parametric estimators of $A_{iw}(t)$. For more details about kernel estimation the readers are referred to (Li and Rachine, 2011).

Our procedure is described as follows. Consider a k -system $\{(U_i, D_i), 1 \leq i \leq k\}$. Let $\{(U_i^{(j)}, D_i^{(j)}), 1 \leq i \leq k, 1 \leq j \leq n\}$ be samples of $\{(U_i, D_i), 1 \leq i \leq k\}$. According to the method of kernel estimation we can obtain the kernel estimators $F_i^{(n)}(t)$ and $H_i^{(n)}(t)$ of CDFs $F_i(t)$ and $H_i(t)$ as

$$F_i^{(n)}(t) \equiv \frac{1}{n} \sum_{j=1}^n K\left(\frac{t - U_i^{(j)}}{h_n}\right), \quad 1 \leq i \leq k$$

$$H_i^{(n)}(t) \equiv \frac{1}{n} \sum_{j=1}^n K\left(\frac{t - U_i^{(j)} - D_i^{(j)}}{h_n}\right), \quad 1 \leq i \leq k,$$

here the selected kernel function $K(t)$ is a CDF and satisfies $\eta \equiv \int_0^\infty \bar{K}(t)dt < \infty$, and $h_n > 0$ is the selected bandwidth parameter satisfying $h_n \rightarrow 0$ as $n \rightarrow +\infty$.

Our study will focus on $A_w(t) \equiv A_{1w}(t)$ at this moment. Using $F_i^{(n)}(t)$ and $H_i^{(n)}(t)$, we further define $c^{(n)}(t)$ and $H^{(n)}(t)$ as estimators of $c(t)$ and $H(t)$ in Theorem 1.1. Finally, let $A_w^{(n)}(t)$ be the solution of the corresponding system of integral equations obtained from replacing all other functions in (1.1) by their estimators mentioned above. The rest of this section is devoted to the discussion on $\lim_{n \rightarrow \infty} A_w^{(n)}(\infty) = \lim_{n \rightarrow \infty} \lim_{t \rightarrow \infty} A_w^{(n)}(t)$. We will see later on that the result is exactly the same for any other $A_{iw}^{(n)}(t), i = 2, \dots, k$. In order to derive the desired limit some auxiliary results are needed. First, in the same manner as Lemma 2 of (Huang and Mi, 2018), the following inequality can be proved. The outline of its proof is given in Appendix.

Lemma 2.2.

$$\begin{aligned} & \max_{0 \leq t \leq x} |A_w^{(n)}(t) - A_w(t)| [1 - H^{(n)}(x)] \\ & \leq \max_{0 \leq t \leq x} |c^{(n)}(t) - c(t)| + [1 + V_0^x(A_w)] \max_{0 \leq t \leq x} |H^{(n)}(t) - H(t)|, \end{aligned}$$

where $V_0^x(A_w)$ is the total variation of $A_w(t)$ on interval $[0, x]$.

Theorem 2.3. Suppose that for any $x \in [0, \infty)$ and $1 \leq i \leq k$, $F_i^{(n)}(t)$ and $H_i^{(n)}(t)$ satisfy

$\max_{0 \leq t \leq x} |F_i^{(n)}(t) - F_i(t)| \rightarrow 0$ and $\max_{0 \leq t \leq x} |H_i^{(n)}(t) - H_i(t)| \rightarrow 0$, a.s. respectively, as $n \rightarrow \infty$, then $\lim_{n \rightarrow \infty} \max_{0 \leq t \leq x} |A_w^{(n)}(t) - A_w(t)| = 0$, a.s.

Proof. Since $\max_{0 \leq t \leq x} |F_i^{(n)}(t) - F_i(t)| \rightarrow 0$ and $\max_{0 \leq t \leq x} |H_i^{(n)}(t) - H_i(t)| \rightarrow 0$, it is true that $c^{(n)}(t) \rightarrow c(t)$ and $H^{(n)}(t) \rightarrow H(t)$, a.s. and the convergence is uniform on any finite interval of t . The result hence follows from Lemma 2.2. $\square$

In the discussion of the following lemmas it is assumed that $\{X_i \geq 0, i = 1, \dots, n\}$ is a random sample from a population with CDF $F_X(x)$ that satisfies $\int_0^\infty \bar{F}_X(x) dx < \infty$. The proofs of these lemmas are quite technical and are therefore given in the Appendix.

Lemma 2.4. For any $w > 0$,

$$\frac{1}{n} \sum_{i=1}^n X_i \bar{K} \left(\frac{w - X_i}{h_n} \right) I_{(X_i \leq w)} \rightarrow 0, \text{ a.s.}$$

when $n \rightarrow +\infty$.

Lemma 2.5. The following limit is true

$$\frac{1}{n} \sum_{i=1}^n X_i \bar{K} \left(\frac{w - X_i}{h_n} \right) \rightarrow \int_w^{+\infty} y dF_X(y) = \mathbb{E}(X I_{(X > w)}), \text{ a.s.}$$

Lemma 2.6. It holds that

$$\lim_{n \rightarrow +\infty} \int_w^{+\infty} \bar{F}_X^{(n)}(t) dt = \int_w^{+\infty} \bar{F}_X(t) dt, \text{ a.s.}$$

Now we can establish the following result.

Theorem 2.7. Suppose that the kernel function $K(t)$ satisfies $\eta \equiv \int_0^\infty y dK(y) < \infty$, and window width $h_n \rightarrow 0$ as $n \rightarrow +\infty$, then

$$A_w(+\infty) = \lim_{n \rightarrow +\infty} \lim_{t \rightarrow +\infty} \hat{A}_w(t, h_n) = \frac{\sum_{j=1}^k \int_w^{+\infty} \bar{F}_j(y) dy}{\sum_{j=1}^k (\mu_j + \nu_j)}, \text{ a.s.}$$

Proof. From Theorem 2.1 it follows that

$$A_w^{(n)}(+\infty) = \lim_{t \rightarrow +\infty} A_w^{(n)}(t) = \frac{\sum_{j=1}^k \int_w^{+\infty} \bar{F}_j^{(n)}(t) dt}{\sum_{j=1}^k (\mu_j^{(n)} + \nu_j^{(n)})}, \quad (2.5)$$

where

$$\mu_j^{(n)} = \int_w^\infty \bar{F}_j^{(n)}(t) dt \text{ and } \nu_j^{(n)} = \int_w^\infty \bar{G}_j^{(n)}(t) dt.$$

According to Lemmas 2.4, 2.5, and 2.6, we see that as $n \rightarrow \infty$

$$\int_w^{+\infty} \bar{F}_j^{(n)}(t) dt \rightarrow \int_w^{+\infty} \bar{F}_j(y) dy \quad a.s., j = 1, \dots, k.$$

Therefore, the claimed result holds. $\square$

3 Numerical Method

Suppose that $F_i^{(n)}(t) \equiv F_i^{(n)}(t, h_n)$ and $H_i^{(n)}(t) \equiv H_i^{(n)}(t, h_n)$ have been obtained based on sample $\{(u_i^{(j)}, d_i^{(j)}), 1 \leq i \leq k+1, 1 \leq j \leq n\}$. Let $A_i^{(n)}(t) \equiv A_i^{(n)}(t, h_n)$ be the availability function determined by $F_i^{(n)}(t)$ and $H_i^{(n)}(t)$. That is, $A_i^{(n)}(t, h_n)$ is the unique solution of the integral equations

$$A_i^{(n)}(t) = \bar{F}_{iw}^{(n)}(t) + H_i^{(n)} * A_{i+1}^{(n)}(t), \quad 1 \leq i \leq k.$$

In order to avoid heavy computation of convolutions, we will not use (1.1) for computing values of $A_n(t)$. Instead, we will use the Block-by-Block method for solving the renewal integral equation (2.1) directly to obtain values of $A_i^{(n)}(t)$. The quadrature rule for the Stieltjes integral used in the Block-by-Block method is described as follows. Consider the following Stieltjes integral on an interval $[a, b]$

$$\int_a^b x(t) dy(t).$$

Denote the values of the function $x(t)$ at $t = a$, $(a+b)/2$, and b by $x_1 = x(a)$, $x_{1.5} = x((a+b)/2)$, and $x_2 = x(b)$. Similar notations will be used for $y(t)$ and other functions involved later. The Stieltjes integral is approximated by

$$\int_a^b x(t) dy(t) \approx \frac{x_1 + 4x_{1.5} + x_2}{6} (y_2 - y_1) + \frac{x_2 - x_1}{3} (y_1 - 2y_{1.5} + y_2). \quad (3.1)$$

In the following section, we will denote $A_i^{(n)}(t)$, $F_{iw}^{(n)}(t)$ and $H_i^{(n)}(t)$ as $A_i(t)$, $F_i(t)$ and $H_i(t)$ respectively for simplicity of notations. Let m be an even number, the interval $[0, c]$ is divided into m subintervals of equal size by

$$0 = t_0 < \dots < t_j = j\delta < \dots < t_m = m\delta = c,$$

and the value of $A_i(t)$ at grid point t_j (or half-grid point $t_{j+1/2}$) will be denoted by $A_{i,j}$ (or $A_{i,j+1/2}$). The same notation will be used for $\bar{F}_i(t_j)$ and $H_i(t_j)$. From the integral equation (2.1) clearly

$$A_i(t_2) = J_i(t_2) + \int_{t_0}^{t_2} A_{i+1}(t_2 - s) dH_i(s), \quad i = 1, \dots, k.$$

With $t_2 = t_0 = 0$, we have $A_{i,0} = J_i(0)$.

Applying the quadrature rule (3.1) with nodes $t_0 = 0, t_1 = \delta$ and $t_2 = 2\delta$ on the interval $[t_0, t_2]$,

$$\begin{aligned} A_{i,2} = & J_{i,2} + \frac{A_{i+1,0} + 4A_{i+1,1} + A_{i+1,2}}{6} (H_{i,2} - H_{i,0}) \\ & + \frac{A_{i+1,0} - A_{i+1,2}}{3} (H_{i,2} - 2H_{i,1} + H_{i,0}) \end{aligned}$$

Rearranging the equation, we have

$$\begin{aligned} & A_{i,2} - \frac{2(H_{i,2} - H_{i,0})}{3} A_{i+1,1} + \frac{3H_{i,0} - 4H_{i,1} + H_{i,2}}{6} A_{i+1,2} \\ = & J_{i,2} + \frac{H_{i,0} - 4H_{i,1} + 3H_{i,2}}{6} A_{i+1,0}. \end{aligned} \quad (3.2)$$

Next, we apply the quadrature rule (3.1) to the interval $[t_0, t_1]$ with nodes $t_0 = 0, t_{1/2} = \delta/2$ and $t_1 = \delta$, and from

$$A_i(t_1) = J_i(t_1) + \int_{t_0}^{t_1} A_{i+1}(t_1 - s) dH_i(s),$$

we have

$$\begin{aligned} A_{i,1} = & J_{i,1} + \frac{A_{i+1,0} + 4A_{i+1,1/2} + A_{i+1,1}}{6} (H_{i,1} - H_{i,0}) \\ & + \frac{A_{i+1,0} - A_{i+1,1}}{3} (H_{i,1} - 2H_{i,1/2} + H_{i,0}) \end{aligned}$$

To deal with the unknown value $A_{i,1/2}$, we approximate it by quadratic interpolation, using values $A_{i,0}$, $A_{i,1}$ and $A_{i,2}$, that is, we replace $A_{i,1/2}$ by

$$A_{i,1/2} \approx \frac{3}{8} A_{i,0} + \frac{3}{4} A_{i,1} - \frac{1}{8} A_{i,2},$$

so that we have

$$\begin{aligned} & A_{i,1} - \frac{3H_{i,0} - 2H_{i,1/2} - H_{i,1}}{3} A_{i+1,1} + \frac{H_{i,1} - H_{i,0}}{12} A_{i+1,2} \\ = & J_{i,1} - \frac{H_{i,0} + 8H_{i,1/2} - 9H_{i,1}}{12} A_{i,0} \end{aligned} \quad (3.3)$$

Combining the $(2k)$ linear equations from (3.2) and (3.3), we have a linear system on $(2k)$ unknowns $A_{i,1}, A_{i,2}, 1 \leq i \leq k$.

It is clear that the coefficient matrix in this linear system is diagonal dominant and nonsingular for sufficiently small $\delta > 0$. Hence the system of equations (3.2) and (3.3) has a unique pair of solutions $A_{i,1}$ and $A_{i,2}$, i.e., we can obtain the values of $A_i(t_1)$ and $A_i(t_2)$.

The general procedure for computing $A_{i,2j+1}$ and $A_{i,2j+2}$ is similar. Assume that the values $A_{i,0}, A_{i,1}, \dots, A_{i,2j}$ are known. We then treat the interval $[t_{2j}, t_{2j+2}]$ with nodes t_{2j}, t_{2j+1} and t_{2j+2} in the same way as we did for the interval $[t_0, t_2]$ with nodes t_0, t_1 and t_2 . Moreover, the same procedure is used in $[t_{2j}, t_{2j+1}]$ with nodes $t_{2j}, t_{2j+1/2}$ and t_{2j+1} . The value $A_{i,2j+1/2}$ at $t_{2j+1/2} = (t_{2j} + t_{2j+1})/2$ is approximated by quadratic interpolation in terms of $A_{i,2j}, A_{i,2j+1}$ and $A_{i,2j+2}$ just like what was done for approximating $A_{i,1/2}$, i.e.

$$A_{i,2j+1/2} = \frac{3}{8}A_{i,2j} + \frac{3}{4}A_{i,2j+1} - \frac{1}{8}A_{i,2j+2}. \quad (3.4)$$

The above process will yield the same linear equations for $A_{i,2j+1}$ and $A_{i,2j+2}$, $1 \leq i \leq k$ similar to (3.2) and (3.3). In fact, the coefficient matrix of the new system of linear equations is exactly equal to that in previous steps, with different right hand sides. This property can significantly improve computation time (See (Huang and Mi, 2012) and (Linz, 1985) for details). By solving this system, we can obtain the values $A_{i,2j+1}$ and $A_{i,2j+2}$.

The integral system (2.1) can be transformed into Laplace type integral system where h_i is the pdf corresponding to the random variable $U + D$.

$$A_{i,w}(t) = J_i(t+w) + \int_0^t A_{i+1,w}(t-s)h_i(s) ds, \quad 1 \leq i \leq k \quad (3.5)$$

For simplification, we consider only an integral equation, not an integral system here:

$$A(t) = \overline{F}(t) + \int_0^t A(s)h(t-s)ds$$

At $t = t_2 = 2\delta$,

$$A(t_2) = \overline{F}(t_2) + \int_0^{t_2} A(s)h(t_2-s)ds$$

By Simpson's rule, with nodes $t = t_0, t_1$ and t_2 , we have

$$A_2 = \overline{F}_2 + \frac{\delta}{3} [A_0h_2 + 4A_1h_1 + A_2h_0]$$

$$\left(-\frac{\delta}{3}4h_1\right) A_1 + \left(1 - \frac{\delta}{3}h_0\right) A_2 = \overline{F}_2 + \frac{\delta}{3}A_0h_2, \quad (3.6)$$

here by convention $A_0 = 1$ since $A(0) = \overline{F}(0) = 1$. At $t = t_1 = \delta$,

$$A(t_1) = \overline{F}(t_1) + \int_0^{t_1} A(s)h(t_1-s)ds$$

we use Simpson's rule with nodes $t = t_0, t_1$ and half-node $t_{1/2} = \delta/2$,

$$A_1 = \bar{F}_1 + \frac{\delta}{6} [A_0 h_1 + 4A_{1/2} h_{1/2} + A_1 h_0].$$

To approximate the unknown $A_{1/2}$ at half-node $t_{1/2} = \delta/2$, we apply quadratic interpolation using values at $t = t_0, t_1$ and t_2 :

$$A_{1/2} \approx \frac{3}{8}A_0 + \frac{3}{4}A_1 - \frac{1}{8}A_2$$

and consequently

$$\begin{aligned} A_1 &= \bar{F}_1 + \frac{\delta}{6} \left[A_0 h_1 + 4 \left(\frac{3}{8}A_0 h_0 + \frac{3}{4}A_1 h_1 - \frac{1}{8}A_2 h_2 \right) + A_1 h_0 \right] \\ \left(1 - \frac{\delta}{2}h_1 - \frac{\delta}{6}h_0 \right) A_1 + \left(\frac{\delta}{12}h_2 \right) A_2 &= \bar{F}_1 + \frac{\delta}{6} \left(h_1 + \frac{3}{2}h_0 \right) A_0. \end{aligned} \quad (3.7)$$

From (3.7) and (3.6), we have the following linear system,

$$\begin{pmatrix} 1 - \delta h_1/2 - \delta h_0/6 & \delta h_2/12 \\ -4\delta h_1/3 & 1 - \delta h_0/3 \end{pmatrix} \begin{pmatrix} A_1 \\ A_2 \end{pmatrix} = \begin{pmatrix} \bar{F}_1 + \delta (h_1 + 3h_0/2) A_0/6 \\ \bar{F}_2 + \delta h_2 A_0/3 \end{pmatrix}. \quad (3.8)$$

It is easy to see that this system has a unique solution for A_1, A_2 when δ is sufficiently small.

The general procedure to compute the approximate values A_{2m+1} and A_{2m+2} is similar as before provided $A_0, A_1, A_2, \dots, A_{2m-1}, A_{2m}$ are available.

4 Example

Here we will illustrate the algorithm described above and show the performance of the proposed estimator $\hat{A}_w(t)$ with the following examples.

In the first two examples, $A_w(t)$ has a closed-form expression, while in the last one it does not.

The procedures described previously are applied to estimate the values of $A_w(t)$ on the time interval $[0, T]$ where $T = \mathbb{E}(U + D)$ is the average length of a cycle of the corresponding alternative renewal process. We use 101 nodes on the interval $[0, T]$ to solve the integral equation in the Block-by-Block method.

Figures show the curves of the average of $k = 1000$ repetitions of $\hat{A}_w(t)$. In addition, we also show the values of the mean integral squared error (MISE) on the interval $[0, T]$ defined by $MISE = \sum_{j=1}^k \int_0^T |\hat{A}_w^{(j)}(t) - A_w(t)|^2 dt / (kT)$, where each $\hat{A}_w^{(j)}(t)$ is the j -th repetition of the non-parametric estimate of $A_w(t)$ based on the j -th simulated sample data $(u_i^{(j)}, d_i^{(j)})$, $1 \leq i \leq n$, $A_w(t)$ is the analytic solution in the first two examples, and is the numerical solution of integral equation (2.1) given CDFs $F(t)$ and $H(t)$ in the last example. The value of MISE measures the accuracy of the estimation of the value $\int_0^T \mathbb{E}(|\hat{A}_w(t) - A_w(t)|^2) dt / T$ when k is large.

h	Gamma	Pareto	Loglogistic	Log-normal	Uniform	Weibull
$1/\sqrt{n}$	0.002964	0.003011	0.002819	0.002878	0.003074	0.003035
$1/n$	0.003005	0.002889	0.002888	0.003015	0.003054	0.003047
$1/n\sqrt{n}$	0.003131	0.003098	0.003099	0.003134	0.003141	0.003140

Table 1: Example 1– MISE with Different h for 6 Other Kernels

4.1 Example 1

Suppose $U \sim \text{Gamma}(p, \alpha)$ and $D \sim \text{Exp}(\beta)$ with probability density functions

$$f(t) = \alpha^p e^{-\alpha t} t^{p-1} / \Gamma(p) \quad \text{and} \quad g(t) = \beta e^{-\beta t}.$$

$A_w(t)$ can be derived by using Laplace transforms.

In the case of $p = 2$ and $\alpha = \beta$, it can be obtained that

$$F(t) = 1 - (\alpha t + 1)e^{-\alpha t}, \quad G(t) = 1 - e^{-\beta t}, \quad H(t) = 1 - (\alpha^2 t^2 / 2 + \alpha t + 1)e^{-\alpha t}.$$

This example was studied in (Huang and Mi, 2018) for instant availability ($w = 0$).

Moreover, in this case, the system availability $A(t)$ has analytical expression given by

$$A_w(t) = \frac{\sqrt{3} \sin(\sqrt{3}\alpha t / 2) + (1 + 2\alpha w) \cos(\sqrt{3}\alpha t / 2)}{3e^{\alpha(3t+2w)/2}} + \frac{2 + \alpha w}{3e^{\alpha w}}.$$

The values $w = 0.25$ and $\alpha = 0.25$ are used in this example. The mean length of lifetime and mean repair time are $\mathbb{E}(U) = 8$, $\mathbb{E}(D) = 4$, and consequently the mean length of a cycle is $T = \mathbb{E}(U + D) = 12$.

In the left panels of Fig.1, sample size $n = 16$, and in the right panels $n = 64$. From the top to the bottom, the bandwidth parameters are $h = n^{-1/2}$, n^{-1} and $n^{-3/2}$ respectively. From Fig.1, we observe that bandwidth $h = n^{-1}$ gives the best results independent of the sample size, which can be as small as 16 or as large as 64.

Additionally, we tried six other kernel functions: Gamma with shape parameter = 3 and scale parameter = 1, Pareto with shape parameter = 1, Log-logistic with both shaper and scale parameters = 1, Log-normal $\text{LN}(\mu, \sigma)$ with $\mu = 0$ and $\sigma = 1$, Standard Uniform (lower-endpoint = 0, upper-endpoint = 1), and Weibull with scale parameter = 1 and shape parameter = 2 in order to compare the effect of different kernels on the estimation accuracy. Table 1 lists the MISEs with the three different choices of bandwidth: $h = n^{-1/2}$, n^{-1} , $n^{-3/2}$, and n^{-2} with $n = 50$. Overall their MISEs are of the same order.

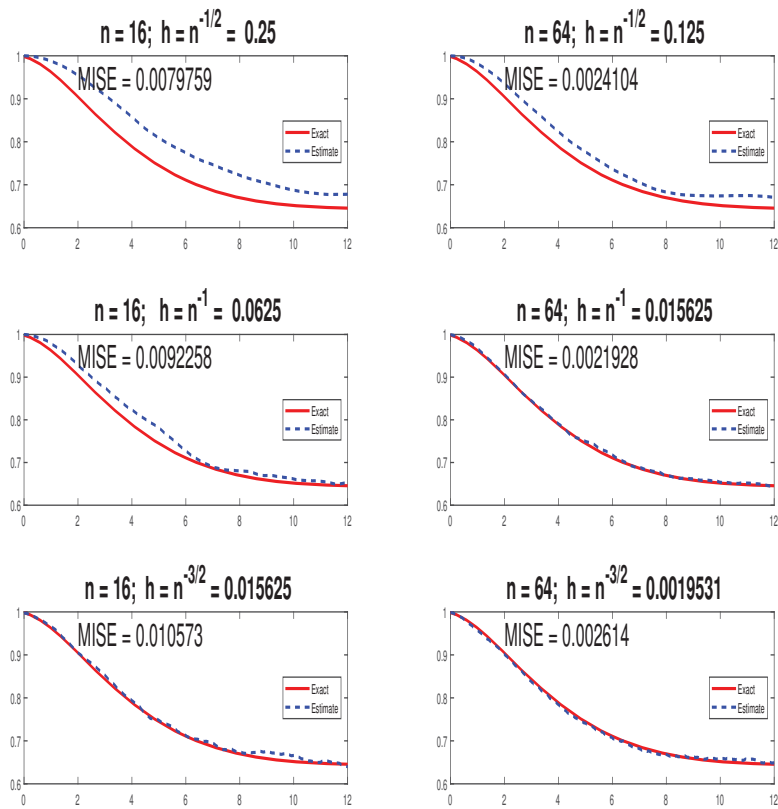


Figure 1: Example 1; Standard Exponential Kernel

4.2 Example 2

We consider the case $k = 2$ and let $U_1 \sim \text{Exp}(2\alpha)$, $U_2 \sim \text{Exp}(3\alpha)$, $D_1 \sim \text{Exp}(5\alpha)$, $D_2 \sim \text{Exp}(4\alpha)$ with the following densities:

$$f_1(t) = 2\alpha e^{-2\alpha t}, \quad f_2(t) = 3\alpha e^{-3\alpha t}, \quad g_1(t) = 5\alpha e^{-5\alpha t}, \quad g_2(t) = 4\alpha e^{-4\alpha t}.$$

This example was used in (Huang and Mi, 2013) to verify the statement of Theorem 2.1 when $i = 1$. We can see that the conclusion is true for any $1 \leq i \leq k$.

The interval availability can be obtained by LS transform as

$$\begin{aligned} A_{1w}(t) &= \frac{30}{77e^{2\alpha w}} + \frac{20}{77e^{3\alpha w}} + \frac{12e^{\alpha w} + 15}{77e^{7\alpha t + 3\alpha w}} \\ &\quad + \frac{5[39 \cos(\sqrt{39}\alpha t/2)(e^{\alpha w} - 1)]}{429e^{7\alpha t/2 + 3\alpha w}} + \frac{5[\sqrt{39} \sin(\sqrt{39}\alpha t/2)(3e^{\alpha w} - 1)]}{429e^{7\alpha t/2 + 3\alpha w}}, \\ A_{2w}(t) &= \frac{30}{77e^{2\alpha w}} + \frac{20}{77e^{3\alpha w}} + \frac{12e^{\alpha w} + 15}{77e^{7\alpha t + 3\alpha w}} \\ &\quad - \frac{2[39 \cos(\sqrt{39}\alpha t/2)(e^{\alpha w} - 1)]}{143e^{7\alpha t/2 + 3\alpha w}} - \frac{2[\sqrt{39} \sin(\sqrt{39}\alpha t/2)(3e^{\alpha w} - 1)]}{143e^{7\alpha t/2 + 3\alpha w}}. \end{aligned}$$

From this expression the limit $\lim_{t \rightarrow \infty} A_w(t)$ is

$$A_{1w}(\infty) = A_{2w}(\infty) = \frac{30}{77e^{2\alpha w}} + \frac{20}{77e^{3\alpha w}}. \quad (4.1)$$

On the other hand, this can also be verified by Theorem 2.1 as follows. Clearly

$$\int_w^\infty J_1(x)dx = \frac{1}{2\alpha e^{2\alpha w}}, \quad \int_w^\infty J_2(x)dx = \frac{1}{3\alpha e^{3\alpha w}}.$$

In addition, we have $\mu_1 + \mu_2 + \nu_1 + \nu_2 = 77/(60\alpha)$. Then formula (2.4) gives:

$$A_w(\infty) = \left(\frac{1}{2\alpha e^{2\alpha w}} + \frac{1}{3\alpha e^{3\alpha w}} \right) \frac{60\alpha}{77} = \frac{30}{77e^{2\alpha w}} + \frac{20}{77e^{3\alpha w}},$$

which is the same as the one in (4.1).

When the densities f_1, f_2, g_1 and g_2 are known, this example was used to show the accuracy and efficiency of the Block-by-Block algorithm for Laplacian type integral equations (3.5) in (Huang and Mi, 2013).

Now we compute the interval availability with simulated sample data $(u_i^{(j)}, d_i^{(j)})$, $1 \leq i \leq n$ by solving the Stieltjes type integral equations (2.1) via estimating the cumulative distribution functions F_1, F_2, H_1 and H_2 . The numerical results are shown in Figure 2.

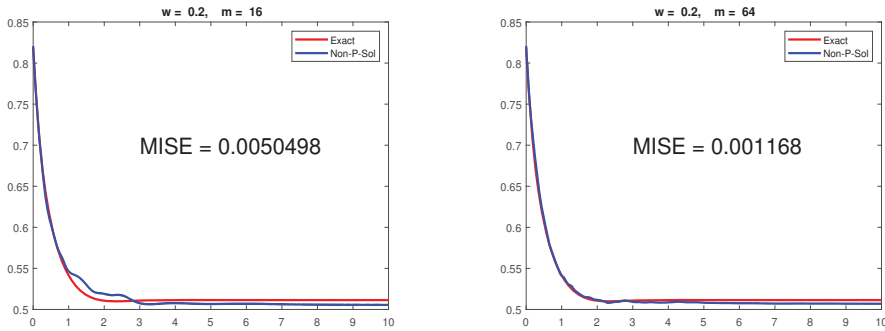


Figure 2: Example 2; $w = 0.2$; $m = 16$ (left), $m = 64$ (right)

4.3 Example 3

The integral system (2.1) can be transformed into Laplacian type integral system where h_i is the pdf corresponding to the random variable $U + D$.

$$A_{iw}(t) = J_i(t + w) + \int_0^t A_{i+1,w}(t - s)h_i(s) ds, \quad 1 \leq i \leq k \quad (3.5)$$

Similarly, the h_i can be obtained by kernel estimate from the data $u_i + d_i, 1 \leq i \leq k$.

In general, the CDF of a random variable has nicer properties than the PDF, and then it is easier to have reliable estimate of a CDF than a PDF.

Suppose that lifetimes U_1, U_2 follow Log-normal distribution, and repair times D_1, D_2 follow Weibull distribution with densities

$$f = \frac{1}{t\sigma\sqrt{2\pi}} e^{-\frac{(\ln t - \mu)^2}{2\sigma^2}}; \quad g = \frac{k}{\lambda} \left(\frac{x}{\lambda}\right)^{k-1} e^{-(x/\lambda)^k}$$

Let $w = 2.5$ and the parameters be $\mu_1 = 1.5, \mu_2 = 1.25; \sigma_1 = 0.3, \sigma_2 = 0.6; k_1 = 1.25, k_2 = 1.5; \lambda_1 = 1, \lambda_2 = 1.5$. The numerical solutions of $A_w(t)$ for (2.1) and (3.5) obtained by using the Block-by-Block method are shown in Figure 3-5.

Compare the estimate by the numerical solutions of (2.1) and (3.5), it is clear that the estimate of $A_w(t)$ based on the estimations of CDF is more reliable than that based on the estimations of PDF.

4.4 Discussions and Conclusions

The selection of the bandwidth $h = h_n$ is always an issue in the kernel estimation method. That is, too small h results in undersmoothed estimator which causes much roughness; on the other hand, too large h will result in oversmoothed estimator which cannot catch the structure of the function to be estimated. In Example 3, we can see that the estimate of $A_w(t)$ based on the estimations of CDF is more reliable than that

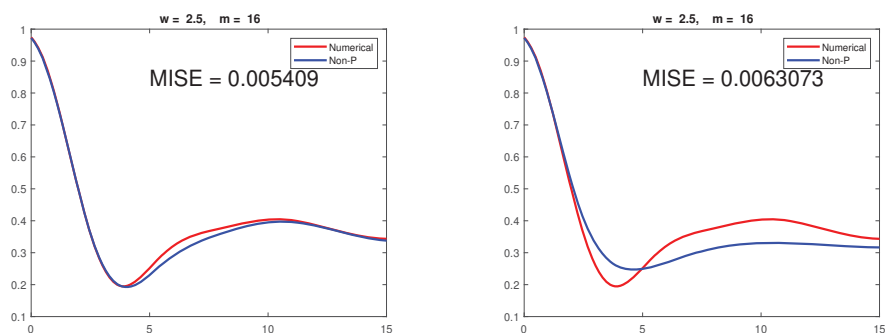


Figure 3: Example 3; $m = 16$; Stieltjes (left) and Lapalace (right) Integral Equation

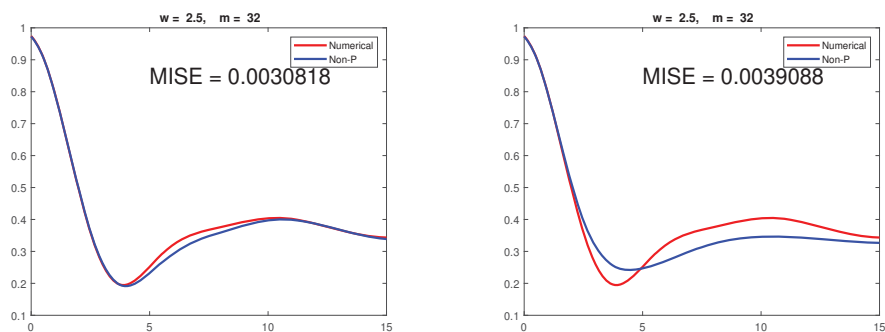


Figure 4: Example 3; $m = 32$; Stieltjes (left) and Lapalace (right) Integral Equation

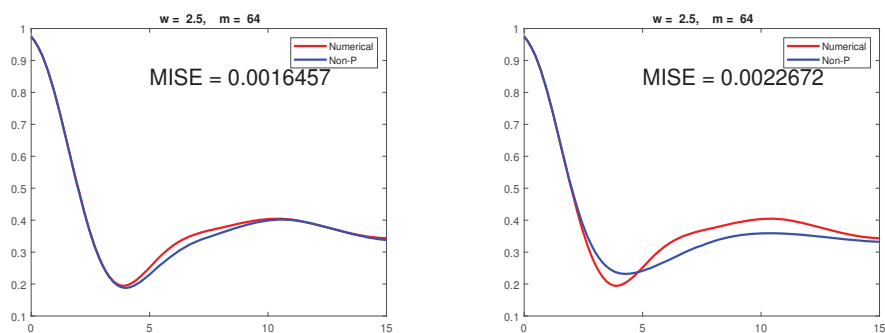


Figure 5: Example 3; $m = 64$; Stieltjes (left) and Lapalace (right) Integral Equation

based on the estimations of PDF. There has not been much work on the bandwidth selection for estimating CDF, but there were considerable studies on the topic for kernel estimation of probability density function (pdf). In practice, the so-called normal scale bandwidths described in (Wand and Jones, 1995) are often applied.

In this paper, our goal is to estimate the interval availability function $A_w(t)$. We study the interval availability $A_w(t)$ of a repairable system through the system of integral equations. To achieve this goal we need to estimate $F(t)$ and $H(t)$.

In our numerical studies, we simulated many other lifetime and repair time distributions and sample sizes, with other kernel CDFs. The same pattern that appeared in Examples 1 and 2 is always observed. The magnitudes of the MISEs are of the same order. Among the three simulated bandwidths, $h_n = 1/n$ has the best result: the obtained curve of $\hat{A}_n(t)$ is pretty smooth, the magnitude of the bias of the estimation is acceptable, and the MISE on $[0, T]$ can always compete with that using $h = n^{-1/2}$ or $h_n = n^{-3/2}$.

A Proof of Lemmas 1-4

Lemma A.1.

$$\begin{aligned} & \max_{0 \leq t \leq x} |A_w^{(n)}(t) - A_w(t)| [1 - H^{(n)}(x)] \\ & \leq \max_{0 \leq t \leq x} |c^{(n)}(t) - c(t)| + [1 + V_0^x(A_w)] \max_{0 \leq t \leq x} |H^{(n)}(t) - H(t)|, \end{aligned}$$

where $V_0^x(A_w)$ is the total variation of $A_w(t)$ on interval $[0, x]$.

Proof. From Eq. (1.1) it holds that

$$\begin{aligned} A_w^{(n)}(t) - A_w(t) &= [c^{(n)}(t) - c(t)] + \int_0^t [A_w^{(n)}(t-s) - A_w(t-s)] dH^{(n)}(s) \\ &+ \left[\int_0^t A_w(t-s) dH^{(n)}(s) - \int_0^t A_w(t-s) dH(s) \right]. \end{aligned}$$

Then the desired result can be obtained by following the same way as the proof of Lemma 2 in (Huang and Mi, 2018). $\square$

Lemma A.2. For any $w > 0$,

$$\frac{1}{n} \sum_{i=1}^n X_i \bar{K} \left(\frac{w - X_i}{h_n} \right) I_{(X_i \leq w)} \rightarrow 0, a.s.$$

when $n \rightarrow +\infty$.

Proof. For any given $\epsilon > 0$, $\exists t_0$, such that $\bar{K}(t) < \epsilon, \forall t > t_0$

$$\frac{w - x_i}{h_n} > t_0 \iff x_i < w - h_n t_0 \quad \text{and} \quad \frac{w - x_i}{h_n} < t_0 \iff x_i > w - h_n t_0$$

For the same $\epsilon > 0$, $\exists \delta \in (0, w)$ such that $\int_{w-\delta}^w y dF_X(y) < \epsilon$.
 Let N be such that $w - h_n t_0 > w - \delta$, $\forall n \geq N$.

$$\begin{aligned} & \frac{1}{n} \sum_{i=1}^n X_i \bar{K} \left(\frac{w - X_i}{h_n} \right) I_{(X_i \leq w)} \\ &= \frac{1}{n} \sum_{i=1}^n X_i \bar{K} \left(\frac{w - X_i}{h_n} \right) I_{(X_i \leq w)} I_{(X_i < w - h_n t_0)} \\ & \quad + \frac{1}{n} \sum_{i=1}^n X_i \bar{K} \left(\frac{w - X_i}{h_n} \right) I_{(X_i \leq w)} I_{(X_i \geq w - h_n t_0)} \end{aligned}$$

Consider the first term in the above expression, it is true that

$$0 \leq \frac{1}{n} \sum_{i=1}^n X_i \bar{K} \left(\frac{w - X_i}{h_n} \right) I_{(X_i \leq w)} I_{(X_i < w - h_n t_0)} \leq \epsilon \cdot \frac{1}{n} \sum_{i=1}^n X_i \rightarrow \epsilon \mathbb{E}(X), \quad a.s.$$

As for the second term,

$$\begin{aligned} & \frac{1}{n} \sum_{i=1}^n X_i \bar{K} \left(\frac{w - X_i}{h_n} \right) I_{(X_i \leq w)} I_{(w - h_n t_0 \leq X_i)} \\ &= \frac{1}{n} \sum_{i=1}^n X_i \bar{K} \left(\frac{w - X_i}{h_n} \right) I_{(w - h_n t_0 \leq X_i \leq w)} \\ &\leq \frac{1}{n} \sum_{i=1}^n X_i \bar{K} \left(\frac{w - X_i}{h_n} \right) I_{(w - \delta \leq X_i \leq w)} \leq \frac{1}{n} \sum_{i=1}^n X_i I_{(w - \delta \leq X_i \leq w)} \end{aligned}$$

then,

$$\begin{aligned} & \limsup_{n \rightarrow +\infty} \frac{1}{n} \sum_{i=1}^n X_i \bar{K} \left(\frac{w - X_i}{h_n} \right) I_{(X_i \leq w)} I_{(w - h_n t_0 \leq X_i)} \\ &\leq \limsup_{n \rightarrow +\infty} \frac{1}{n} \sum_{i=1}^n X_i I_{(w - \delta \leq X_i \leq w)} = \int_{w-\delta}^w y dF_X(y) < \epsilon \quad a.s. \end{aligned}$$

Combining all of these, we obtain

$$\limsup_{n \rightarrow +\infty} \frac{1}{n} \sum_{i=1}^n X_i \bar{K} \left(\frac{w - X_i}{h_n} \right) I_{(X_i \leq w)} \leq \epsilon \mathbb{E}(X) + \epsilon \quad a.s.$$

Due to the arbitrariness of $\epsilon > 0$, it is therefore proved that

$$\lim_{n \rightarrow +\infty} \frac{1}{n} \sum_{i=1}^n X_i \bar{K} \left(\frac{w - X_i}{h_n} \right) I_{(X_i \leq w)} = 0 \quad a.s.$$

□

Lemma A.3. *The following limit is true*

$$\frac{1}{n} \sum_{i=1}^n X_i \bar{K} \left(\frac{w - X_i}{h_n} \right) \rightarrow \int_w^{+\infty} y dF_X(y) = \mathbb{E}(X I_{(X > w)}), a.s.$$

Proof. Since

$$\begin{aligned} X_i \bar{K} \left(\frac{w - X_i}{h_n} \right) &= X_i \bar{K} \left(\frac{w - X_i}{h_n} \right) I_{(X_i > w)} + X_i \bar{K} \left(\frac{w - X_i}{h_n} \right) I_{(X_i \leq w)} \\ &= X_i I_{(X_i > w)} + X_i \bar{K} \left(\frac{w - X_i}{h_n} \right) I_{(X_i \leq w)} \end{aligned}$$

and

$$\frac{1}{n} \sum_{i=1}^n X_i \bar{K} \left(\frac{w - X_i}{h_n} \right) = \frac{1}{n} \sum_{i=1}^n X_i I_{(X_i > w)} + \frac{1}{n} \sum_{i=1}^n X_i \bar{K} \left(\frac{w - X_i}{h_n} \right) I_{(X_i \leq w)}.$$

It is clear that the first term

$$\frac{1}{n} \sum_{i=1}^n X_i I_{(X_i > w)} \rightarrow \mathbb{E}(X I_{(X > w)}) \quad a.s.$$

and the second term converges to 0 *a.s.* by Lemma A.2, hence the desired result is proved. $\square$

Lemma A.4. *It holds that*

$$\lim_{n \rightarrow +\infty} \int_w^{+\infty} \bar{F}_X^{(n)}(t) dt = \int_w^{+\infty} \bar{F}_X(t) dt, \quad a.s.$$

Proof. With the integration by parts,

$$\begin{aligned} \int_w^{+\infty} \bar{K} \left(\frac{t - X_i}{h_n} \right) dt &= t \bar{K} \left(\frac{t - X_i}{h_n} \right) \Big|_{t=w}^{t=+\infty} - \int_w^{+\infty} t d\bar{K} \left(\frac{t - X_i}{h_n} \right) \\ &= -w \bar{K} \left(\frac{w - X_i}{h_n} \right) + \int_w^{+\infty} t dK \left(\frac{t - X_i}{h_n} \right) \end{aligned}$$

From the definition of non-parametric kernel estimators $\bar{F}_X^{(n)}(t)$,

$$\begin{aligned} \int_w^{+\infty} \bar{F}_X^{(n)}(t) dt &= \frac{1}{n} \sum_{i=1}^n \int_w^{+\infty} \bar{K} \left(\frac{t - X_i}{h_n} \right) dt \\ &= -\frac{w}{n} \sum_{i=1}^n \bar{K} \left(\frac{w - X_i}{h_n} \right) + \frac{1}{n} \sum_{i=1}^n \int_w^{+\infty} t dK \left(\frac{t - X_i}{h_n} \right) \quad (A.1) \end{aligned}$$

For the first term in (A.1), it is clear that

$$-w \cdot \frac{1}{n} \sum_{i=1}^n \bar{K} \left(\frac{w - X_i}{h_n} \right) \rightarrow -w \bar{F}_X(w) \quad a.s. \quad n \rightarrow +\infty$$

For the second term in (A.1), we have

$$\begin{aligned} &\int_w^{+\infty} t dK \left(\frac{t - X_i}{h_n} \right) \\ &= \int_{\frac{w - X_i}{h_n}}^{+\infty} (X_i + h_n y) dK(y) = X_i \int_{\frac{w - X_i}{h_n}}^{+\infty} dK(y) + h_n \int_{\frac{w - X_i}{h_n}}^{+\infty} y dK(y) \end{aligned}$$

$$\begin{aligned}
 & \frac{1}{n} \sum_{i=1}^n \int_w^{+\infty} t dK \left(\frac{t - X_i}{h_n} \right) \\
 &= \frac{1}{n} \sum_{i=1}^n X_i \bar{K} \left(\frac{w - X_i}{h_n} \right) + h_n \cdot \frac{1}{n} \sum_{i=1}^n \int_{\frac{w - X_i}{h_n}}^{+\infty} y dK(y) \quad (\text{A.2}) \\
 & 0 \leq h_n \cdot \frac{1}{n} \sum_{i=1}^n \int_{\frac{w - X_i}{h_n}}^{+\infty} y dK(y) \leq h_n \eta,
 \end{aligned}$$

By the lemma A.3, the first term in (A.2) has limit $\int_w^{+\infty} y dF_X(y) = \mathbb{E}(XI_{(X>w)})$ and the second term in (A.2) has limit 0.

Combining the above, we conclude that

$$\begin{aligned}
 \int_w^{+\infty} \bar{F}_X^{(n)}(t) dt &= \frac{1}{n} \sum_{i=1}^n \int_w^{+\infty} \bar{K} \left(\frac{t - X_i}{h_n} \right) dt \\
 &\rightarrow -w \bar{F}_X(w) + \mathbb{E}(XI_{(X>w)}) = \int_w^{+\infty} \bar{F}_X(t) dt, \quad a.s.
 \end{aligned}$$

□

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