

On The Darboux Frame Motion of a Developable Surface

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Abstract

Developable ruled surfaces and the rotation-minimizing frames (RMF) have a crucial importance in computer-aided geometric design (CAGD) because of being unfoldable (developable) into a plane without distortion. Also, the RMF of curves are generally used in computer graphics. In this paper, we discuss the relationship between spherical motion and a developable ruled surface that has a line of curvature as a base curve. On the other hand, we use spherical linear interpolation (SLERP) to analyze the relationship between the trajectory generated on the unit sphere due to this spherical motion and the inflection curve. Moreover, we examine some invariant values of this developable ruled surface. We also present the S.Frenet frame of this trajectory based on general quaternion frame equations.

Keywords: Developable ruled surface, Lines of curvature, Rotation-minimizing frame, Darboux frame motion, Invariant derivatives.

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1 Introduction

CAGD provides stylish tools for many tasks in computer graphics, kinematics, mechanics, and robotics. Ravani presented the Cartesian motion of an end-effector using a set of key control configurations in (Ravani, 2009). In (Mohamed, Radwan and Abd Elmegeed, 2016), they focused on the implementation of the skin surface decomposition based on 3D-Delaunay algorithm and quadratic patches.

A ruled surface is a surface consists of a family of straight lines and given as

$$\mathbf{y}(u, v) = \mathbf{r}(v) + u\mathbf{d}(v) \quad (1.1)$$

where $\mathbf{r}(v)$ represents the base curve or the directrix. In addition, for the constant v , the equation $\mathbf{L}_v(u) = \mathbf{r}(v_0) + u\mathbf{d}(v_0)$ indicates a line that corresponds to the generator or the ruling of the ruled surface. If the tangent plane is constant at all points along a given ruling, then this

surface transforms to a developable ruled surface. Developable ruled surfaces are mostly used in marine craft surfaces (Frey and Mancewicz, 1992), and aircraft skins (Pegna and Wolter, 1992) and architectural structures. The Gaussian curvature of every point of these surfaces is zero (Kreyszig, 1991).

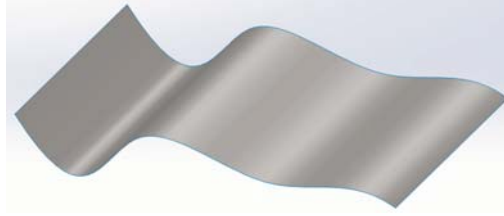


Figure 1: A developable ruled surface

A Gauss map shows the relationship of every point of a surface in R^3 to a corresponding point on the unit sphere, and developable ruled surfaces transform into the curve or a network of curves via the Gauss map. If this curve is a single curve, then the base curve of the developable ruled surface is a single continuous curve (Pottmann and Wallner, 2001).

Quaternions, especially unit quaternions, are commonly used in many fields, including robotics, animation and computer graphics. Spherical linear interpolation (SLERP) can be utilized to make 3D rotations in computer graphics because it guarantees a fixed angle or, in other words, a constant velocity. Therefore, SLERP is different from linear interpolation (Hanson, 2006). Rotation-minimizing frames (RMFs) of space curves are used for modeling the sweep surfaces in CAD (Juttler, 1998).

In this paper, we express the spherical motion of a developable ruled surface in which a line of curvature as a base curve and use SLERP to evaluate the relationship between the trajectory on the unit sphere which is formed as a result of this motion and the inflection curve. We also express the S.Frenet frame of this trajectory using general quaternion frame equations. The spherical motion of curves has been presented before, but in this study, we tackle the issue of the spherical motion of a developable ruled surface, which in conjunction with spherical motion and SLERP, contributes to various areas of research in robotics, computer graphics, and kinematics. Additionally, we examined some invariants of this developable ruled surface whose base curve is a line of curvature.

2 Lines of Curvature

The S. Frenet frame of a space curve $r(v)$ is given as,

$$r'(v) = T, \quad \frac{r' \times r''}{\|r' \times r''\|} \times T = N, \quad \frac{r' \times r''}{\|r' \times r''\|} = B, \quad (2.1)$$

where the unit vectors T , N and B are the tangent, principal and binormal vectors of the curve $r(v)$. $\kappa = \frac{\|r' \times r''\|}{\|r'\|^3}$ and $\tau = \frac{(r' \times r'') \cdot r'''}{\|r' \times r''\|^2}$ represent the curvature and torsion of the curve $r(v)$,

respectively (Kreyszig, 1991). For the curve $r(v)$ on the surface $y(u, v)$ the Darboux frame can be given as

$$T = \frac{v' y_v + u' y_u}{\|v' y_v + u' y_u\|}, \quad g = \frac{y_v \times y_u}{\|y_v \times y_u\|} \times T, \quad n = \frac{y_v \times y_u}{\|y_v \times y_u\|},$$

where T is the tangent vector, n is the surface normal along the curve and g is the tangent normal. The Darboux derivative formulas are

$$\begin{bmatrix} T' \\ g' \\ n' \end{bmatrix} = \sigma \begin{bmatrix} 0 & \kappa_g & \kappa_n \\ -\kappa_g & 0 & -\tau_g \\ -\kappa_n & \tau_g & 0 \end{bmatrix}, \quad (2.2)$$

with $\sigma = \|v' y_v + u' y_u\|$, $\kappa_n = \frac{n \cdot T'}{\sigma}$, $\kappa_g = \frac{g \cdot T'}{\sigma}$, $\tau_g = \frac{g \cdot n'}{\sigma}$ are the parametric velocity, normal curvature, geodesic curvature, and geodesic torsion, respectively. Additionally, a curve on the surface is called as a line of curvature if and only if $\tau_g = 0$ (Kreyszig, 1991), (Struik, 1988).

Definition: A moving frame that does not rotate about the instantaneous tangent of the spine curve $r(v)$ is an RMF of $r(v)$ (Wang, Juttler, Zheng and Liu, 2008).

Proposition 1: If the Darboux frame of a curve on a regular surface is an RMF with respect to the curve tangent, then this curve is called a line of curvature (Biard, Farouki and N., 2010).

3 Generating Developable Ruled Surfaces Using Lines of Curvature

There are some existing theorems that involve lines of curvature on a surface.

Theorem 1: A curve on a surface is a line of curvature if and only if the surface normals along the curve form a developable surface (Li, Wang and Zhu, 2013).

Theorem 2: When a space curve $r(v)$ is given, there exists a developable ruled surface $y(u, v)$ that possesses this curve as a line of curvature.

Proof: Let the normal surface of the curve $r(v)$ be,

$$y_1(u, v) = r(v) + un(v), \quad (3.1)$$

where $n(v) = N \cos \theta + B \sin \theta$ and $\theta = -\int_{v_0}^v \tau dv + \theta_0$. Then, using the equation

$$d(v) = n(v) \times T(v) = -g(v)$$

the surface is given as

$$y(u, v) = r(v) + ud(v). \quad (3.2)$$

Hence, $y(0, v) = r(v)$, and the surface $y(u, v)$ interpolates the curve $r(v)$ as in the following equation

$$\begin{aligned} y_u(u, v) \times y_v(u, v)|_{t=0} &= \left(r'(v) + u d'(v) \right) \times d(v) |_{u=0} \\ &= T(v) \times d(v) \end{aligned}$$

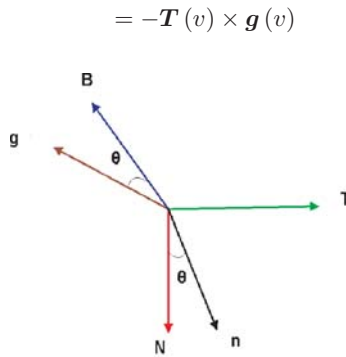


Figure 2: The S. Frenet Frame T , N , and B , and the vectors n and g

From Figure (2), we get $y_u(u, v) \times y_v(u, v)|_{u=0} = n(v)$. Therefore, the surface $y_1(u, v)$ is the normal surface of $y(u, v)$. Moreover, the surface $y_1(u, v)$ is a developable surface, and according to Theorem 1, it is also a developable surface. Since

$$d'(v) = -g'(v)$$

$$= B\theta' \sin \theta - B' \cos \theta + N\theta' \cos \theta + N' \sin \theta$$

$$= \tau B \sin \theta + \tau N \cos \theta - \tau N \cos \theta + (-\kappa T + \tau B) \sin \theta$$

$$= -\kappa \sin \theta T,$$

it is true that

$$(r', d, d') = (T, n \times T, -\kappa T \sin \theta) = 0.$$

Therefore, the surface $y(u, v)$ is a developable ruled surface.

4 Rotation-Minimizing Spherical Motions

The RMF of the curve is generated as the resulting motion when the unit tangent of this curve forms a the spherical trajectory. For a curve segment $r = r(v)$, $v \in [0, 1]$ is given on the unit sphere. The components of $r(v) = \begin{bmatrix} r_1 & r_2 & r_3 \end{bmatrix}^T$ are supposed to be C^1 functions.

In addition the curve $r(v)$ is shown as the trajectory drawn by $e_3 = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix}^T$ (Bottema and Roth, 1979). We then find that the orthogonal matrix $M(v)$ whose components are C^1 functions, satisfies the condition

$$r(v) = M(v) e_3. \quad (4.1)$$

The components of the curve $r(v)$ are in the third column of the matrix. In addition, the velocity of the point R_0 on the trajectory which is formed by the spherical motion $M(v)$ is $R' = \omega \times R$. In this equation, $\omega = \begin{bmatrix} \omega_1 & \omega_2 & \omega_3 \end{bmatrix}^T$, which shows the angular velocity of the spherical motion $M(v)$.

Proposition 2: The spherical motion $M(v)$ is a rotation-minimizing motion (RMM) with the trajectory $r(v)$ if and only if its angular velocity is

$$\omega(v) = r \times r'.$$

Corollary: The RMM $M(v)$ with the trajectory $r(v)$ does not depend on the choice of the parameterization of the given spherical curve (Juttler, 1998).

5 Quaternion Interpolation

A quaternion in R^4 is a 4-tuple. Thus, for the quaternion

$$q = a + bi + cj + dk,$$

where $a, b, c,$ and d are real numbers, $i, j,$ and k are the standart orthonormal base in R^3 and $i^2 = j^2 = k^2 = ijk = -1$. For the quaternions $q_1 = a_1 + u$ and $q_2 = a_2 + v$, the quaternion multiplication is

$$q_1 \star q_2 = (a_1 a_2 - u \cdot v, a_1 v + a_2 u + u \times v)$$

or in component form

$$q_1 \star q_2 = \begin{bmatrix} [q_1 \star q_2] \\ [q_1 \star q_2]_i \\ [q_1 \star q_2]_j \\ [q_1 \star q_2]_k \end{bmatrix} = \begin{bmatrix} a_1 a_2 - b_1 b_2 - c_1 c_2 - d_1 d_2 \\ b_1 a_2 + a_1 b_2 + c_1 d_2 - d_1 c_2 \\ c_1 a_2 + a_1 c_2 + d_1 b_2 - b_1 d_2 \\ d_1 a_2 + a_1 d_2 + b_1 c_2 - c_1 b_2 \end{bmatrix}.$$

A unit quaternion can be defined as $|q| = |\bar{q}| = 1$. Furthermore, if a quaternion rotates θ degrees around a vector n , it can be shown as (Hast, 2005), (Kuipers, 1999)

$$q = (\cos(\theta/2), n \sin(\theta/2)).$$

Unit quaternions can also be used to express rotations (Bottema and Roth, 1979) and they can be found for any rotation matrix $M(v)$ (Juttler, 1998).

A SLERP used in computer graphics is a geodesic arc on S^3 . It also provides the unit length of the interpolated vector and leaves it on the unit sphere. Also SLERP ensures smoothness and constant angular velocity. If the unit quaternions q_1 and q_2 are chosen on the sphere and are taken as the base, the equation for the spherical geodesic curve can be obtained from

$$q(v) = q_1 \frac{\sin(1-v)\varphi}{\sin\varphi} + q_2 \frac{\sin v\varphi}{\sin\varphi}, \quad (5.1)$$

in which $v \in [0, 1]$ and φ is the angle between q_1 and q_2 . In turn, this yields computed as

$$\varphi = \cos^{-1}(q_1 \cdot q_2) \quad (5.2)$$

Therefore, equation (5.1) gives big circles which correspond to the geodesics of the sphere (Hanson, 2006), (Kim and Nam, 1995).

6 The Spherical Motion of Developable Ruled Surfaces

The spherical curve is a trajectory drawn by a moving point on the unit sphere. The shortest distance between two points on the sphere is the arc of a big circle that connects these two points. The geodesic curvature of the big circles on the surface of a sphere is zero. On the other hand, if the geodesic curvature of a spherical curve is zero, then this spherical curve corresponds to the inflection curve. Therefore, an inflection curve forms a path in the shape of a circle arc (Kamphuis, 1969), (Roth and Yang, 1973).

Let the curve $\mathbf{r}(v)$ represent the base curve for the ruled surface $\mathbf{y}(u, v) = \mathbf{r}(v) + u\mathbf{d}(v)$, and be a line of curvature. Then, the surface $\mathbf{y}(u, v)$ can be taken as a developable ruled surface whose base curve is $\mathbf{r}(v)$ according to Theorem 1. As mentioned in Section 6, if the trajectory on the unit sphere is generated by the tangent vector of the base curve, then this motion will be the RMM of the curve $\mathbf{r}(v)$. Since the RMF of the line of curvature corresponds to the Darboux frame, the trajectory generated by the spherical motion will also be the trajectory generated by the Darboux frame of the base curve. Therefore, the RMM will correspond to the Darboux frame motion.

If the unit quaternion $q(v) = bi + cj + dk$ is chosen instead of the tangent vector of the base curve $\mathbf{r}(v)$, using SLERP, the generated motion will be the Darboux frame motion generated by this unit quaternion. Hence, the trajectory formed on the unit sphere corresponds to the big circle arc and the inflection curve as calculated via SLERP.

7 The S.Frenet Frame of the Trajectory Generated on the Unit Sphere by the Darboux Frame Motion

The correspondence between 3x3 matrices R_j^i and quaternions q can be given as

$$R_q(V)^i = \sum_j R_j^i V^j = q \star (0, V^i) \star q^{-1}. \quad (7.1)$$

Each orthonormal frame component is expressed as a column of R using an arbitrary quaternion as

$$\begin{aligned} \hat{T} &= q \star (0, \hat{x}) \star q^{-1}, \\ \hat{N} &= q \star (0, \hat{y}) \star q^{-1}, \\ \hat{B} &= q \star (0, \hat{z}) \star q^{-1}. \end{aligned} \quad (7.2)$$

All of these can be represented as (Hanson, 2006)

$$\begin{bmatrix} \hat{T} \\ \hat{N} \\ \hat{B} \end{bmatrix} = \begin{pmatrix} a^2 + b^2 - c^2 - d^2 & 2bc - 2ad & 2bd + 2ac \\ 2bc + 2ad & a^2 - b^2 + c^2 - d^2 & 2cd - 2ab \\ 2bd - 2ac & 2cd + 2ab & a^2 - b^2 - c^2 + d^2 \end{pmatrix} \quad (7.3)$$

Since the trajectory on the unit sphere generated by the Darboux frame motion of a developable ruled surface $\mathbf{y}(u, v)$ whose base curve is a line of curvature $\mathbf{r}(v)$ corresponds to the inflection curve, the equation of the inflection curve is

$$\begin{aligned} q(v) &= q_1 \frac{\sin(1-v)\varphi}{\sin\varphi} + q_2 \frac{\sin v\varphi}{\sin\varphi} \\ &= (b = q_{1b} \frac{\sin(1-v)\varphi}{\sin\varphi} + q_{2b} \frac{\sin v\varphi}{\sin\varphi}, \\ c &= q_{1c} \frac{\sin(1-v)\varphi}{\sin\varphi} + q_{2c} \frac{\sin v\varphi}{\sin\varphi}, \\ d &= q_{1d} \frac{\sin(1-v)\varphi}{\sin\varphi} + q_{2d} \frac{\sin v\varphi}{\sin\varphi}, \\ &= b\mathbf{i} + c\mathbf{j} + d\mathbf{k}. \end{aligned}$$

Thus, the components of this curve are

$$\begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} = \begin{bmatrix} 0 \\ q_{1b} \frac{\sin(1-t)\varphi}{\sin\varphi} + q_{2b} \frac{\sin t\varphi}{\sin\varphi} \\ q_{1c} \frac{\sin(1-t)\varphi}{\sin\varphi} + q_{2c} \frac{\sin t\varphi}{\sin\varphi} \\ q_{1d} \frac{\sin(1-t)\varphi}{\sin\varphi} + q_{2d} \frac{\sin t\varphi}{\sin\varphi} \end{bmatrix}.$$

We can also obtain the S.Frenet frame of the inflection curve from equation (7.3) as

$$\begin{bmatrix} \hat{T} \\ \hat{N} \\ \hat{B} \end{bmatrix} = \begin{bmatrix} b^2 - c^2 - d^2 & 2bc & 2bd \\ 2bc & b^2 + c^2 - d^2 & 2cd \\ 2bd & 2cd & -b^2 - c^2 + d^2 \end{bmatrix}, \quad (7.4)$$

or

$$\begin{aligned} \begin{bmatrix} \hat{T} \end{bmatrix} &= \begin{bmatrix} b^2 - c^2 - d^2 \\ 2bc \\ 2bd \end{bmatrix}, \quad \begin{bmatrix} \hat{N} \end{bmatrix} = \begin{bmatrix} 2bc \\ b^2 + c^2 - d^2 \\ 2cd \end{bmatrix}, \\ \begin{bmatrix} \hat{B} \end{bmatrix} &= \begin{bmatrix} 2bd \\ 2cd \\ -b^2 - c^2 + d^2 \end{bmatrix}. \end{aligned}$$

8 Invariant Derivatives along The Lines of Curvature

Let us take the developable ruled surface as $\mathbf{y}(u, v)$ and make parameter transformation as

$$u = u(u^*, v^*), \quad v = v(u^*, v^*). \quad (8.1)$$

After this transformation, we can rewrite the developable ruled surface $\mathbf{y}(u, v)$ as in the following equation:

$$\mathbf{y}(u, v) = \mathbf{y}(u[u^*, v^*], v[u^*, v^*]) = \mathbf{y}^*(u^*, v^*).$$

Our purpose is to find invariants related to only curvature properties of our developable ruled surface $\mathbf{y}(u, v)$. Certain statements are invariants based on geometric means, for example, principal curvature radii R_1 and R_2 , Gauss curvature K and mean curvature H . However, we want to obtain an idea about all existing invariants of our surface systematically only by means of computation. If we can relate our surface to the curve network, which has geometric (invariant) meaning, then we can get closer our goal. Let's take lines of curvature network, since they are always real. In this case, we do not take umbilic points. If we take lines of curvature as parameter curves, then from the first fundamental form, we have $F = M = 0$. Directions of the vectors $\mathbf{y}_u$ and $\mathbf{y}_v$ have geometric (invariant) meaning depend only on the developable ruled surface $\mathbf{y}(u, v)$. However, lengths of the vectors $\mathbf{y}_u$ and $\mathbf{y}_v$ are not invariant. We can make transformations without changing parameter curves network as

$$u = f(u^*), \quad v = g(v^*) \quad \left[\frac{df}{du^*} \neq 0, \frac{dg}{dv^*} \neq 0 \right]. \quad (8.2)$$

In this case, it can be found that $\partial \mathbf{y}^* : \partial u^* = [\partial \mathbf{y} : \partial u] \cdot f'$, where $f' = df : du^*$. Hence, changing both of the scales, which attach only curve teams, corresponds to the transformation in (8.2). This transformation does not generally change the curve network.

Now, we can determine the change in every quantity that belongs to the developable ruled surface $\mathbf{y}(u, v)$ according to the transformation in (8.2). First of all, let's determine S as the absolute invariant of the surface $\mathbf{y}(u, v)$. For example, we can take S as the invariants Gauss curvature K or mean curvature H . In this case, even if the lines of curvature parameters are used, derivatives S_u and S_v are not invariants anymore. Because they turn into the status below.

$$\frac{\partial S^*}{\partial u^*} = S_u \cdot f' \quad \text{and} \quad \frac{\partial S^*}{\partial v^*} = S_v \cdot g'. \quad (8.3)$$

However, we can obtain two new invariants by taking derivative of the invariant S . The quantities S_1 and S_2 , which are given with the following equations respectively, are invariants that correspond to the S_u and S_v which are not invariant.

$$S_1 = \frac{S_u}{\sqrt{E}}, \quad \text{and} \quad S_2 = \frac{S_v}{\sqrt{G}}. \quad (8.4)$$

Since, $E = y_u^2$, $F = y_u y_v$ and $G = y_v^2$, we can write the relations as below

$$E^* = E.f'^2 \quad \text{and} \quad G^* = G.g'^2. \quad (8.5)$$

Hence, the following equations can be expressed using (8.3) and (8.5) with

$$\frac{1}{\sqrt{E^*}} \frac{\partial S^*}{\partial u^*} = \frac{S_u}{\sqrt{E}} \quad \text{and} \quad \frac{1}{\sqrt{G^*}} \frac{\partial S^*}{\partial v^*} = \frac{S_v}{\sqrt{G}}. \quad (8.6)$$

The quantities S_1 and S_2 are called invariant derivatives of S . Geometrically, S_1 and S_2 are derivatives formed only through lines of curvature of S , which is given as location function on the surface, and according to the arc length of them. Arc elements of the lines of curvature u and v can be given respectively as

$$ds_1 = \sqrt{E} du \quad \text{and} \quad ds_2 = \sqrt{G} dv. \quad (8.7)$$

Therefore, since $dS = S_u du + S_v dv$, we can obtain the following relations.

$$S_1 = \frac{(dS)_{dv=0}}{ds_1} \quad \text{and} \quad S_2 = \frac{(dS)_{du=0}}{ds_2}. \quad (8.8)$$

The arc elements ds_1 and ds_2 are two invariant differentials of the developable ruled surface $y(u, v)$. The values ds_1 and ds_2 for a line element of the developable ruled surface $y(u, v)$, where the line element does not exist on the lines of curvature and joins (u, v) to $(u + du, v + dv)$, are only the projections on the two lines of curvature that pass through the points u and v (Blaschke, 1950).

9 Isometric Mapping Provided That Lines of Curvature are Preserved

First of all, we will give the theorem below to clarify isometric mapping.

Theorem 3: If the linear forms of a surface with

$$ds_1 = n_1 du + n_2 dv \quad (9.1)$$

$$ds_2 = \bar{n}_1 du + \bar{n}_2 dv$$

and the mean curvature $2H = r + \bar{r}$ are given as a function of arbitrary parameters, then this surface is determined with movement difference. Here, r and $\bar{r}$ are principal curvatures of the surface, and, n_i and $\bar{n}_i$, ($i = 1, 2$) are defined as

$$n_i = \sqrt{\frac{R_2}{R_1 - R_2}} p_i \quad \bar{n}_i = \sqrt{\frac{R_1}{R_2 - R_1}} \bar{p}_i, \quad (i = 1, 2)$$

where R_1 and R_2 are principal curvature radii. If we take lines of curvature as parameter curves then we can write the values p_1 , p_2 , $\bar{p}_1$ and $\bar{p}_2$ as

$$p_1 = \sqrt{\frac{G}{N} : \frac{E}{L} - 1} \cdot \sqrt{E}, \quad p_2 = 0$$

$$\bar{p}_1 = 0, \quad \bar{p}_2 = \sqrt{\frac{E}{L} : \frac{G}{N} - 1} \cdot \sqrt{G}.$$

If the forms in (9.1) are found same for two surfaces that are made equate with each other as point to point with the same values of u and v , then, the first fundamental forms of arc elements of these surfaces in accordance with $[I] = ds_1^2 + ds_2^2$ will also be equal. Hence, we have two surfaces which are isometric or interconvertible with a twist. Also, since the lines of curvatures of the surfaces are obliged to correspond to each other, this mapping is an isometric mapping that preserves the lines of curvature. Vice versa is also true.

Lemma 1: Generally, a surface is determined as only in one way with the linear form given in (9.1) by a movement.

Principal curvatures r and $\bar{r}$, and also principal curvature radii R_1 and R_2 are already determined for the surface by only one common sign difference. Therefore, in order to determine of the surface from its own linear forms given in (9.1). For example, it is enough to compute the Gauss curvature

$$r\bar{r} = K \quad (9.2)$$

and the quantity,

$$r^2 = s \quad (9.3)$$

from these forms, where s is used for r^2 . On the other hand, there are following relationships between invariants of the surface using Gauss's Theorema Egregium and Codazzi Equations as

$$-\bar{q}_1 - q_2 - \bar{q}^2 - q^2 = r\bar{r} = K \quad (9.4)$$

$$r_2 = q(\bar{r} - r) \quad (9.5)$$

$$\bar{r}_1 = \bar{q}(r - \bar{r}), \quad (9.6)$$

where the values q and $\bar{q}$ are the geodesic curvatures of curvature strips. According to the equation (9.4), $K = r\bar{r}$ can be determined by using the forms given in (9.1) and quantities q , $\bar{q}$, $\bar{q}_1$ and q_2 which can be extracted from (9.1).

Let us a developable ruled surface $y(u, v)$ is given and for this situation, for example, we can accept

$$\bar{r} = 0 \quad (r \neq 0). \quad (9.7)$$

In this case, the equations $\bar{q} = 0$ and $q_2 + q^2 = 0$ are obtained from (9.4) and (9.6), respectively. Therefore, for r , only the equation

$$\frac{r_2}{r} = -q \quad (9.8)$$

or for s , only the equation

$$(\lg s)_2 = -2q \quad (9.9)$$

remains, where s is not determined as univocal. If we use parameters of lines of curvature, then we can express ds_1 and ds_2 as

$$ds_1 = \sqrt{E}du \quad \text{and} \quad ds_2 = \sqrt{G}dv. \quad (9.10)$$

We can rewrite the equation (9.9) by using the equations (8.4) and

$$q = \frac{1}{2} \frac{E_v}{E\sqrt{G}} \quad \text{and} \quad \bar{q} = \frac{1}{2} \frac{G_u}{G\sqrt{E}},$$

as

$$(\lg s)_v = -(\lg E)_v. \quad (9.11)$$

Hence, the following equation is found as

$$s = \frac{1}{E} U(u), \quad (9.12)$$

where U is only a random function of u , and s can be determined by comprising an arbitrary function of a variable. Since, s is an invariant, developable ruled surfaces $y(u, v)$ and others, which belong to the several U , can not be obtained from each other by movement. Now then, corresponding to a developable ruled surface $y(u, v)$, there are always a surface family, which belong to an arbitrary function, such that, this family and developable ruled surface $y(u, v)$ have same forms from point to point in a convenient way given in (9.1). Since, the surfaces, which have the same forms given in (9.1), can be converted to each other with twist, these surfaces are inherently developable ruled surfaces, too.

Since, the rulings in developable ruled surfaces constitute a lines of curvature group, their orthogonal trajectories give the other group. Therefore, we obtain an isometric mapping of two developable ruled surfaces whose rulings correspond to each other.

A developable ruled surface $y(u, v)$ can be a cylinder or a cone or a surface formed from the tangents of a space curve. Cylinders are characterized with the equations, which are applicable for all developable ruled surfaces, as

$$\bar{r} = 0, \quad \bar{q} = 0, \quad q_2 + q^2 = 0 \quad (9.13)$$

and also, with the equation below

$$q = 0. \quad (9.14)$$

Since, $q = 0$ and $\bar{q} = 0$, then all invariant derivatives of q and $\bar{q}$ are also zero. Hence, all invariants, which can be computed from the forms given in (9.1), are equal to each other in all cylinders. On the other hand, since the relationship (9.14) is a condition which is shown only with the forms given in (9.1), cylinders can have equal linear forms only with cylinders.

In general cones, the condition below

$$q_1 = 0 \quad (9.15)$$

is taken in addition to the relationships in (9.13). A cone has only one invariant that can be calculated only from the forms given in (9.1), and this invariant is $q = 0$ because of the relationship $q_2 = -q^2$ given in (9.13). Therefore, all cones, which have same invariant q , have same forms given in (9.1), when their surface parameters in similar points correspond in an appropriate manner. The quantity q is the geodesic curvature of the lines of curvature, which are not lines, of a cone, i.e. q is the inverse of the radii of circles that lines of curvatures transform when the cone is unfolded (developed) into a plane.

Now, if we take the surface $y(u, v)$ formed from the tangents of a space curve, then

$$q_1 \neq 0. \quad (9.16)$$

In this case, the developable ruled surface $y(u, v)$ is a surface composed of the tangents of the space curve

$$z = x - \frac{1}{q}x_2, \quad (9.17)$$

where $x_2 = \sum \bar{n}^i \frac{\partial x}{\partial w^i}$. The curvature κ and the torsion τ of this curve are given respectively as

$$\kappa^2 = \frac{(z_{11}^2)(z_1^2) - (z_1 z_{11})^2}{(z_1^2)^3}$$

and

$$\frac{1}{\tau} = \frac{|z_1 z_{11} z_{111}|}{(z_{11}^2)(z_1^2) - (z_1 z_{11})^2}.$$

Using the relationships below,

$$x_{11} = -qx_2 + rn, \quad x_{12} = \bar{q}x_2$$

$$x_{22} = -\bar{q}x_1 + \bar{r}n, \quad x_{21} = qx_1$$

$$n_1 = -rx_1, \quad n_2 = -\bar{r}x_2,$$

and (9.13), we can find the following relationships as

$$z_1 = \frac{q_1}{q^2}x_2 \quad \text{and} \quad z_{11} = \left(\frac{q_1}{q^2}\right)_1 x_2 + \frac{q_1}{q}x_1$$

$$z_{111} = \left[2(\lg q)_{11} - (\lg q)_1^2\right]x_1 + [\dots]x_2 + r\frac{q_1}{q}n.$$

Hence, the relationships given with

$$\frac{1}{\rho^2} = \frac{q^6}{q_1^2} \quad \text{and} \quad \frac{1}{\tau} = -\frac{rq^2}{q_1}$$

are obtained. Therefore, the curvature κ can be determined only from the forms given in (9.1), but the torsion τ can be completely an arbitrary function.

Lemma 2: Two surfaces are composed of the tangents of two space curves, which correspond to each other as having same arc lengths and equal curvatures in similar points, have same linear forms given in (9.1), when the parameters correspond to each other in a convenient way. Moreover, surfaces composed of the tangents of space curves, which have equal curvature, can have same linear form (Blaschke, 1950).

10 Application

Example 1: Let a line of curvature be given as $\mathbf{r}(v) = (3 \cos v, 3 \sin v, v)$. A developable ruled surface (see Figure((3))) which has this curve as the base curve is

$$\begin{aligned} \mathbf{y}(u, v) = & (3 \cos v, 3 \sin v, v) \\ & + u \left[-\frac{1}{\sqrt{10}} \sin \left(\frac{1}{\sqrt{10}} v + \frac{\pi}{3} \right) \sin v \right. \\ & - \sin \left(-\frac{1}{\sqrt{10}} v + \frac{\pi}{6} \right) \cos v, \\ & - \sin v \sin \left(-\frac{1}{\sqrt{10}} v + \frac{\pi}{6} \right) \\ & + \frac{1}{\sqrt{10}} \sin \left(\frac{1}{\sqrt{10}} v + \frac{\pi}{3} \right) \cos v, \\ & \left. - \frac{3}{\sqrt{10}} \sin \left(\frac{1}{\sqrt{10}} v + \frac{\pi}{3} \right) \right] \end{aligned}$$

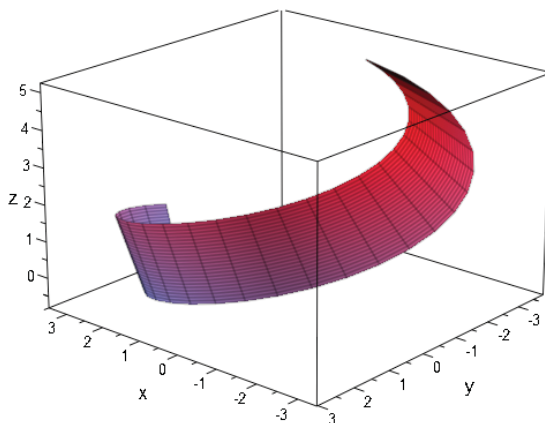


Figure 3: The developable ruled surface $\mathbf{y}(u, v)$.

The unit tangent vector, unit principal, and unit binormal vectors of the curve $\mathbf{r}(v)$ can then be given as

$$\mathbf{T}(v) = \frac{(-3 \sin v, 3 \cos v, 1)}{\sqrt{10}}, \quad \mathbf{N}(v) = (-\cos v, -\sin v, 0),$$

and

$$B(v) = \frac{(\sin v, -\cos v, 3)}{\sqrt{10}}$$

Next, let q_1 and q_2 be two unit quaternions taken as the base, with v being a scalar, $q(v)$ being at q_1 for $v = 0.0$ and at q_2 for $v = 1.0$, and φ being the angle between q_1 and q_2 . For q_i ($i = 1, 2$) and φ , the values are the following:

$$q_1 = 0\mathbf{i} + \frac{3}{\sqrt{10}}\mathbf{j} + \frac{1}{\sqrt{10}}\mathbf{k}, \quad \|q_1\| = 1$$

$$q_2 = -\frac{3}{\sqrt{10}}\mathbf{i} + 0\mathbf{j} + \frac{1}{\sqrt{10}}\mathbf{k}, \quad \|q_2\| = 1$$

$$\varphi = \arccos \frac{1}{10}, \quad v \in [0, 1].$$

We can see different trajectories on the unit sphere generated by the Darboux frame motion of the developable ruled surface $\mathbf{y}(u, v)$ from the Figures (4) and (5). Additionally, the S. Frenet frame for $v = 0$ from equation (7.4) can then be identified as

$$\begin{bmatrix} [\hat{T}] & [\hat{N}] & [\hat{B}] \end{bmatrix} = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 0.8 & 0.6 \\ 0 & 0.6 & -0.8 \end{bmatrix},$$

$$\begin{bmatrix} \hat{T} \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \\ 0 \end{bmatrix}, \quad \begin{bmatrix} \hat{N} \end{bmatrix} = \begin{bmatrix} 0 \\ 0.8 \\ 0.6 \end{bmatrix}, \quad \begin{bmatrix} \hat{B} \end{bmatrix} = \begin{bmatrix} 0 \\ 0.6 \\ -0.8 \end{bmatrix}.$$

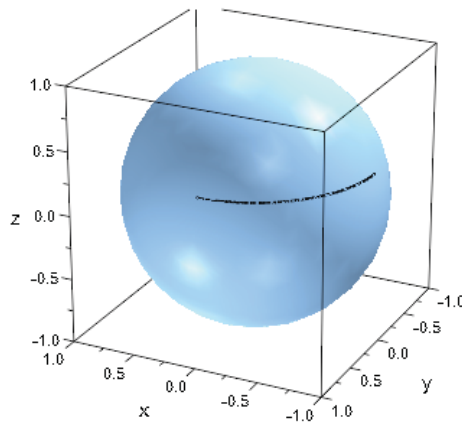


Figure 4: The trajectory on the unit sphere generated by the Darboux frame motion of the developable ruled surface $\mathbf{y}(u, v)$.

Example 2: Let a quadratic Bezier curve $\mathbf{r}(v) = (2v^2, 4v, 4v - 4v^2)$ represent a line of curvature with the control points of the curve being $P_0 = (0, 0, 0)$, $P_1 = (0, 2, 2)$, and $P_2 = (2, 4, 0)$. A

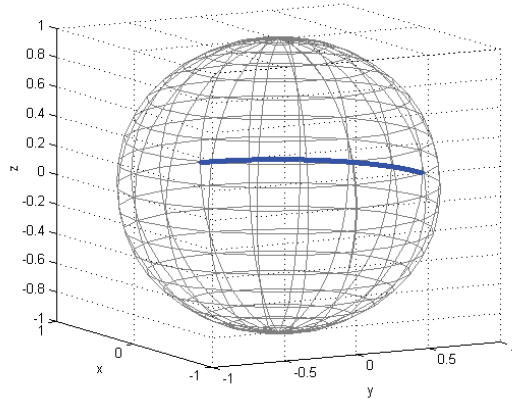


Figure 5: The trajectory on the unit sphere generated by the Darboux frame motion of the developable ruled surface $y(u, v)$ by using SLERP.

developable ruled surface (see Figure(6)) with this curve as the base curve can then be written as

$$y(u, v) = (2v^2, 4v, 4v - 4v^2) + u \left(\frac{2}{\sqrt{6}}, -\frac{1}{\sqrt{6}}, \frac{1}{\sqrt{6}} \right).$$

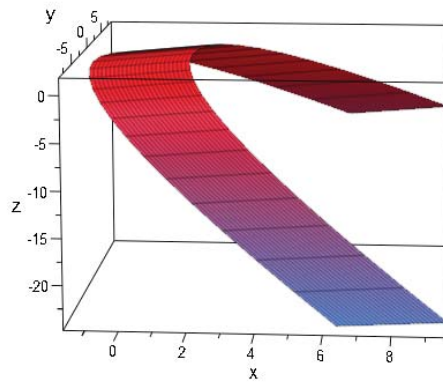


Figure 6: The developable ruled surface $y(u, v)$.

The unit tangent vector, unit principal and unit binormal vectors of the curve $r(v)$ are

$$T(v) = \frac{(v, 1, 1 - 2v)}{\sqrt{5v^2 - 4v + 2}}, \quad N(v) = \frac{(-2v + 2, -5v + 2, -v - 2)}{\sqrt{6(5v^2 - 4v + 2)}},$$

and

$$B(v) = \frac{(-2, 1, -1)}{\sqrt{6}}.$$

Let q_1 and q_2 be two unit quaternions taken as basis, v be a scalar, $q(v)$ be at q_1 for $v = 0.0$, at q_2 for $v = 1.0$ and φ be the angle between q_1 and q_2 . For q_i ($i = 1, 2$) and φ , the values are taken as

$$q_1 = 0\mathbf{i} + \frac{1}{\sqrt{2}}\mathbf{j} + \frac{1}{\sqrt{2}}\mathbf{k}, \quad \|q_1\| = 1$$

$$q_2 = \frac{1}{\sqrt{3}}\mathbf{i} + \frac{1}{\sqrt{3}}\mathbf{j} - \frac{1}{\sqrt{3}}\mathbf{k}, \quad \|q_2\| = 1$$

$$\varphi = \arccos 0, \quad v \in [0, 1].$$

We can see different trajectories on the unit sphere generated by the Darboux frame motion of the developable ruled surface $\mathbf{y}(u, v)$ from the Figures (7) and (8). Furthermore, the unit tangent vector, unit principal, and unit binormal vectors of the curve $\mathbf{r}(v)$ are

$$\begin{bmatrix} \hat{\mathbf{T}} \\ \hat{\mathbf{N}} \\ \hat{\mathbf{B}} \end{bmatrix} = \begin{bmatrix} -1/3 & 2/3 & -2/3 \\ 2/3 & -1/3 & -2/3 \\ -2/3 & -2/3 & -1/3 \end{bmatrix},$$

$$\begin{bmatrix} \hat{\mathbf{T}} \end{bmatrix} = \begin{bmatrix} -1/3 \\ 2/3 \\ -2/3 \end{bmatrix}, \quad \begin{bmatrix} \hat{\mathbf{N}} \end{bmatrix} = \begin{bmatrix} 2/3 \\ -1/3 \\ -2/3 \end{bmatrix}, \quad \begin{bmatrix} \hat{\mathbf{B}} \end{bmatrix} = \begin{bmatrix} -2/3 \\ -2/3 \\ -1/3 \end{bmatrix}.$$

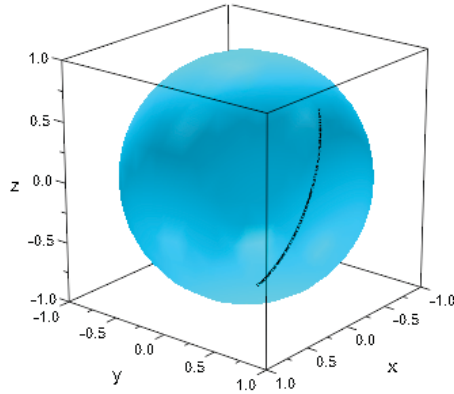


Figure 7: The trajectory on the unit sphere generated by the Darboux frame motion of the developable ruled surface $\mathbf{y}(u, v)$.

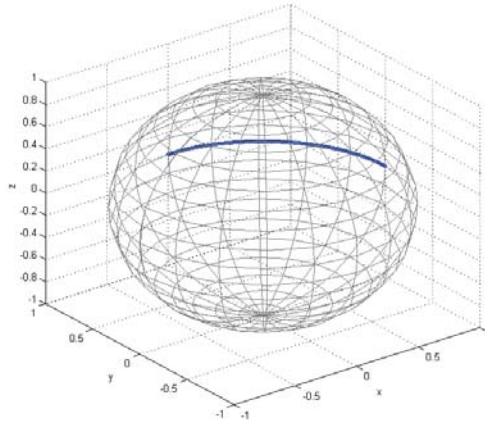


Figure 8: The trajectory on the unit sphere generated by the Darboux frame motion of the developable ruled surface $y(u, v)$ by using SLERP.

11 Conclusion

In this paper, we determined that the RMM of the developable ruled surface whose base curve is a line of curvature corresponds to the Darboux frame motion of this surface. In addition, we also identified the relationship between the trajectory formed by this motion and the inflection curve on the unit sphere using SLERP. On the other hand, we examined some invariants of this developable ruled surface that has a line of curvature base curve. Furthermore, we presented the S. Frenet frame of the trajectory generated by the Darboux frame motion on the unit sphere utilizing general quaternion frame equations and other applications which were provided.

We believe that this study offers a different perspective regarding the relationship between developable ruled surfaces which are very important in CAGD and spherical motion which is widely used in fields such as spherical mechanisms, path planning and animation.

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