

Generalized two-dimensional quaternionic fractional Mellin transform

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ABSTRACT

In this study, authors have established two-dimensional quaternionic fractional Mellin transforms in a distributional sense. The testing function space and some properties are studied. The analyticity, boundedness, and inversion formula are analyzed. Operational properties and Mellin-type convolution properties are established. The applications related to Mathematical Physics are given in the concluding section.

Keywords: quaternion fractional Mellin transform, analyticity, testing function space, boundedness, inversion, convolution.

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1 Introduction

W. R. Hamilton developed quaternions in 1853. The necessity of enlarging the operations on three-dimensional vectors to include multiplication and division led Hamilton to introduce the four-dimensional algebra of quaternions. In 1998, authors (Akay and Boudreaux-Bartels, 1998) established fractional Mellin transform as an extension of the fractional frequency concept for scale. Digital computation of the fractional Mellin transform was analyzed in (Biner and Akay, 2005). In (Klimek and Dziembowski, 2008), authors developed right-sided derivative and variable potential to fractional differential equations. In (Sharma and Deshmukh, 2009), a two-dimensional fractional Mellin transform was extended to the distribution of compact support. Extensions of the distributional Mellin transform to Boehmian spaces were studied in (Al-Omari, 2011). Generalized functions were represented in (Al-Omari and Kiliçman, 2013). The author in (Hitzer, 2012) introduced the quaternionic Fourier-Mellin transform. The class of quaternions was investigated by (Al-Omari and Baleanu, 2016) for Mellin transform. In (Nefzi, Brahim and Fitouhi, 2018), the finite Mellin transforms in quantum calculus and application was established. In a recent study (Lakshmi Gorty and Parmar, 2019), the two-dimensional quaternionic fractional Mellin transform is defined and various results were derived. In this study, the authors have defined generalized two-dimensional quaternionic fractional

Mellin transforms. The testing spaces, analyticity, boundedness, and inversion formula are established. Operational properties are also proved. In the concluding part of the study, Mellin-type convolution properties and applications are studied.

2 Preliminary results

Let q be a quaternion defined in

$$\mathbb{H} = \{q = x_0 + ix_1 + jx_2 + kx_3 : x_0, x_1, x_2, x_3 \in \mathbb{R}\} \quad (2.1)$$

be the division ring of quaternions, where i, j, k satisfy Hamilton's multiplication rules (see, e.g. (Ell, 1993))

$$ij = -ji = k, jk = -kj = i, ki = -ik = j, i^2 = j^2 = k^2 = ijk = -1. \quad (2.2)$$

The quaternion conjugate of q is defined by

$$\bar{q} = x_0 - ix_1 - jx_2 - kx_3; \quad x_0, x_1, x_2, x_3 \in \mathbb{R}. \quad (2.3)$$

The norm of $q \in \mathbb{H}$ is defined as

$$|q| = \sqrt{q\bar{q}} = \sqrt{x_0^2 + x_1^2 + x_2^2 + x_3^2}. \quad (2.4)$$

If $f \in L^2(\mathbb{R}^2; \mathbb{H})$, then the function is expressed as

$$f(x, y) = f_0(x, y) + if_1(x, y) + jf_2(x, y) + kf_3(x, y). \quad (2.5)$$

Definition 2.1. For every $f \in L^2(\mathbb{R}^2, \mathbb{H})$, two-dimensional quaternionic fractional Mellin transform (2D-QFrMT) of particular order α, β is defined in (Lakshmi Gorty and Parmar, 2019) as

$$\mathcal{M}_{\alpha, \beta}[f(u, v); s_\alpha, s_\beta] = \int_0^\infty \int_0^\infty u^{s_\alpha-1} f(u, v) v^{s_\beta-1} dudv \quad (2.6)$$

where $s_\alpha = a - iw_1^{\frac{1}{\alpha}}; s_\beta = b - jw_2^{\frac{1}{\beta}}$ with $a_1 \leq a \leq a_2, b_1 \leq b \leq b_2$ and $0 < \alpha, \beta \leq 1$.

3 Main Results

Testing Function Space: $B(c, d)$ is the space of infinitely differentiable quaternion-valued smooth function $\phi(x, y)$ on $\mathbb{R}^n$ such that for each compact set $I \subset S_c, K \subset S_d$; where $S_c = \{x : x \in \mathbb{R}^n, |x| \leq c, c > 0\}, S_d = \{y : y \in \mathbb{R}^n, |y| \leq d, d > 0\}$

$$\gamma_{D, q, \lambda}(\phi(x, y)) = \sup_{x \in I, y \in K} \left| D_x^q D_y^\lambda \phi(x, y) \right| < \infty. \quad (3.1)$$

$B(c, d)$ is a linear space under the usual definition of addition and multiplication. Thus $B(c, d)$ will denote the space of all $\phi(x, y)$ with support contained in S_c and S_d . Each $\gamma_{D, q, \lambda}$ is a seminorm on $B(c, d)$ and the collection $\{\gamma_{D, q, \lambda}\}$ is a multinorm. The topology of $B(c, d)$ is generated by $\{\gamma_{D, q, \lambda}\}$. By lemma 3.2-1 of (Zemanian, 1968) $B(c, d)$ is complete and therefore a Fréchet space. Let $B^*(c, d)$ denote the dual space of $B(c, d)$.

Definition 3.1. The generalized 2D-QFrMT of $f(x, y) \in B^*(c, d)$ can be defined as

$$FrMT\{f(x, y)\} = F_{\alpha, \beta}(s_{\alpha}, s_{\beta}) = \langle f(x, y), K_{\alpha, \beta}(x, y, s_{\alpha}, s_{\beta}) \rangle \quad (3.2)$$

where $K_{\alpha, \beta}(x, y, s_{\alpha}, s_{\beta}) = x^{s_{\alpha}-1}y^{s_{\beta}-1}$ is the kernel of the transform and $K_{\alpha, \beta}(x, y, s_{\alpha}, s_{\beta}) \in B(c, d)$.

Theorem 3.1 (Analyticity Theorem). *Let $f \in B^*(c, d)$ and its quaternion fractional Mellin transform be defined by (3.2). Then $F_{\alpha, \beta}(s_{\alpha}, s_{\beta})$ is analytic on $\mathbb{C}^n$. Moreover $F_{\alpha, \beta}(s_{\alpha}, s_{\beta})$ is differentiable and*

$$D_{s_{\alpha}}^q D_{s_{\beta}}^l F_{\alpha, \beta}(s_{\alpha}, s_{\beta}) = \left\langle f(x, y), D_{s_{\alpha}}^q D_{s_{\beta}}^l K_{\alpha, \beta}(x, y, s_{\alpha}, s_{\beta}) \right\rangle. \quad (3.3)$$

Proof. For fixed $s_{\alpha j} \neq 0$ choose two concentric circles C and C_1 with centre at $s_{\alpha j}$ and radius r and r_{α} respectively. Let Δs_{α} be a small increment. Also for fixed $s_{\beta j} \neq 0$ choose two concentric circles C' and C'_1 with centre at $s_{\beta j}$ and radius r' and r_{β} respectively. Let Δs_{β} be a small increment.

Consider

$$\frac{F_{\alpha, \beta}(s_{\alpha j} + \Delta s_{\alpha}, s_{\beta j}) - F_{\alpha, \beta}(s_{\alpha j}, s_{\beta j})}{\Delta s_{\alpha}} \frac{F_{\alpha, \beta}(s_{\alpha j}, s_{\beta j} + \Delta s_{\beta}) - F_{\alpha, \beta}(s_{\alpha j}, s_{\beta j})}{\Delta s_{\beta}} - \left\langle f(x, y), \frac{\partial}{\partial s_{\alpha j}} \frac{\partial}{\partial s_{\beta j}} K_{\alpha, \beta}(x, y, s_{\alpha}, s_{\beta}) \right\rangle = \langle f(x, y), \psi_{\Delta s_{\alpha} \Delta s_{\beta}}(x, y) \rangle$$

where

$$\begin{aligned} \psi_{\Delta s_{\alpha} \Delta s_{\beta}}(x, y) &= \frac{1}{\Delta s_{\alpha}} \{K_{\alpha, \beta}(x, y, s_{\alpha j} + \Delta s_{\alpha}, s_{\beta j}) - K_{\alpha, \beta}(x, y, s_{\alpha j}, s_{\beta j})\} \\ &\quad \times \frac{1}{\Delta s_{\beta}} \{K_{\alpha, \beta}(x, y, s_{\alpha j}, s_{\beta j} + \Delta s_{\beta}) - K_{\alpha, \beta}(x, y, s_{\alpha j}, s_{\beta j})\} \\ &\quad - \frac{\partial}{\partial s_{\alpha j}} \frac{\partial}{\partial s_{\beta j}} K_{\alpha, \beta}(x, y, s_{\alpha}, s_{\beta}). \end{aligned} \quad (3.4)$$

For all $z \in C$; $w \in C_1$ and for any fixed integer q and l , $D_x^q D_y^l K_{\alpha, \beta}(x, y, \tilde{s}_{\alpha}, \tilde{s}_{\beta})$ is analytic inside and on C, C_1 respectively.

By Cauchy's integral formula, we have

$$\begin{aligned} &D_x^q D_y^l \psi_{\Delta s_{\alpha} \Delta s_{\beta}}(x, y) \\ &= \frac{1}{4\pi^2 i j} D_x^q D_y^l \int_C \int_{C_1} K_{\alpha, \beta}(x, y, \tilde{s}_{\alpha}, \tilde{s}_{\beta}) \\ &\quad \times \left[\frac{1}{\Delta s_{\alpha}} \left(\frac{1}{z - s_{\alpha j} - \Delta s_{\alpha}} - \frac{1}{z - s_{\alpha j}} - \frac{1}{(z - s_{\alpha j})^2} \right) \right] \\ &\quad \times \left[\frac{1}{\Delta s_{\beta}} \left(\frac{1}{w - s_{\beta j} - \Delta s_{\beta}} - \frac{1}{w - s_{\beta j}} - \frac{1}{(w - s_{\beta j})^2} \right) \right] dz dw \\ &= \frac{\Delta s_{\alpha} \Delta s_{\beta}}{4\pi^2 i j} \int_C \int_{C_1} \frac{D_x^q D_y^l K_{\alpha, \beta}(x, y, \tilde{s}_{\alpha}, \tilde{s}_{\beta}) dz dw}{(z - s_{\alpha j} - \Delta s_{\alpha})(z - s_{\alpha j})^2 (w - s_{\beta j} - \Delta s_{\beta})(w - s_{\beta j})^2}. \end{aligned}$$

Let $D_x^q D_y^l K_{\alpha, \beta}(x, y, \tilde{s}_{\alpha}, \tilde{s}_{\beta})$ be bounded by a constant Q .

Also $|(z - s_{\alpha j} - \Delta s_{\alpha})| > r_{\alpha} - r > 0$, $|z - s_{\alpha j}| = r_{\alpha}$, $|(w - s_{\beta j} - \Delta s_{\beta})| > r_{\beta} - r > 0$ and

$$|w - s_{\beta j}| = r_{\beta}.$$

$$\begin{aligned} \left| D_x^q D_y^l \psi_{\Delta s_{\alpha} \Delta s_{\beta}}(x, y) \right| &\leq \frac{|\Delta s_{\alpha}| |\Delta s_{\beta}|}{4\pi^2} \int_C \int_{C_1} \frac{Q |dz| |dw|}{(r_{\alpha} - r) r_{\alpha}^2 (r_{\beta} - r) r_{\beta}^2} \\ &\leq \frac{|\Delta s_{\alpha}| |\Delta s_{\beta}| Q}{(r_{\alpha} - r) r_{\alpha} (r_{\beta} - r) r_{\beta}}. \end{aligned}$$

Thus as $|\Delta s_{\alpha}| \rightarrow 0$ and $|\Delta s_{\beta}| \rightarrow 0$, $D_x^q D_y^l \psi_{\Delta s_{\alpha} \Delta s_{\beta}}(x, y)$ tends to zero uniformly. Hence we conclude that (3.4) also tends to zero.

Therefore $F_{\alpha, \beta}(s_{\alpha}, s_{\beta})$ is analytic and hence the proof. $\square$

Theorem 3.2 (Boundedness Property). *Let $f \in B^*(c, d)$ and its quaternion fractional Mellin transform be defined by (3.2)*

$$|F_{\alpha, \beta}(s_{\alpha}, s_{\beta})| \leq T \max_{|g| \leq q, |h| \leq l} \sum_{n=0}^g \sum_{m=0}^h \binom{g}{n} \binom{h}{m} \sup |e(s_{\alpha}) r(s_{\beta}) x^{-n} y^{-m} x^{s_{\alpha}-1} y^{s_{\beta}-1}|. \quad (3.5)$$

Proof. By the boundedness property of the generalized functions, there exists a constant T such that

$$\begin{aligned} |F_{\alpha, \beta}(s_{\alpha}, s_{\beta})| &= |\langle f(x, y), K_{\alpha, \beta}(x, y, s_{\alpha}, s_{\beta}) \rangle| \\ &\leq T \max_{|g| \leq q, |h| \leq l} \sup_{x \in \mathbb{R}^n, y \in \mathbb{R}^n} |D_x^g D_y^h K_{\alpha, \beta}(x, y, s_{\alpha}, s_{\beta})|. \end{aligned}$$

As given in (Zemanian, 1968, note III, p. 53),

$$\begin{aligned} &\leq T \max_{|g| \leq q, |h| \leq l} \sum_{n=0}^g \sum_{m=0}^h \binom{g}{n} \binom{h}{m} \sup \left| \frac{\Gamma(s_{\alpha})}{\Gamma(s_{\alpha} - n)} x^{s_{\alpha}-n-1} \frac{\Gamma(s_{\beta})}{\Gamma(s_{\beta} - m)} y^{s_{\beta}-m-1} \right| \\ &\leq T \max_{|g| \leq q, |h| \leq l} \sum_{n=0}^g \sum_{m=0}^h \binom{g}{n} \binom{h}{m} \sup |e(s_{\alpha}) r(s_{\beta}) x^{-n} y^{-m} x^{s_{\alpha}-1} y^{s_{\beta}-1}| \end{aligned}$$

where $e(s_{\alpha})$ and $r(s_{\beta})$ are functions of s_{α} and s_{β} respectively. $\square$

Theorem 3.3 (Inversion Formula). *Let $\mathcal{M}\{f(u, v)\} = F_{\alpha, \beta}(s_{\alpha}, s_{\beta})$*

for $a_1 \leq a \leq a_2$, $b_1 \leq b \leq b_2$. Also, let r_1 and r_2 be real numbers. Then

$$f(u, v) = \lim_{r_1 \rightarrow \infty} \lim_{r_2 \rightarrow \infty} \frac{1}{(2\pi)^2 i j} \int_{a-ir_1}^{a+ir_1} \int_{b-jr_2}^{b+jr_2} F_{\alpha, \beta}(s_{\alpha}, s_{\beta}) u^{-s_{\alpha}} v^{-s_{\beta}} ds_{\alpha} ds_{\beta}. \quad (3.6)$$

Proof. We have to show that

$$\lim_{r_1 \rightarrow \infty} \lim_{r_2 \rightarrow \infty} \left\langle \frac{1}{(2\pi)^2 i j} \int_{a-ir_1}^{a+ir_1} \int_{b-jr_2}^{b+jr_2} \mathcal{M}\{f(u, v)\} u^{-s_{\alpha}} v^{-s_{\beta}} ds_{\alpha} ds_{\beta}, \phi(u, v) \right\rangle = \langle f, \phi \rangle.$$

By definition of inner product, we get

$$\begin{aligned} &\left\langle \frac{1}{(2\pi)^2 i j} \int_{a-ir_1}^{a+ir_1} \int_{b-jr_2}^{b+jr_2} \mathcal{M}\{f\} u^{-s_{\alpha}} v^{-s_{\beta}} ds_{\alpha} ds_{\beta}, \phi \right\rangle \\ &= \frac{1}{(2\pi)^2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \phi \left(\int_{-r_1}^{+r_1} \int_{-r_2}^{+r_2} \mathcal{M}\{f\} u^{-s_{\alpha}} v^{-s_{\beta}} dw_1 dw_2 \right) du dv. \end{aligned}$$

By changing the order of integration, we get

$$\begin{aligned} & \left\langle \frac{1}{(2\pi)^2 ij} \int_{a-ir_1}^{a+ir_1} \int_{b-jr_2}^{b+jr_2} \mathcal{M}\{f\} u^{-s_\alpha} v^{-s_\beta} ds_\alpha ds_\beta, \phi \right\rangle \\ &= \frac{1}{(2\pi)^2} \int_{-r_1}^{+r_1} \int_{-r_2}^{+r_2} \langle f, x^{s_\alpha-1} y^{s_\beta-1} \rangle \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \phi u^{-s_\alpha} v^{-s_\beta} du dv dw_1 dw_2 \end{aligned}$$

and by lemma 3.5-1 of (Zemanian, 1968),

$$\begin{aligned} &= \left\langle f, \frac{1}{(2\pi)^2} \int_{-r_1}^{+r_1} \int_{-r_2}^{+r_2} x^{s_\alpha-1} y^{s_\beta-1} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \phi u^{-s_\alpha} v^{-s_\beta} du dv dw_1 dw_2 \right\rangle \\ &= \left\langle f, \frac{1}{(2\pi)^2} \int_{-r_1}^{+r_1} \int_{-r_2}^{+r_2} \phi \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} x^{s_\alpha-1} y^{s_\beta-1} u^{-s_\alpha} v^{-s_\beta} du dv dw_1 dw_2 \right\rangle. \end{aligned}$$

Substituting $x = e^{-t_1}$, $y = e^{-t_2}$, $u = e^{\tau_1}$, $v = e^{\tau_2}$ it follows

$$\begin{aligned} &= \left\langle f, \frac{1}{(2\pi)^2} \int_{-r_1}^{+r_1} \int_{-r_2}^{+r_2} \phi \int_0^\infty \int_0^\infty e^{-(s_\alpha-1)t_1} e^{-(s_\beta-1)t_2} e^{s_\alpha \tau_1} e^{s_\beta \tau_2} e^{-\tau_1} e^{-\tau_2} \right. \\ &\quad \left. d\tau_1 d\tau_2 dw_1 dw_2 \right\rangle \\ &= \left\langle f, \frac{1}{(2\pi)^2} \int_{-r_1}^{+r_1} \int_{-r_2}^{+r_2} \phi \int_0^\infty \int_0^\infty e^{(s_\alpha-1)(\tau_1-t_1)} e^{(s_\beta-1)(\tau_2-t_2)} d\tau_1 d\tau_2 dw_1 dw_2 \right\rangle \\ &= \left\langle f, \frac{1}{(2\pi)^2} \int_{-r_1}^{+r_1} \int_{-r_2}^{+r_2} \phi \int_0^\infty \int_0^\infty e^{(a-iw_1)(\tau_1-t_1)} e^{(b-jw_2)(\tau_2-t_2)} e^{-(\tau_1-t_1)} e^{-(\tau_2-t_2)} \right. \\ &\quad \left. d\tau_1 d\tau_2 dw_1 dw_2 \right\rangle \\ &= \left\langle f, \frac{1}{(\pi)^2} \int_0^\infty \int_0^\infty \phi e^{(a-1)t_1} e^{(b-1)t_2} \frac{\sin r_1 t_1}{t_1} \frac{\sin r_2 t_2}{t_2} d\tau_1 d\tau_2 \right\rangle. \end{aligned}$$

By lemma 3.5-2 of (Zemanian, 1968), the second term of the inner product tends to ϕ as $r_1, r_2 \rightarrow \infty$.

Hence

$$f(u, v) = \lim_{r_1 \rightarrow \infty} \lim_{r_2 \rightarrow \infty} \frac{1}{(2\pi)^2 ij} \int_{a-ir_1}^{a+ir_1} \int_{b-jr_2}^{b+jr_2} F_{\alpha, \beta}(s_\alpha, s_\beta) u^{-s_\alpha} v^{-s_\beta} ds_\alpha ds_\beta.$$

□

Theorem 3.4 (Uniqueness). If $\mathcal{M}\{f(x, y)\} = F_{\alpha, \beta}(s_\alpha, s_\beta)$,

$\mathcal{M}\{g(x, y)\} = G_{\alpha, \beta}(s_\alpha, s_\beta)$ and $\text{supp } f \subset S_c \cap S_d$, $\text{supp } g \subset S_c \cap S_d$.

If $F_{\alpha, \beta}(s_\alpha, s_\beta) = G_{\alpha, \beta}(s_\alpha, s_\beta)$, then $f = g$ in the sense of equality in $B^*(c, d)$.

Proof. By inversion formula, we get

$$f - g = \lim_{r_1 \rightarrow \infty} \lim_{r_2 \rightarrow \infty} \frac{1}{(2\pi)^2 ij} \int_{a-ir_1}^{a+ir_1} \int_{b-jr_2}^{b+jr_2} [F_{\alpha, \beta}(s_\alpha, s_\beta) - G_{\alpha, \beta}(s_\alpha, s_\beta)] u^{-s_\alpha} v^{-s_\beta} ds_\alpha ds_\beta.$$

$$f - g = 0.$$

Thus $f = g$.

Hence the proof.

□

4 Operational properties

Property 4.1. For $f, g \in B^*(c, d)$ and $\lambda_1, \lambda_2 \in \mathbb{R}$ then

$$\begin{aligned} & \langle \lambda_1 f + \lambda_2 g, K_{\alpha, \beta}(x, y, s_\alpha, s_\beta) \rangle \\ &= \lambda_1 \langle f, K_{\alpha, \beta}(x, y, s_\alpha, s_\beta) \rangle + \lambda_2 \langle g, K_{\alpha, \beta}(x, y, s_\alpha, s_\beta) \rangle. \end{aligned} \quad (4.1)$$

Proof. Let λ_1, λ_2 be any real numbers and $f, g \in B^*(c, d)$, then

$$\begin{aligned} & \langle \lambda_1 f + \lambda_2 g, K_{\alpha, \beta}(x, y, s_\alpha, s_\beta) \rangle \\ &= \langle \lambda_1 f, K_{\alpha, \beta}(x, y, s_\alpha, s_\beta) \rangle + \langle \lambda_2 g, K_{\alpha, \beta}(x, y, s_\alpha, s_\beta) \rangle \\ &= \lambda_1 \langle f, K_{\alpha, \beta}(x, y, s_\alpha, s_\beta) \rangle + \lambda_2 \langle g, K_{\alpha, \beta}(x, y, s_\alpha, s_\beta) \rangle. \end{aligned}$$

□

Property 4.2. Let p, q be quaternion numbers and $f \in B^*(c, d)$, then

$$\langle x^p f(x, y) y^q, K_{\alpha, \beta}(x, y, s_\alpha, s_\beta) \rangle = \langle f(x, y), K_{\alpha, \beta}(x, y, s_\alpha + p, s_\beta + q) \rangle. \quad (4.2)$$

Proof. Let p, q be quaternion numbers. Analogous to (Zemanian, 1968, Sec. 2.5), the operator $f \mapsto x^p f(x, y) y^q$ is defined on $f \in B^*(c, d)$ by

$$\langle x^p f(x, y) y^q, K_{\alpha, \beta}(x, y, s_\alpha, s_\beta) \rangle = \langle f(x, y), x^p K_{\alpha, \beta}(x, y, s_\alpha, s_\beta) y^q \rangle. \quad (4.3)$$

Here $K_{\alpha, \beta} \mapsto x^p K_{\alpha, \beta} y^q$ and $f \mapsto x^p f(x, y) y^q$ are isomorphisms. Now for any given s_α and s_β , we may choose $s_\alpha + p$ and $s_\beta + q$ such that

$$\begin{aligned} \langle x^p f(x, y) y^q, x^{s_\alpha-1} y^{s_\beta-1} \rangle &= \langle f(x, y), x^{s_\alpha+p-1} y^{s_\beta+q-1} \rangle \\ &= \langle f(x, y), K_{\alpha, \beta}(x, y, s_\alpha + p, s_\beta + q) \rangle. \end{aligned}$$

Hence the proof. □

Property 4.3. For $f \in B^*(c, d)$, $\mathcal{D}_x^n = \frac{\partial^n}{\partial x^n}, \mathcal{D}_y^m = \frac{\partial^m}{\partial y^m}$ and from (Zemanian, 1968)

$$(s_\alpha - n)_n = (s_\alpha - n)(s_\alpha - n + 1) \cdots (s_\alpha - 1);$$

$$(s_\beta - m)_m = (s_\beta - m)(s_\beta - m + 1) \cdots (s_\beta - 1) \text{ where } n, m = 0, 1, 2, \dots$$

The operational property is given by:

$$\langle \mathcal{D}_x^n f, K_{\alpha, \beta}(x, y, s_\alpha, s_\beta) \rangle = \langle f, (-1)^n (s_\alpha - n)_n K_{\alpha, \beta}(x, y, s_\alpha - n, s_\beta) \rangle. \quad (4.4)$$

$$\langle \mathcal{D}_y^m f, K_{\alpha, \beta}(x, y, s_\alpha, s_\beta) \rangle = \langle f, (-1)^m (s_\beta - m)_m K_{\alpha, \beta}(x, y, s_\alpha, s_\beta - m) \rangle. \quad (4.5)$$

$$\begin{aligned} & \langle \mathcal{D}_x^n \mathcal{D}_y^m f, K_{\alpha, \beta}(x, y, s_\alpha, s_\beta) \rangle \\ &= \langle f, (-1)^{n+m} (s_\alpha - n)_n (s_\beta - m)_m K_{\alpha, \beta}(x, y, s_\alpha - n, s_\beta - m) \rangle. \end{aligned} \quad (4.6)$$

Proof. Let the operator $K_{\alpha, \beta} \mapsto (-\mathcal{D}_x)^n K_{\alpha, \beta}$ be a continuous linear mapping. Define an adjoint operator $f \mapsto \mathcal{D}_x^n f(x, y)$ on $f \in B^*(c, d)$ as

$$\langle \mathcal{D}_x^n f, K_{\alpha, \beta}(x, y, s_\alpha, s_\beta) \rangle = \langle f, (-\mathcal{D}_x)^n K_{\alpha, \beta}(x, y, s_\alpha, s_\beta) \rangle. \quad (4.7)$$

This is also a continuous linear mapping. Now for any given s_α and s_β , we may choose $s_\alpha - n$ and s_β such that

$$\begin{aligned} \langle \mathcal{D}_x^n f, x^{s_\alpha-1} y^{s_\beta-1} \rangle &= \langle f, (-\mathcal{D}_x)^n x^{s_\alpha-1} y^{s_\beta-1} \rangle \\ &= \langle f, (-1)^n (s_\alpha - n)_n x^{s_\alpha-n-1} y^{s_\beta-1} \rangle \\ &= \langle f, (-1)^n (s_\alpha - n)_n K_{\alpha, \beta}(x, y, s_\alpha - n, s_\beta) \rangle \end{aligned}$$

where $(s_\alpha - n)_n = (s_\alpha - n)(s_\alpha - n + 1) \cdots (s_\alpha - 1)$ for $n = 1, 2, 3, \dots$.

Similarly, let the operator $K_{\alpha, \beta} \mapsto (-\mathcal{D}_y)^m K_{\alpha, \beta}$ be a continuous linear mapping. Define an adjoint operator $f \mapsto \mathcal{D}_y^m f(x, y)$ on $f \in B^*(c, d)$ as

$$\langle \mathcal{D}_y^m f, K_{\alpha, \beta}(x, y, s_\alpha, s_\beta) \rangle = \langle f, (-\mathcal{D}_y)^m K_{\alpha, \beta}(x, y, s_\alpha, s_\beta) \rangle. \quad (4.8)$$

This is also a continuous linear mapping. Now for any given s_α and s_β , we may choose s_α and $s_\beta - m$ such that

$$\begin{aligned} \langle \mathcal{D}_y^m f, x^{s_\alpha-1} y^{s_\beta-1} \rangle &= \langle f, (-\mathcal{D}_y)^m x^{s_\alpha-1} y^{s_\beta-1} \rangle \\ &= \langle f, (-1)^m (s_\beta - m)_m x^{s_\alpha-1} y^{s_\beta-m-1} \rangle \\ &= \langle f, (-1)^m (s_\beta - m)_m K_{\alpha, \beta}(x, y, s_\alpha, s_\beta - m) \rangle \end{aligned}$$

where $(s_\beta - m)_m = (s_\beta - m)(s_\beta - m + 1) \cdots (s_\beta - 1)$ for $m = 1, 2, 3, \dots$.

On combining the operators (4.7) and (4.8) of (4.4) and (4.5), the result (4.6) is obtained. $\square$

Property 4.4. For $f \in B^*(c, d)$, $\mathcal{D}_x^n = \frac{\partial^n}{\partial x^n}$, $\mathcal{D}_y^m = \frac{\partial^m}{\partial y^m}$ and from (Zemanian, 1968)

$$(s_\alpha)_n = s_\alpha(s_\alpha + 1) \cdots (s_\alpha + n - 1); \quad (s_\beta)_m = (s_\beta)(s_\beta + 1) \cdots (s_\beta + m - 1)$$

where $n, m = 0, 1, 2, \dots$.

The operational property is given by:

$$\langle x^n \mathcal{D}_x^n f, K_{\alpha, \beta}(x, y, s_\alpha, s_\beta) \rangle = \langle f, (-1)^n (s_\alpha)_n K_{\alpha, \beta}(x, y, s_\alpha, s_\beta) \rangle. \quad (4.9)$$

$$\langle y^m \mathcal{D}_y^m f, K_{\alpha, \beta}(x, y, s_\alpha, s_\beta) \rangle = \langle f, (-1)^m (s_\beta)_m K_{\alpha, \beta}(x, y, s_\alpha, s_\beta) \rangle. \quad (4.10)$$

Proof. By combining the operators (4.3) for $p = n$, $q = 0$ and (4.7), we define the operator $f \mapsto x^n \mathcal{D}_x^n f(x, y)$ on $f \in B^*(c, d)$ as

$$\langle x^n \mathcal{D}_x^n f, K_{\alpha, \beta}(x, y, s_\alpha, s_\beta) \rangle = \langle f, (-\mathcal{D}_x)^n x^n K_{\alpha, \beta}(x, y, s_\alpha, s_\beta) \rangle. \quad (4.11)$$

This is a continuous linear mapping. So for any s_α and s_β we have

$$\begin{aligned} \langle x^n \mathcal{D}_x^n f, x^{s_\alpha-1} y^{s_\beta-1} \rangle &= \langle f, (-\mathcal{D}_x)^n x^{s_\alpha+n-1} y^{s_\beta-1} \rangle \\ &= \langle f, (-1)^n (s_\alpha)_n x^{s_\alpha-1} y^{s_\beta-1} \rangle \\ &= \langle f, (-1)^n (s_\alpha)_n K_{\alpha, \beta}(x, y, s_\alpha, s_\beta) \rangle \end{aligned}$$

where $(s_\alpha)_n = s_\alpha(s_\alpha + 1) \cdots (s_\alpha + n - 1)$ for $n = 1, 2, 3, \dots$.

Similarly, by combining the operators (4.3) for $p = 0$, $q = m$ and (4.7), we define the operator $f \mapsto y^m \mathcal{D}_y^m f(x, y)$ on $f \in B^*(c, d)$ as

$$\langle y^m \mathcal{D}_y^m f, K_{\alpha, \beta}(x, y, s_\alpha, s_\beta) \rangle = \langle f, (-\mathcal{D}_y)^m y^m K_{\alpha, \beta}(x, y, s_\alpha, s_\beta) \rangle. \quad (4.12)$$

This is a continuous linear mapping. So for any s_α and s_β we have

$$\begin{aligned} \langle y^m \mathcal{D}_y^m f, x^{s_\alpha-1} y^{s_\beta-1} \rangle &= \langle f, (-\mathcal{D}_y)^m x^{s_\alpha-1} y^{s_\beta+m-1} \rangle \\ &= \langle f, (-1)^m (s_\beta)_m x^{s_\alpha-1} y^{s_\beta-1} \rangle \\ &= \langle f, (-1)^m (s_\beta)_m K_{\alpha, \beta}(x, y, s_\alpha, s_\beta) \rangle \end{aligned}$$

where $(s_\beta)_m = (s_\beta)(s_\beta + 1) \cdots (s_\beta + m - 1)$ for $m = 1, 2, 3, \dots$. □

Example 1. Consider a partial differential equation

$$r \frac{\partial u}{\partial r} + t \frac{\partial u}{\partial t} = \mu(t-1) \quad (4.13)$$

where $\mu(t-1)$ is a unit step function with r and t having its usual meaning.

When applying generalized 2D-QFrMT, it follows:

$$U_{\alpha, \beta}(s_\alpha, s_\beta) = \langle u(r, t), K_{\alpha, \beta}(r, t, s_\alpha, s_\beta) \rangle = \langle u(r, t), r^{s_\alpha-1} t^{s_\beta-1} \rangle. \quad (4.14)$$

On applying generalized 2D-QFrMT and using property 4.4 on (4.13), we get

$$(-1)s_\alpha U_{\alpha, \beta}(s_\alpha, s_\beta) + (-1)s_\beta U_{\alpha, \beta}(s_\alpha, s_\beta) = -\frac{1}{s_\beta}.$$

Thus

$$U_{\alpha, \beta}(s_\alpha, s_\beta) = \frac{1}{s_\beta(s_\alpha + s_\beta)}.$$

Applying the inversion formula (3.6),

$$u(r, t) = \lim_{r_1 \rightarrow \infty} \lim_{r_2 \rightarrow \infty} \frac{1}{(2\pi)^2 i j} \int_{a-ir_1}^{a+ir_1} \int_{b-jr_2}^{b+jr_2} \frac{1}{s_\beta(s_\alpha + s_\beta)} r^{-s_\alpha} t^{-s_\beta} ds_\alpha ds_\beta$$

which is the required solution of (4.13).

5 Mellin-type convolution theorem

Definition 5.1. If f and g are well-behaved functions, then the Mellin-type convolution from (Zemanian, 1968) is defined as

$$f * g = \int_0^\infty f(y) g\left(\frac{x}{y}\right) \frac{dy}{y} \quad (5.1)$$

and

$$f \circ g = \int_0^\infty f(y) g(xy) dy. \quad (5.2)$$

Theorem 5.1 (Convolution theorem). For $f, g \in B^*(c, d)$;

$$\langle f * g, K_{\alpha, \beta}(x, y, s_{\alpha}, s_{\beta}) \rangle = \langle f, K_{\alpha, \beta}(x, y, s_{\alpha}, s_{\beta}) \rangle \langle g, K_{\alpha, \beta}(x, y, s_{\alpha}, s_{\beta}) \rangle. \quad (5.3)$$

$$\langle f \circ g, K_{\alpha, \beta}(x, y, s_{\alpha}, s_{\beta}) \rangle = \langle f, K_{\alpha, \beta}(x, y, 1 - s_{\alpha}, 1 - s_{\beta}) \rangle \langle g, K_{\alpha, \beta}(x, y, s_{\alpha}, s_{\beta}) \rangle. \quad (5.4)$$

Proof. For $f, g \in B^*(c, d)$, we define the functional $f * g$ on $B^*(c, d)$ as given in (Zemnian, 1966)

$$\langle f * g, K_{\alpha, \beta}(x, y, s_{\alpha}, s_{\beta}) \rangle = \langle f(x, y), \langle g(u, v), K_{\alpha, \beta}(xu, yv, s_{\alpha}, s_{\beta}) \rangle \rangle.$$

If f and g are convergent in $B^*(c, d)$, $f * g$ is continuous and convergent in $B^*(c, d)$. For any s_{α} and s_{β} we have

$$\begin{aligned} \langle f * g, x^{s_{\alpha}-1} y^{s_{\beta}-1} \rangle &= \langle f(x, y), \langle g(u, v), (xu)^{s_{\alpha}-1} (yv)^{s_{\beta}-1} \rangle \rangle \\ &= \langle f(x, y), x^{s_{\alpha}-1} y^{s_{\beta}-1} \langle g(u, v), u^{s_{\alpha}-1} v^{s_{\beta}-1} \rangle \rangle \\ &= \langle f(x, y), x^{s_{\alpha}-1} y^{s_{\beta}-1} \rangle \langle g(u, v), u^{s_{\alpha}-1} v^{s_{\beta}-1} \rangle \\ &= \langle f, K_{\alpha, \beta}(x, y, s_{\alpha}, s_{\beta}) \rangle \langle g, K_{\alpha, \beta}(x, y, s_{\alpha}, s_{\beta}) \rangle. \end{aligned}$$

Now we define the operator $f \mapsto x^{-1} f y^{-1}$ which is an isomorphism.

For $f, g \in B^*(c, d)$, we define the functional $f \circ g$ on $B^*(c, d)$ as given in (Zemnian, 1966)

$$\begin{aligned} \langle f \circ g, K_{\alpha, \beta}(x, y, s_{\alpha}, s_{\beta}) \rangle &= \langle x^{-1} f(x^{-1}, y^{-1}) y^{-1}, \langle g(u, v), K_{\alpha, \beta}(xu, yv, s_{\alpha}, s_{\beta}) \rangle \rangle \\ &= \langle f(x, y), \langle g(u, v), x^{-1} K_{\alpha, \beta}(u/x, v/y, s_{\alpha}, s_{\beta}) y^{-1} \rangle \rangle. \end{aligned}$$

Now for any given s_{α} and s_{β} , we may choose $1 - s_{\alpha}$ and $1 - s_{\beta}$ such that

$$\begin{aligned} \langle f \circ g, x^{s_{\alpha}-1} y^{s_{\beta}-1} \rangle &= \langle f(x, y), \langle g(u, v), x^{-1} (u/x)^{s_{\alpha}-1} (v/y)^{s_{\beta}-1} y^{-1} \rangle \rangle \\ &= \langle f(x, y), \langle g(u, v), x^{1-s_{\alpha}-1} u^{s_{\alpha}-1} v^{s_{\beta}-1} y^{1-s_{\beta}-1} \rangle \rangle \\ &= \langle f(x, y), x^{1-s_{\alpha}-1} y^{1-s_{\beta}-1} \langle g(u, v), u^{s_{\alpha}-1} v^{s_{\beta}-1} \rangle \rangle \\ &= \langle f(x, y), x^{1-s_{\alpha}-1} y^{1-s_{\beta}-1} \rangle \langle g(u, v), u^{s_{\alpha}-1} v^{s_{\beta}-1} \rangle \\ &= \langle f, K_{\alpha, \beta}(x, y, 1 - s_{\alpha}, 1 - s_{\beta}) \rangle \langle g, K_{\alpha, \beta}(x, y, s_{\alpha}, s_{\beta}) \rangle. \end{aligned}$$

Hence the proof. □

6 Application

Consider the Dirichlet's problem for a wedge as given is (Zemnian, 1968)

$$r^2 \frac{\partial^2 u}{\partial r^2} + r \frac{\partial u}{\partial r} + \frac{\partial^2 u}{\partial \theta^2} = 0; \quad 0 < r < \infty, \quad 0 < \theta < \gamma \quad (6.1)$$

where $0 < \gamma < 2\pi$.

The boundary conditions are:

i) As $\theta \rightarrow 0+$, $u(r, \theta)$ converges to quaternionic Mellin transformable generalized function $f(r)$.

ii) As $\theta \rightarrow \gamma -$, $u(r, \theta)$ converges to zero uniformly.

Solution: When applying generalized 2D-QFrMT, we have

$$U_{\alpha, \beta}(s_{\alpha}, s_{\beta}) = \langle u(r, \theta), K_{\alpha, \beta}(r, \theta, s_{\alpha}, s_{\beta}) \rangle = \langle u(r, \theta), r^{s_{\alpha}-1} \theta^{s_{\beta}-1} \rangle. \quad (6.2)$$

By using property 4.4, we have

$$s_{\alpha}(s_{\alpha} + 1)U_{\alpha, \beta}(s_{\alpha}, s_{\beta}) - s_{\alpha}U_{\alpha, \beta}(s_{\alpha}, s_{\beta}) + \frac{\partial^2}{\partial \theta^2}U_{\alpha, \beta}(s_{\alpha}, s_{\beta}) = 0.$$

On assuming that $\partial^2/\partial \theta^2$ is interchangeable with $\mathcal{M}$, we get

$$U_{\alpha, \beta}(s_{\alpha}, s_{\beta}) = Ae^{is_{\alpha}\theta} + Be^{-is_{\alpha}\theta}. \quad (6.3)$$

Considering the boundary conditions i) and ii), we get

$$A + B = F(s_{\alpha}).$$

$$Ae^{is_{\alpha}\gamma} + Be^{-is_{\alpha}\gamma} = 0.$$

Further gives

$$A = \frac{F(s_{\alpha})}{1 - e^{i2\gamma s_{\alpha}}}.$$

$$B = \frac{F(s_{\alpha})}{1 - e^{-i2\gamma s_{\alpha}}}.$$

Substituting A and B in (6.3), we get

$$U_{\alpha, \beta}(s_{\alpha}, s_{\beta}) = \frac{\sin(\gamma s_{\alpha} - \theta s_{\alpha})}{\sin(\gamma s_{\alpha})} F(s_{\alpha}). \quad (6.4)$$

Using the inversion formula (3.6), we get

$$u(r, \theta) = \lim_{r_1 \rightarrow \infty} \lim_{r_2 \rightarrow \infty} \frac{1}{(2\pi)^2 i j} \int_{a-ir_1}^{a+ir_1} \int_{b-jr_2}^{b+jr_2} \frac{\sin(\gamma s_{\alpha} - \theta s_{\alpha})}{\sin(\gamma s_{\alpha})} F(s_{\alpha}) r^{-s_{\alpha}} \theta^{-s_{\beta}} ds_{\alpha} ds_{\beta}. \quad (6.5)$$

This is the required solution to (6.1).

7 Conclusion

Authors have established a two-dimensional quaternionic fractional Mellin transform in a distributional sense. The testing function space and some properties are studied. The analyticity, boundedness, and inversion formula are also demonstrated. Operational properties and Mellin-type convolution properties are established. The theory of generalized functions is supported with applications related to Mathematical Physics.

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