

Fluid Queue controlled by Markovian Queue with Matrix Approach

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ABSTRACT

This study aims to analyze a fluid queue with infinite capacity controlled by Markovian queue with the matrix approach. The Markovian queue is an M/M/1/N queue, where customers arrive and get service with constant rates. After that, customers inflow to the fluid queue buffer with net effective input constant rates. For this scenario, we could develop a mathematical model and obtain closed-form formulas. Further, numerical illustration is shown the effectiveness and accuracy of the results and the proposed approach.

Keywords: Fluid queue, M/M/1/N queue, Matrix Approach.

Mathematics Subject Classification: 60k30

1. INTRODUCTION

Fluid queueing models are considered an important topic in queueing theory that is widely used in many life applications such as manufacturing, communications, and computer systems. Fluid models deal with models that have a continuous customer. It is difficult to study these models with classic queueing models which manipulate the customers in serving and leaving stages individually see, Ramanath and Lakshmi (2003), Aboul-Hassan et al. (2010), Maragathasundari and Srinivasan(2013). In fluid models, the customers reach a storage facility where it is stored and gradually released in a stream of nature, see, Mitra (1988), Elwalid, and Mitra (1991).

Many authors and researchers analyze fluid models which controlled by M/M/1 queue that modulate input fluid rates in the buffer as Virtamo, and Norros (1994), Adan, and Resing (1996), but a few of them analyze fluid models which controlled by Markovian queue with a finite state space as Lenin, and Parthasarathy (2000), Vijayashree, and Anjuka (2018).

In Lenin, and Parthasarathy (2000), they used some tri-diagonal identities of determinates to find analytically the buffer occupancy distribution function. In Vijayashree, and Anjuka (2018), the generating function approach is used to find the buffer content under subsequent repair and disaster.

To the best of our knowledge, the matrix approach has not been used before in studying a fluid queue controlled by $M/M/1/N$ queue. Our paper aims to find a formula for the stationary joint probability distribution of the buffer occupancy by using matrix approach in a closed-form. Further, we obtained other closed-form expressions that describe the model behavior. Besides, we provided numerical illustrations to show the validity of the obtained results.

2. PRELIMINARIES AND MODEL ANALYSIS

In the proposed model, the customers arrive at $M/M/1/N$ queue where the arrival and service are according to exponential distribution, and this queue is called the background queue. After customers get out of the queue, the discrete behavior of customers convert to continuous and the customers flow into another queue which is called a fluid queue. The previous scenario can be characterized by two stochastic processes $\{Y(t), Q(t), t \geq 0\}$ where $Y(t)$ denotes the number of customers need to be served by the server in the system at time t and $Q(t)$ refers to the buffer content at time t . The parameters of the first stochastic process are constant parameters λ and μ , and the parameters of the second stochastic process are input flow rate r and output flow rate c .

The difference between the two stochastic processes is that in the first one it's easy to distinguish between the customers in spite of each customer may be considered as a discrete or continuous stream of flow, but in the second one, all customers are transmitted as fluid flow to the storage server.

We study the general model under these considerations $r - c > 0$, $\mu > \lambda$ and $u \geq 0$, such that r and c are the input flow rate and output flow rate respectively.

In the fluid queue, we let that whenever the reservoir is empty the buffer content cannot decrease. So,

$$\frac{dQ(t)}{dt} = \begin{cases} 0, & \text{if } Q(t) = 0, Y(t) = 0 \\ r - c, & \text{elsewhere.} \end{cases}$$

We restrict our analysis on a fluid queue in stationary state case. Therefore, the stability condition is satisfied and can be written as:

$$\sum_{j=0}^N p_j (r - c) < 0, \text{ and } j \in S,$$

such that p_j refers to the background stationary probabilities, and boundary conditions are given as:

$$F_0(0) = 1 \text{ and } F_j(0) = 0, \text{ where } p_0 = \frac{1 - \rho}{1 - \rho^{N+1}} \text{ and } \rho = \frac{\lambda}{\mu}.$$

Assume that

$$F_j(t, u) = \text{Prob}\{Y(t) = j, Q(t) \leq u\}, \quad j \in S, \quad u \geq 0,$$

such that $F_j(t, u)$ is the distribution of the buffer occupancy and $Prob$ denotes to the probability function.

For isfluid queue in steady state asystem of differential equations for , the $u > 0$

$$\frac{dF(u)}{du} = AF(u), \quad (1)$$

where

$$F(u) = [F_0(u), F_1(u), \dots, F_N(u)]^T_{(N+1) \times 1},$$

and

$$A = \begin{pmatrix} \frac{-\lambda}{r-c} & \frac{\mu}{r-c} & & & \\ \frac{\lambda}{r-c} & \frac{-(\lambda+\mu)}{r-c} & \frac{\mu}{r-c} & & \\ & \ddots & \ddots & \ddots & \\ & & & \frac{\mu}{r-c} & \\ & & & \frac{\lambda}{r-c} & \frac{-\mu}{r-c} \end{pmatrix}_{(N+1) \times (N+1)}.$$

Define

$$\psi_j(\theta) = \int_0^\infty e^{-\theta u} F_j(u) du.$$

Take Laplace transformation for the previous system, we get

$$M\psi(\theta) = B, \quad (2)$$

where

$$\psi(\theta) = [\psi_0(\theta), \psi_1(\theta), \dots, \psi_N(\theta)]^T_{(N+1) \times 1}, \quad B = [1, 0, \dots, 0]^T_{(N+1) \times 1} \text{ and}$$

$$M = \begin{pmatrix} \alpha & \gamma & & & \\ \alpha & \beta & \gamma & & \\ & \ddots & \ddots & \ddots & \\ & & & \gamma & \\ & & & \alpha & d \end{pmatrix}_{(N+1) \times (N+1)},$$

such that

$$a = \theta + \frac{\lambda}{r-c}, \alpha = \frac{-\lambda}{r-c}, \beta = \theta + \frac{(\lambda+\mu)}{r-c}, \gamma = \frac{-\mu}{r-c} \text{ and } d = \theta + \frac{\mu}{r-c} \quad (3)$$

3. MODEL SOLUTION

The solution for Eq. (2) can be written as

$$\psi(\theta) = \frac{c^*}{\det(A)}, \quad (4)$$

where

$$c^* = [c_{0j}]_{(N+1) \times 1}^T \text{ and } 0 \leq j \leq N,$$

such that c_{0j} elements are called the adjoint matrix elements of matrix M .

Theorem 1 For the proposed model, the determinate of the matrix M has the following form

$$\det(M) = (g-h)^{-1} \theta [(g^{N+1} - h^{N+1})],$$

where the eigenvalues of the underlying matrix M are g and h have the following form

$$g = \frac{\sqrt{\lambda\mu}}{r-c} e^{J\varphi}, \quad h = \frac{\sqrt{\lambda\mu}}{r-c} e^{J\varphi}, \quad J = \sqrt{-1} \text{ and } \varphi = \cos^{-1} \left(\frac{\beta}{2\sqrt{\alpha\gamma}} \right).$$

Proof.

The determinate of matrix A can be written as

$$\det(A) = aD_N - \alpha\gamma D_{N-1}, \quad (5)$$

where

$$D_N = \begin{pmatrix} \beta & \gamma & & & \\ \alpha & \beta & \gamma & & \\ & \ddots & \ddots & \ddots & \\ & & & \gamma & \\ & & & \alpha & b \end{pmatrix}_{(N) \times (N)},$$

after some calculation, D_N has the following form:

$$D_N = (g-h)^{-1} [(g^{N+1} - h^{N+1}) - \frac{\lambda}{r-c} (g^N - h^N)]. \quad (6)$$

For general form, we get

$$D_{N-j} = (g-h)^{-1} [(g^{N-j+1} - h^{N-j+1}) - \frac{\lambda}{r-c} (g^{N-j} - h^{N-j})]. \quad (7)$$

Substituting Eqs. (3), (6) and Eq. (10) with $j = 1$ in Eq. (5) and after some calculations we get

$$\det(A) = (g-h)^{-1} \theta(g^{N+1} - h^{N+1}), \quad (8)$$

where

$$g, h = \frac{\beta \pm \sqrt{\beta^2 - 4\alpha\gamma}}{2}, \quad (9)$$

such that g, h are real values

Substituting Eq. (3) in Eq. (9), we get after some calculations

$$g = \frac{\sqrt{\lambda\mu}}{r-c} e^{J\varphi}, h = \frac{\sqrt{\lambda\mu}}{r-c} e^{-J\varphi}, J = \sqrt{-1} \text{ and } \varphi = \arcsin\left(\frac{\beta}{2\sqrt{\alpha\gamma}}\right). \quad (10)$$

Lemma 1 For any state j , the adjoint of matrix A has the following form

$$c_{0j} = (-1)^j \alpha^j D_{N-j}, \quad j \in S.$$

Proof.

After simple calculations we get $c_{00} = D_N$, $c_{01} = -\alpha D_{N-1}$, $c_{02} = \alpha^2 D_{N-2}$ and $c_{03} = -\alpha^3 D_{N-3}$,

In general

$$c_{0j} = (-1)^j \alpha^j D_{N-j}, \quad (11)$$

Theorem 2 The distribution of the buffer occupancy for every $j \in S$, has the following form

$$F_j(u) = \frac{(1-\rho)\rho^j}{(1-\rho^{N+1})} - \frac{\rho^2}{(N+1)} \sum_{i=1}^N \frac{\sin u [\sin jv - \sqrt{\rho} \sin(j+1)v]}{(1+\rho-2\sqrt{\rho} \cos v)} e^{\left[\frac{-\lambda}{r-c} - \frac{\mu}{r-c} + 2\frac{\sqrt{\lambda\mu}}{r-c} \cos v\right]u},$$

where $\rho = \frac{\lambda}{\mu}$ is traffic intensity and $v = \frac{\pi i}{N+1}$.

Proof.

First, we derive a new form of Eq. (4) as the following:

$$(g^{N+1} - h^{N+1}) = (g-h) \sum_{i=0}^N g^i h^{N-i}, \quad (12)$$

also

$$\sum_{i=0}^N g^i h^{N-i} = \prod_{i=1}^N (g + h - 2y_{Ni} \sqrt{gh}). \quad (13)$$

After comparing Eq. (12) with Eq. (13), the Chebyshev's polynomial roots as follows

$$y_{Ni} = \cos(v), \quad v = \frac{\pi i}{N+1}, \quad i = 1, 2, \dots, k, \dots, N, \quad (14)$$

such that these roots of the second kind, real and distinct.

Hence, the new form of Eq. (8) is

$$\det(M) = \theta \prod_{i=1}^N (g + h - 2y_{N,i} \sqrt{gh}). \quad (15)$$

So, according to partial fraction method Eq. (4) can be written as

$$\psi_j(\theta) = \frac{L_0}{\theta} + \sum_{k=1}^N \frac{L_k}{(x + y - 2y_{N,k} \sqrt{xy})}. \quad (16)$$

Such that

$L_0, L_1, L_2, \dots, L_N$ are the unknown coefficients of partial fraction method.

L_0 can be defined as follows

$$L_0 = \lim_{\theta \rightarrow 0} [\theta \psi_j(\theta)], \quad (17)$$

when θ convergence to zero we consider $\theta = 0$, so substituting in Eq. (9) by $\theta = 0$ and after some calculation, we get

$$g = \frac{\lambda}{r-c}, \text{ and } h = \frac{\mu}{r-c}. \quad (18)$$

Substituting Eq.s (3), (7), (8), (11) and Eq. (18) in Eq. (17), we obtain after some calculation

$$L_0 = \frac{(1-\rho)\rho^j}{(1-\rho^{N+1})}. \quad (19)$$

Also, L_k can be defined as the following:

$$L_k = \frac{1}{r-c} \left\{ \lim_{\theta \rightarrow \theta_k} [\theta(r-c) + \lambda + \mu - 2y_{N,k} \sqrt{\lambda\mu}] \psi_j(\theta) \right\}, \quad (20)$$

where

$$\theta_k = \frac{-\lambda - \mu + 2y_{N,k} \sqrt{\lambda\mu}}{r-c}, \text{ and } y_{N,k} = \cos\left(\frac{\pi k}{N+1}\right). \quad (21)$$

Substituting Eq.s (3), (7), (11) and Eq. (15) in Eq. (20), we get after simple calculations

$$L_k = \left[\frac{(-1)^j (\alpha)^j}{r-c} \right] \left[\frac{\lim_{\theta \rightarrow \theta_k} \left[\frac{D_{N-j}}{g-h} \right]}{\theta(g-h) \prod_{i=1}^N (g + h - 2y_{N,i} \sqrt{gh})} \right], \quad (22)$$

$$\lim_{\theta \rightarrow \theta_k} \left[\frac{1}{\theta(r-c) + \lambda + \mu - 2y_{N,k} \sqrt{\lambda\mu}} \right]$$

also

$$\lim_{\theta \rightarrow \theta_k} \left[\frac{\theta(g-h) \prod_{i=1}^N (g+h-2y_{Ni} \sqrt{gh})}{\theta(r-c) + \lambda + \mu - 2y_{Nk} \sqrt{\lambda\mu}} \right] = \frac{1}{(r-c)^N} [(\lim_{\theta \rightarrow \theta_k} \theta)(\lim_{\theta \rightarrow \theta} (g-h))(\lim_{\theta \rightarrow \theta} \prod_{i=1, i \neq k}^N \theta(r-c) + \lambda + \mu - 2y_{Ni} \sqrt{\lambda\mu})], \quad (23)$$

where

$$\lim_{\theta \rightarrow \theta_k} \left[\frac{D_{N-j}}{g-h} \right] = \frac{2J(-1)^{k+1} \mu^{N-j+1} \rho^{\frac{N-j+1}{2}} [\sin j\nu - \sqrt{\rho} \sin(j+1)\nu]}{(r-c)^{N-j+1}}. \quad (24)$$

Substituting Eq. (23), Eq. (24) in Eq. (22) we have after some simplification

$$I_k = \frac{-\rho^{\frac{j+1}{2}} \sin\nu [\sin j\nu - \sqrt{\rho} \sin(j+1)\nu]}{(N+1)(1 + \rho - 2\sqrt{\rho} \cos\nu)}. \quad (25)$$

Substituting Eq. (19) and Eq. (25), in Eq. (16), and after taking Laplace transform inverse, we get

$$F_j(u) = \frac{(1-\rho)\rho^j}{(1-\rho^{N+1})} - \frac{\rho^{\frac{j+1}{2}}}{(N+1)} \sum_{k=1}^N \frac{\sin\nu [\sin j\nu - \sqrt{\rho} \sin(j+1)\nu]}{(1 + \rho - 2\sqrt{\rho} \cos\nu)} e^{\left[\frac{-\lambda}{r-c} - \frac{\mu}{r-c} + 2\frac{\sqrt{\lambda\mu}}{r-c} \cos\nu \right] u} \quad (26)$$

4. NUMERICAL ILLUSTRATION

Depending on the formula of Eq. (26), we will give some numerical illustrations for different relations.

At first, we plot the relation between $F_j(u)$ and u with different values of j , the figure shows that the result curve is according to curve of cumulative function as it is shown in Fig. (1)

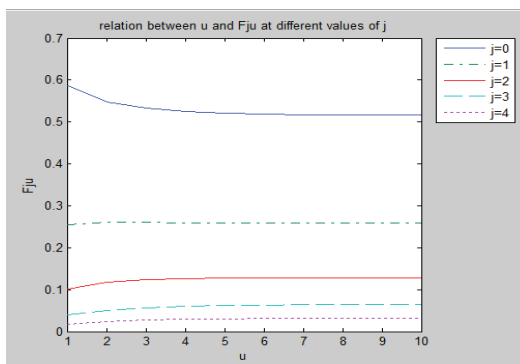


Figure 1. Relation between $F_j(u)$ vs. u at some values of j .

The marginal distribution $P(C \leq u)$ can be defined directly by using Eq. (26) as the following

$$P(C \leq u) = 1 - \sum_{j=0}^N F_j(u), \quad (27)$$

We will plot the relation between marginal distribution u in Figure 2, as the following

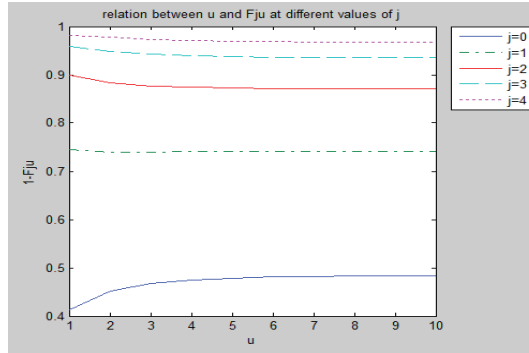


Figure 2. Relation between $P(C \leq u)$ vs. u for different values of j .

The density function of the buffer content $b(u)$ can be defined directly by differentiation Eq. (26) according to u , so the relation between $b(u)$ and u is shown in Figure 3 as the following

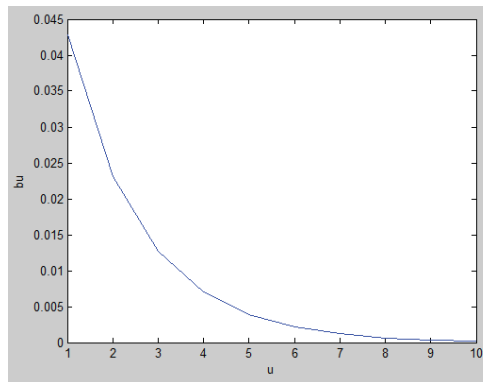


Figure 3. Relation between $b(u)$ vs. u

5. DISCUSSION AND CONCLUSION

A few authors discussed the problem of fluid queue controlled by Markovian queue with finite state space. Different techniques are used to develop this problem as tri-diagonal determinates identities and generating function approach. In the present paper, matrix approach is used to obtain analytical results. Moreover, matrix approach is a new approach in tackling the proposed problem. Different

results are given as stationary distribution, marginal distribution of the buffer occupancy, and the buffer content. Numerical illustrations are given to show the effectiveness and validity of the obtained results.

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