

A Stochastic Approach to a Combined Parallel System

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ABSTRACT

This paper is aimed to introduce a parallel system consisting of random number of components according to a bivariate geometric distribution. The structure of this system is that it initially includes two components combined with parallel and then there are random component additions to these two core components in every discrete time. This situation gives a specific system structure and the stochastic process of this system structure is examined.

Keywords: Bivariate geometric distribution; exponentially distributed components; reliability analyses; stochastic process; Markov chain.

Mathematics Subject Classification: 62E10, 60J05.

1. INTRODUCTION

In industrial engineering, studying in reliability and lifetimes of systems is very popular. In many situations this issue has vital importance. In addition, there are many studies with bivariate geometric distribution, and many of them include shock modeling. In a study a new class is determined by phase-type distribution and Eryilmaz (2017) choose bivariate geometric distribution as a discrete phase-type.

Furthermore, determination of the compound distribution is common problem. By special choosing of compound random variable like sum, maxima and minima, this issue is partially overcome. Some studies in this area are in Marshall and Olkin (1997), Li and Zuo (2004), Ahmad and Kayid (2007), Kundu and Gupta (2014), Eryilmaz (2017).

The discrete phase-type distributions are commonly used in modeling waiting time distributions like geometric, mixed geometric, bivariate geometric and negative binomial. In a Markov process, discrete

phase-type distribution can be defined with d transient states and an absorbing state may be defined as "0".

Krishna and Singh (2009) assessed applications of reliabilities in bivariate geometric distribution and gained characteristics of this valuable distribution. They also studied Bayes estimation for parameters in bivariate geometric distribution. Miaomiao et al. (2014) studied extreme shock models in phase-type and by this optimal component changes determined. Haijun (2003) combined multivariate Poisson process with Markov process by usability of phase-type in shock models.

In this study a new statistical distribution is gained by statistical definition of a combined parallel system and a stochastic process of this system examined either.

2. METARIAL AND METHODS

2.1. Binary Geometric Distribution

Bivariate geometric distribution was examined in a study and the formulation is as follow (Jayakumar & Mundassery 2007).

$$Pr(N = n, M = m) = \sum \sum \binom{m+n}{n} p_1^n p_2^m p_0 \quad p_0 + p_1 + p_2 = 1 \quad n, m = 0, 1, 2, \dots$$

p_0 : probability of no adding,

p_1 : probability of adding to upper part,

p_2 : probability of adding to lower part.

To decide the number of new components in system marginal distribution functions are needed. We obtain marginals as follow.

$$Pr(N = n) = \sum_{m=0}^{\infty} \frac{(m+n)!}{m!n!} p_1^n p_2^m p_0 = \frac{p_1^n p_0}{(1-p_2)^{n+1}} \sum_{m=0}^{\infty} \frac{(m+n)!}{m!n!} p_2^m (1-p_2)^{n+1}$$

With using negative binomial distribution the sum is 1. So the first marginal distribution is in equation (1).

$$Pr(N = n) = \frac{p_1^n p_0}{(1-p_2)^{n+1}} \quad (1)$$

Geometric distribution with success rate $\frac{p_0}{p_0+p_1}$ is gained. Second marginal distribution can be obtained

with the same way in equation (2).

$$Pr(M = m) = \frac{p_2^m p_0}{(1 - p_1)^{m+1}} \quad (2)$$

Therefore geometric distribution with $\frac{p_0}{p_0+p_2}$ success rate is gained. In this stage, stochastic dependence changes the statistical structure of this system. If coming of components are independent these marginals will be enough. As coming of components are dependant, conditional probability function is needed. Firstly, n for the new component addings for the upper part is decided. Lately, m for the new component addings for the lower part is decided, with a given condition with n (Yılmaz & Ünözkan 2015).

$$Pr(M = m | N = n) = \frac{(m+n)!}{m! n!} p_2^m (1 - p_2)^{n+1}$$

By this conditional situation we gain negative binomial distribution with unsuccessful try number is m , success number is $n + 1$.

In the number of components deciding process at first number generates from $Pr(N = n)$, later other number is obtained from $Pr(M = m | N = n)$, so (n, m) numbers group gained.

We assumed that each component had exponentially distributed with $1/\beta$ mean before. So we randomly generate $m + n$ times exponential distribution lifetime in which mean is $1/\beta$. After that maximum of maximums is choosen for system lifetime.

$$T_{SYSTEM} = \max\{\max\{X, T_1, T_2, \dots, T_N\}, \max\{Y, Z_1, Z_2, \dots, Z_M\}\}$$

A close form for joint survival function of random variables is defined as below (Eryılmaz 2017).

Proposition 2.1. Let $\bar{F}(x) = P\{X_i > x\}$, $\bar{G}(x) = P\{Y_i > x\}$, $x > 0$. Then the joint survival function of the

two random variables T_1 and T_2 can be defined as;

$$P\{T_1 > x, T_2 > y\} = \frac{\sum_{0 \leq j, k \leq 2d} a_{jk} \bar{F}^j(x) \bar{G}^k(y)}{\sum_{0 \leq j, k \leq 2d} a_{jk} \bar{F}^j(x) \bar{G}^k(y)}$$

3. RESULTS

The structure of the proposed parallel system includes two components at the beginning. Next, random m and n components are added to these two main components.. Component additions are distributed bivariate geometric and components adding to upper part with probability p_1 , components adding to lower part with probability p_2 , and no components adding to upper or lower part with probability p_0 .

Suppose, random number of components parallel combined to upper part is M , random number of components parallel combined to second part is N , then common probability density function is,

$$Pr(N = n, M = m) = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \binom{m+n}{n} p_1^n p_2^m p_0, 0 < p_0, p_1, p_2 < 1, p_0 + p_1 + p_2 = 1$$

and $n, m = 0, 1, 2, \dots$

Main components and adding components have been assumed independent lifetimes and each component lifetime has $1/\beta$ mean exponential distribution. The probability density function of each

component is,

$$f(t) = \beta e^{-t\beta}, t > 0, \beta > 0.$$

3.1. Stochastic approach

In stochastic approach to this system includes probability transition matrices as below;

$$P_{SYSTEM} = \begin{bmatrix} p_0 & p_1 & p_2 \\ 0 & p_0 & 0 \\ 0 & 0 & p_0 \end{bmatrix}$$

If the comings are independent and identically distributed (iid), then;

$$P_{UPPER PART} = \begin{bmatrix} p_0 & p_1 \\ 0 & p_0 \end{bmatrix}$$

$$P_{LOWER PART} = \begin{bmatrix} p_0 & p_2 \\ 0 & p_0 \end{bmatrix}$$

While in the long run,

$$P_{SYSTEM} = \begin{bmatrix} p_0 & p_1 & p_2 & \dots & \vdots & \vdots & \vdots \\ 0 & p_0 & 0 & \dots & \vdots & \vdots & \vdots \\ 0 & 0 & p_0 & \ddots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \end{bmatrix}$$

In this stage, we try to find a definite form of;

$$P_{ij}^n = P(X_{n+1} = j | X_n = i)$$

We may assume transition probabilities are homogeneous and independent.

When the comings are independent, then there are two different marginal geometric distributions which are available for comings. Their probability values are $\frac{p_0}{p_0+p_1}$ and $\frac{p_0}{p_0+p_2}$.

When the comings are dependent, then first marginal distribution is geometric distribution with probability value is $\frac{p_0}{p_0+p_1}$ and the other marginal distribution is negative binomial distribution with unsuccessful try number is m , success number is $n + 1$.

To define easily we assume that comings are independent. Then the in the last position matrices are as follows;

$$P_{UPPER PART} = \begin{bmatrix} \frac{p_0}{p_0+p_1} & \frac{p_1}{p_0+p_1} \\ 0 & \frac{p_0}{p_0+p_1} \end{bmatrix} \quad P_{LOWER PART} = \begin{bmatrix} \frac{p_0}{p_0+p_2} & \frac{p_2}{p_0+p_2} \\ 0 & \frac{p_0}{p_0+p_2} \end{bmatrix}$$

4. APPLICATION

Example 4.1. Assume that parameters are $p_0 = 0.0052$, $p_1 = 0.4974$, $p_2 = 0.4974$. According to these parameter estimations, we calculate probability transition matrices as below;

$$P_{UPPER PART} = \begin{bmatrix} \frac{0.0052}{0.0052+0.4974} & \frac{0.4974}{0.00006+0.4974} \\ 0 & \frac{0.0052}{0.00006+0.4974} \end{bmatrix} = \begin{bmatrix} 0.01 & 0.99 \\ 0 & 0.01 \end{bmatrix}$$

$$P_{LOWER PART} = \begin{bmatrix} \frac{0.0052}{0.0052+0.4974} & \frac{0.4974}{0.00006+0.4974} \\ 0 & \frac{0.0052}{0.00006+0.4974} \end{bmatrix} = \begin{bmatrix} 0.01 & 0.99 \\ 0 & 0.01 \end{bmatrix}$$

$$P_{SYSTEM} = \begin{bmatrix} 0.0052 & 0.4974 & 0.4974 \\ 0 & 0.0052 & 0 \\ 0 & 0 & 0.0052 \end{bmatrix}$$

After obtaining these matrices, we can make up an autoregressive process.

$$P_{00}^1 = P(X_2 = 0 | X_1 = 0) = 0.0052$$

$$P_{01}^1 = P(X_2 = 1 | X_1 = 0) = 0.4974$$

$$P_{02}^1 = P(X_2 = 2 | X_1 = 0) = 0.4974$$

After first step with $P_{00}^1 = 0.0052$ there are two starting component in the system. Respectively, with $P_{01}^1 = 0.4974$ there are three components in the system with two starting and one new adding to upper part, $P_{02}^1 = 0.4974$ there are three components in the system with two starting and one new adding to lower part.

For the second step the transition probability matrix is below;

$$P_{SYSTEM}^2 = \begin{bmatrix} 0.0052 & 0.4974 & 0.4974 & 0 & 0 & 0 & 0 \\ 0 & 0.0052 & 0 & 0.4974 & 0.4974 & 0 & 0 \\ 0 & 0 & 0.0052 & 0 & 0 & 0.4974 & 0.4974 \end{bmatrix}$$

Example 4.2. Assume that parameters are $p_0 = 0.00006, p_1 = 0.49992, p_2 = 0.49992$.

According to these parameter estimations, we may reach probability transition matrices as below;

$$P_{UPPER PART} = \begin{bmatrix} \frac{0.00006}{0.00006 + 0.49992} & \frac{0.49992}{0.00006 + 0.49992} \\ 0 & \frac{0.00006}{0.00006 + 0.49992} \end{bmatrix} = \begin{bmatrix} 0.0002 & 0.9998 \\ 0 & 0.0002 \end{bmatrix}$$

$$P_{LOWER PART} = \begin{bmatrix} \frac{0.00006}{0.00006 + 0.49992} & \frac{0.49992}{0.00006 + 0.49992} \\ 0 & \frac{0.00006}{0.00006 + 0.49992} \end{bmatrix} = \begin{bmatrix} 0.0002 & 0.9998 \\ 0 & 0.0002 \end{bmatrix}$$

$$P_{SYSTEM} = \begin{bmatrix} 0.00006 & 0.49992 & 0.49992 \\ 0 & 0.00006 & 0 \\ 0 & 0 & 0.00006 \end{bmatrix}$$

Then, we can make up an autoregressive process.

$$P_{00}^1 = P(X_2 = 0 | X_1 = 0) = 0.00006$$

$$P_{01}^1 = P(X_2 = 1 | X_1 = 0) = 0.49992$$

$$P_{02}^1 = P(X_2 = 2 | X_1 = 0) = 0.49992$$

After first step with $P_{00}^1 = 0.00006$ there are two starting component in the system. Respectively, with $P_{01}^1 = 0.49992$ there are three components in the system with two starting and one new adding to upper part, $P_{02}^1 = 0.49992$ there are three components in the system with two starting and one new adding to lower part.

For the second step the transition probability matrix is below;

$$P^2_{SYSTEM} = \begin{bmatrix} 0.00006 & 0.49992 & 0.49992 & 0 & 0 & 0 & 0 \\ 0 & 0.00006 & 0 & 0.49992 & 0.49992 & 0 & 0 \\ 0 & 0 & 0.00006 & 0 & 0 & 0.49992 & 0.49992 \end{bmatrix}$$

5.CONCLUSION

Related with our system, at first we introduced combined parallel system. Later we make a study to reach a stochastic approach to system. In the end we try to define the stochastic process for data sets and give two examples. And according to stochastic process for this system, we conclude that this system may be capable in component structures like strength structures, pandemic diseases and mechanical structures.

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