

## A note on some integral inequalities in a generalized framework

Juan Galeano<sup>1</sup> and Juan E. Nápoles<sup>2</sup> and Edgardo Pérez<sup>3</sup>

<sup>1</sup>Facultad de Ciencias e Ingeniería  
Universidad del Sinú Elías Bechara Zainúm  
Montería, Colombia  
juangaleanod@unisinu.edu.co

<sup>2</sup>UNNE, FaCENA  
Ave. Libertad 5450, Corrientes 3400, Argentina  
jnapoles@exa.unne.edu.ar  
and

UTN-FRRE, French 414, Resistencia, Chaco 3500, Argentina  
jnapoles@frre.utn.edu.ar

<sup>3</sup>Universidad del Sinú Elías Bechara Zainúm  
Montería, Colombia  
edgardomath@gmail.com

### ABSTRACT

*In this paper, we present some new integral inequalities, defined within the framework of generalized integrals.*

**Keywords:** generalized integral inequality, cuasi convex functions.

**2000 Mathematics Subject Classification:** 26A33, 26A42, 26A51.

### 1 Preliminars

It is known that fractional calculation basically concerns the integration and differentiation of arbitrary orders and consists of several integration and derivation operators of different natures. To date, the study of this area has attracted the attention of many researchers, not only in Pure Mathematics, but in multiple fields of applied science. Between its own theoretical development and the multiplicity of applications, the field has grown rapidly in recent years, in such a way that a single definition of “fractional derivative or integral” does not exist, or at least is not unanimously accepted, in (Baleanu and Fernandez, 2019) suggests and justifies the idea of a fairly complete classification of the known operators of the Fractional Calculus, on the other hand, in the work (Ortigueira, Martynyuk and Fedula, 2019) the authors study this phenomenon and support the appearance of various operators, both in theoretical and practical research. Let us point out that these developments have been obtained in different contexts, and not with

a single starting point, that is, they are taken as a basis, from the Riemann-Liouville integral, to that of Katugampola, through other formulations such as Weyl's, Hadamard, or Erdelyi-Kober, in this way various authors have defined different integral operators, even from different notions of generalized local derivatives, this last point of view is the one present in our work.

We believe that it is convenient to take into account the historical route that is presented in Chapter 1 of (Atangana, 2016) where a historical route of differential operators, whether local or global, is made, starting from Newton's classical formulation and arriving at Caputo's Definition, which serves as the basis for presenting a differential operator, with a new parameter, and providing a great variety of applications, taking into account the difference between both types of differential operators, global and local. A seminal question is addressed in 1.5.1 (p.24), where sentence "We can therefore conclude that both the Riemann – Liouville and Caputo operators are not derivatives, and then they are not fractional derivatives, but fractional operators. We agree with the result (Umarov and Steinberg, 2009) that, the local fractional operator is not a fractional derivative" (p.24). For all the above, we can affirm that there is a great variety of integral operators, which have proven their usefulness in solving a great variety of applications and in successive theoretical developments.

In the following, we will not remember questions related to the Riemann Integral, since we consider that the reader is familiar with it.

Probably the first integral operator, recognized and accepted as fractional, is the Riemann-Liouville Integral, apart from the above, this operator is the basis of fruitful investigations and extensions that exceed the limits of this work. In (Gorenflo and Mainardi, 1997) you can see the definition of this fractional operator as follows.

**Definition 1.1.** With  $\varphi$  a real function of class  $L^1[a_1, a_2]$ ,  $a_1$  and  $a_2$  real numbers satisfying  $a_1 < a_2$ . The fractional integral of Riemann-Liouville, right and left, of order  $\alpha > 0$  can be defined as follows

$${}^{RL}J_{a_1^+}^\alpha \varphi(\tau) = \frac{1}{\Gamma(\alpha)} \int_{a_1}^{\tau} (\tau - u)^{\alpha-1} \varphi(u) du, \quad \tau > a_1, \quad (1.1)$$

and

$${}^{RL}J_{a_2^-}^\alpha \varphi(\tau) = \frac{1}{\Gamma(\alpha)} \int_{\tau}^{a_2} (u - \tau)^{\alpha-1} \varphi(u) du, \quad \tau < a_2. \quad (1.2)$$

From these integral operators, the differential operators can be formally defined in this way

$$\begin{aligned} D_{a_1^+}^{\alpha} \varphi(\tau) &= \frac{d}{d\tau} \left( {}^{RL}J_{a_1^+}^{1-\alpha} \varphi(\tau) \right) = \frac{1}{\Gamma(1-\alpha)} \frac{d}{d\tau} \int_{a_1}^{\tau} \frac{\varphi(u)}{(\tau - u)^{\alpha}} du. \\ D_{a_2^-}^{\alpha} \varphi(\tau) &= -\frac{d}{d\tau} \left( {}^{RL}J_{a_2^-}^{1-\alpha} \varphi(\tau) \right) = -\frac{1}{\Gamma(1-\alpha)} \frac{d}{d\tau} \int_{\tau}^{a_2} \frac{\varphi(u)}{(u - \tau)^{\alpha}} du. \end{aligned}$$

In a known result, (Nápoles, Guzman, Lugo and Kashuri, 2020) (see also (Zhao and Luo, 2017)), a generalized differential operator was defined that will be the basis of the generalized integral operator that we will use in our work.

**Definition 1.2.** With  $\varphi$  a real function defined on  $[0, +\infty)$ . The generalized N-derivative of  $\varphi$  of order  $\alpha$  is established in this way

$$N_F^\alpha \varphi(\tau) = \lim_{\varepsilon \rightarrow 0} \frac{\varphi(\tau + \varepsilon F(\tau, \alpha)) - \varphi(\tau)}{\varepsilon} \quad (1.3)$$

with  $\tau > 0$ ,  $\alpha \in (0, 1)$  and  $F(\tau, \alpha)$  the kernel function.

The derivative  $N_F^\alpha \varphi(o) = \lim_{\tau \rightarrow 0^+} N_F^\alpha \varphi(\tau)$  is defined for functions  $\alpha$ -differentiable in some  $(0, \alpha)$ , and assuming that  $\lim_{\tau \rightarrow 0^+} N_F^\alpha \varphi(\tau)$  exists. On the other hand, if  $\varphi$  is a differentiable function in the classical sense, then  $N_F^\alpha \varphi(\tau) = F(\tau, \alpha) \varphi'(\tau)$  being  $\varphi'(\tau)$  the classic derivative.

As we said before, based on this Generalized Derivative, we will now present a generalized integral operator, taking the kernel  $F$  as an absolutely continuous function. More details about this operator can be found in (Nápoles, Guzman, Lugo and Vivas, 2020).

**Definition 1.3.** Let's consider some real interval  $I$ ,  $a_1, \tau \in I$ ,  $\alpha$  a real number and  $\varphi$  a locally integrable function on  $I$ . We define the generalized integral operators, right and left, as follows

$$J_{F, a_1+}^\alpha(\varphi)(\tau) = \int_{a_1}^\tau \frac{\varphi(u)}{F(\tau - u, \alpha)} du, \quad \tau > a_1. \quad (1.4)$$

$$J_{F, a_2-}^\alpha(\varphi)(\tau) = \int_\tau^{a_2} \frac{\varphi(u)}{F(u - \tau, \alpha)} du, \quad a_2 > \tau. \quad (1.5)$$

In (Jarad, Abdeljawad and Alzabut, 2017) the following integral operators, left and right, are defined as follows.

**Definition 1.4.** Taking  $\alpha \in \mathbb{C}$ ,  $Re(\alpha) > 0$  and  $\delta \in (0, 1]$ , the GPF integrals of a function  $\varphi$ , are defined as

$${}_{{a_1}}I^{\alpha, \delta} \varphi(\tau) = \frac{1}{\delta^\alpha \Gamma(\alpha)} \int_{a_1}^\tau e^{[\frac{\delta-1}{\delta}(\tau-u)]} (\tau - u)^{\alpha-1} \varphi(u) du, \quad a_1 < \tau. \quad (1.6)$$

Similarly for the right operator.

**Remark 1.1.** Taking into account the previous definition, we see that said operator is a particular case of the generalized integral operator defined in Definition 1.4, other particular cases of our integral operator can be consulted in (Nápoles, Guzman, Lugo and Vivas, 2020).

**Remark 1.2.** In this context, the set of all integrable functions on  $[a_1, a_2]$  according to Definition 1.4, i.e.,  $(J_{F, a_1+}^\alpha[\varphi(\tau)]^p(a_2)) < +\infty$ , is called the  $L_\alpha^p[a_1, a_2]$  space.

**Remark 1.3.** We will also use the “central” integral operator defined by (see (Nápoles, Guzman, Lugo and Vivas, 2020) and (Zhao and Luo, 2017)):

$$J_{F, a_1}^\alpha(\varphi)(a_2) = \int_{a_1}^{a_2} \frac{\varphi(\tau)}{F(\tau, \alpha)} d\tau, \quad a_2 > a_1. \quad (1.7)$$

The following statement is analogous to the one known from the Ordinary Calculus (see (Nápoles, Guzman, Lugo and Vivas, 2020) and (Zhao and Luo, 2017)).

**Theorem 1.1.** Let  $\varphi$  be  $N$ -differentiable function in  $(\tau_0, \infty)$  with  $\alpha \in (0, 1]$ . Then for all  $\tau > \tau_0$  we have

$$a) \quad J_{F, \tau_0}^\alpha(N_F^\alpha \varphi(\tau)) = \varphi(\tau) - \varphi(\tau_0).$$

$$b) N_F^\alpha \left( J_{F,\tau_0}^\alpha \varphi(\tau) \right) = \varphi(\tau).$$

The result that we present below, in addition to being important, is necessary in our work.

**Theorem 1.2.** (Integration by parts) Let  $u$  and  $v$  be  $N$ -differentiable function in  $(\tau_0, \infty)$  with  $\alpha \in (0, 1]$ . Then for all  $\tau > \tau_0$  we have

$$J_{F,\tau_0}^\alpha ((uN_F^\alpha v)(\tau)) = [uv(\tau) - uv(\tau_0)] - J_{F,\tau_0}^\alpha ((vN_F^\alpha u)(\tau)) \quad (1.8)$$

In our work, we will present several integral inequalities, in the context of the generalized integral operator of the Definition 1.4, which are generalizations of various results known from the literature.

## 2 Results

Our first result is the following.

**Theorem 2.1.** Consider two positive continuous functions  $\varphi$  and  $\psi$  on  $[a_1, a_2]$  with  $\varphi \leq \psi$  on  $[a_1, a_2]$ . If  $\frac{\varphi}{\psi}$  is decreasing and  $\varphi$  is increasing on  $[a_1, a_2]$ , then for any convex function  $g$  with  $g(0) = 0$ , we have

$$\frac{J_{F,a_1}^\alpha [\varphi](\tau)}{J_{F,a_1}^\alpha [\psi](\tau)} \geq \frac{J_{F,a_1}^\alpha [g(\varphi)](\tau)}{J_{F,a_1}^\alpha [g(\psi)](\tau)}, \quad a_1 < \tau \leq a_2, \alpha > 0. \quad (2.1)$$

*Proof.* For  $u, v \in [a_1, a_2]$ , and using that  $\frac{\varphi}{\psi}$  is decreasing and  $\frac{g(\varphi(w))}{\varphi(w)}$  is increasing, we have

$$\left( \frac{g(\varphi(v))}{\varphi(v)} - \frac{g(\varphi(u))}{\varphi(u)} \right) \left( \frac{\varphi(u)}{\psi(u)} - \frac{\varphi(v)}{\psi(v)} \right) \geq 0, \quad (2.2)$$

or equivalently

$$\frac{g(\varphi(v))}{\varphi(v)} \frac{\varphi(u)}{\psi(u)} - \frac{g(\varphi(v))}{\varphi(v)} \frac{\varphi(v)}{\psi(v)} - \frac{g(\varphi(u))}{\varphi(u)} \frac{\varphi(u)}{\psi(u)} + \frac{g(\varphi(u))}{\varphi(u)} \frac{\varphi(v)}{\psi(v)} \geq 0. \quad (2.3)$$

Now, multiplying by  $\psi(v)\psi(u)$  we obtain

$$\frac{g(\varphi(v))}{\varphi(v)} \varphi(u)\psi(v) + \frac{g(\varphi(u))}{\varphi(u)} \varphi(v)\psi(u) - g(\varphi(v))\psi(u) - g(\varphi(u))\psi(v) \geq 0 \quad (2.4)$$

Multiplying (2.4) by  $\frac{1}{F(\tau-v, \alpha)}$ ; next, we integrate the resulting inequality with respect to  $v$  over  $[a_1, \tau]$ ,  $a_1 < \tau \leq a_2$ , it holds

$$\begin{aligned} \varphi(u) J_{F,a_1}^\alpha \left[ \frac{g(\varphi(v))}{\varphi(v)} \psi(v) \right] (\tau) + \frac{g(\varphi(u))}{\varphi(u)} \psi(u) J_{F,a_1}^\alpha [\varphi(v)] (\tau) - g(\varphi(u)) J_{F,a_1}^\alpha [\psi(v)] (\tau) \\ - \psi(u) J_{F,a_1}^\alpha [g(\varphi(v))] (\tau) \geq 0. \end{aligned} \quad (2.5)$$

Next, multiplying (2.5) by  $\frac{1}{F(\tau-u, \alpha)}$ ; then, we integrate the result with respect to  $u$  over  $[a_1, \tau]$ ,  $a_1 < \tau \leq a_2$ , we obtain

$$\begin{aligned} J_{F,a_1}^\alpha [\varphi(u)] (\tau) J_{F,a_1}^\alpha \left[ \frac{g(\varphi(v))}{\varphi(v)} \psi(v) \right] (\tau) + J_{F,a_1}^\alpha \left[ \frac{g(\varphi(u))}{\varphi(u)} \psi(u) \right] (\tau) J_{F,a_1}^\alpha [\varphi(v)] (\tau) \\ - J_{F,a_1}^\alpha [g(\varphi(u))] (\tau) J_{F,a_1}^\alpha [\psi(v)] (\tau) - J_{F,a_1}^\alpha [\psi(u)] (\tau) J_{F,a_1}^\alpha [g(\varphi(v))] (\tau) \geq 0. \end{aligned} \quad (2.6)$$

Equation (2.6) is equivalent to

$$\frac{J_{F,a_1^+}^\alpha[\varphi](\tau)}{J_{F,a_1^+}^\alpha[\psi](\tau)} \geq \frac{J_{F,a_1^+}^\alpha[g(\varphi)](\tau)}{J_{F,a_1^+}^\alpha\left[\frac{g(\varphi)}{\varphi}\psi\right](\tau)}. \quad (2.7)$$

Now, using that  $\varphi \leq \psi$  on  $[a_1, a_2]$  and  $\frac{g(w)}{w}$  is increasing on  $[a_1, \tau]$ ,  $a_1 < \tau \leq a_2$ , we obtain

$$\frac{g(\varphi(v))}{\varphi(v)} \leq \frac{g(\psi(v))}{\psi(v)}. \quad (2.8)$$

Then, if we multiply (2.8) by  $\frac{\psi(v)}{F(\tau-v, \alpha)}$  and integrating with respect to  $v$  over  $[a_1, \tau]$ ,  $a_1 < \tau \leq a_2$ , we obtain

$$J_{F,a_1^+}^\alpha\left[\frac{g(\varphi)}{\varphi}\psi\right](\tau) \leq J_{F,a_1^+}^\alpha[g(\psi)](\tau). \quad (2.9)$$

Finally, from (2.7) and (2.9) we get

$$\frac{J_{F,a_1^+}^\alpha[\varphi](\tau)}{J_{F,a_1^+}^\alpha[\psi](\tau)} \geq \frac{J_{F,a_1^+}^\alpha[g(\varphi)](\tau)}{J_{F,a_1^+}^\alpha[g(\psi)](\tau)}. \quad (2.10)$$

□

**Remark 2.1.** Let's look at some particular cases of this Theorem reported in the literature. For example, if  $F \equiv 1$ , that is, we work with the classic Riemann Integral, we obtain Theorem 9 of (Liu, Ngo and Huy, 2009) taking the function  $\phi$  as convex. If we consider the kernel  $F(\tau - u, \alpha)$  of the Generalized Proportional Fractional integral (GPF of (Jarad et al., 2017)) then Theorem 3 of (Rahman, Nisar, Abdeljawad and Ullah, 2020) is obtained from the previous result. Finally, if we consider the Riemann-Liouville Integral, (1.1) then we obtain Theorem 3 of (Rahman et al., 2020).

A more general variation from the previous result, in which two fractional orders are involved, is as follows.

**Theorem 2.2.** Consider two positive continuous functions  $\varphi$  and  $\psi$  on  $[a_1, a_2]$  with  $\varphi \leq \psi$  on  $[a_1, a_2]$ . If  $\frac{\varphi}{\psi}$  is decreasing and  $\varphi$  is increasing on  $[a_1, a_2]$ , then for any convex function  $g$  with  $g(0) = 0$ , we obtain

$$\frac{J_{F,a_1^+}^\alpha[\varphi](\tau)J_{F,a_1^+}^\lambda[g(\psi)](\tau) + J_{F,a_1^+}^\lambda[\varphi](\tau)J_{F,a_1^+}^\alpha[g(\psi)](\tau)}{J_{F,a_1^+}^\alpha[\psi](\tau)J_{F,a_1^+}^\lambda[g(\varphi)](\tau) + J_{F,a_1^+}^\lambda[\psi](\tau)J_{F,a_1^+}^\alpha[g(\varphi)](\tau)} \geq 1, \quad (2.11)$$

where  $\alpha, \lambda > 0$ ,  $a_1 < \tau \leq a_2$ .

**Proof.** From (2.5) we obtain

$$\begin{aligned} & J_{F,a_1^+}^\lambda[\varphi(u)](\tau)J_{F,a_1^+}^\alpha\left[\frac{g(\varphi(v))}{\varphi(v)}\psi(v)\right](\tau) + J_{F,a_1^+}^\lambda\left[\frac{g(\varphi(u))}{\varphi(u)}\psi(u)\right](\tau)J_{F,a_1^+}^\alpha[\varphi(v)](\tau) \\ & - J_{F,a_1^+}^\lambda[g(\varphi)](\tau)J_{F,a_1^+}^\alpha[\psi(v)](\tau) - J_{F,a_1^+}^\lambda[\psi(u)](\tau)J_{F,a_1^+}^\alpha[g(\varphi(v))](\tau) \geq 0. \end{aligned} \quad (2.12)$$

Now, as  $\varphi \leq \psi$  on  $[a_1, a_2]$ , and the function  $\frac{g(w)}{w}$  is increasing, it holds

$$J_{F,a_1^+}^\lambda\left[\frac{g(\varphi(v))}{\varphi(v)}\psi(v)\right](\tau) \leq J_{F,a_1^+}^\lambda\left[\frac{g(\psi(v))}{\psi(v)}\psi(v)\right](\tau). \quad (2.13)$$

Hence, thanks to (2.9), (2.12) and (2.13), we get (2.11). □

**Remark 2.2.** If  $\lambda = \alpha$  this result becomes Theorem 2.1. On the other hand, if we consider the  $F$  kernel to be the GPF integral kernel, we obtain Theorem 4 of (Rahman et al., 2020). Finally, if  $F \equiv 1$  our theorem is reduced to Theorem 3.3 of (Dahmani, 2012).

A more general conclusion to Theorem 2.1, where a positive, continuous and increasing function  $\rho$  in  $[a_1, a_2]$  is added is given in the following result.

**Theorem 2.3.** Consider three positive continuous functions  $\varphi, \psi$  and  $\rho$  on  $[a_1, a_2]$  and  $\varphi \leq \psi$  on  $[a_1, a_2]$ . If  $\frac{\varphi}{\psi}$  is decreasing,  $\varphi$  and  $\rho$  are increasing on  $[a_1, a_2]$  and  $g$  is a convex function with  $g(0) = 0$ . Then, for any  $\alpha > 0$ ,  $a_1 < \tau \leq a_2$ , we have

$$\frac{J_{F,a_1^+}^\alpha[\varphi](\tau)}{J_{F,a_1^+}^\alpha[\psi](\tau)} \geq \frac{J_{F,a_1^+}^\alpha[g(\varphi)\rho](\tau)}{J_{F,a_1^+}^\alpha[g(\psi)\rho](\tau)}, \quad a_1 < \tau \leq a_2, \alpha > 0. \quad (2.14)$$

**Proof.** Since the function  $\frac{g(\varphi(w))\rho(w)}{\varphi(w)}$  is increasing, and  $\frac{\varphi(w)}{\psi(w)}$  is decreasing, thus, for  $u, v \in [a_1, \tau]$ ,  $a_1 < \tau \leq a_2$  we have

$$\left( \frac{g(\varphi(v))\rho(v)}{\varphi(v)} - \frac{g(\varphi(u))\rho(u)}{\varphi(u)} \right) \left( \frac{\varphi(u)}{\psi(u)} - \frac{\varphi(v)}{\psi(v)} \right) \geq 0, \quad (2.15)$$

or equivalently

$$\begin{aligned} \frac{g(\varphi(v))\rho(v)}{\varphi(v)}\varphi(u)\psi(v) + \frac{g(\varphi(u))\rho(u)}{\varphi(u)}\varphi(v)\psi(u) \\ - \frac{g(\varphi(v))\rho(v)}{\varphi(v)}\varphi(v)\psi(u) - \frac{g(\varphi(u))\rho(u)}{\varphi(u)}\varphi(u)\psi(v) \geq 0. \end{aligned} \quad (2.16)$$

Now, we multiply both sides of (2.16) by  $\frac{1}{F(\tau-v, \alpha)}$  and integrating with respect to  $v$  over  $[a_1, \tau]$ ; and, next multiplying by  $\frac{1}{F(\tau-u, \alpha)}$  and integrating with respect to  $u$  over  $[a_1, \tau]$  we have

$$J_{F,a_1^+}^\alpha[\varphi(v)](\tau)J_{F,a_1^+}^\alpha\left[\frac{g(\varphi(v))\rho(v)}{\varphi(v)}\psi(v)\right](\tau) - J_{F,a_1^+}^\alpha[\psi(v)](\tau)J_{F,a_1^+}^\alpha[g(\varphi(v))\rho(v)](\tau) \geq 0. \quad (2.17)$$

On the other hand using that  $\frac{g(w)}{w}$  is increasing and  $\varphi \leq \psi$  we have

$$J_{F,a_1^+}^\alpha\left[\frac{g(\varphi(v))\rho(v)}{\varphi(v)}\psi(v)\right](\tau) \leq J_{F,a_1^+}^\alpha[g(\psi(v))\rho(v)](\tau). \quad (2.18)$$

Thus from (2.17) and (2.18) we obtain (2.14).  $\square$

**Remark 2.3.** If in this result we use the GPF integral kernel, we get Theorem 5 from (Rahman et al., 2020). If, in addition,  $\alpha \equiv 1$  our theorem is reduced to Theorem 3.5 of (Dahmani, 2012) and if we work with the classic Riemann Integral, that is, if  $F \equiv 1$ , then we obtain Theorem 10 of (Liu et al., 2009).

The following consequence, which is a generalization to Theorem 2.2, includes as the previous Theorem, a certain positive function  $\rho$ , continuous and increasing in  $[a_1, a_2]$  and is expressed in the following result.

**Theorem 2.4.** Consider three positive continuous functions  $\varphi, \psi$  and  $\rho$  with  $\varphi \leq \psi$  on  $[a_1, a_2]$ . If  $\frac{\varphi}{\psi}$  is decreasing,  $\varphi$  and  $\rho$  are increasing on  $[a_1, a_2]$ , and  $g$  is a convex function with  $g(0) = 0$ . Then, for any  $\alpha, \lambda > 0$ ,  $a_1 < \tau \leq a_2$ , we get

$$\frac{J_{F,a_1^+}^\alpha[\varphi(v)](\tau)J_{F,a_1^+}^\lambda[g(\psi(v))\rho(v)](\tau) + J_{F,a_1^+}^\lambda[\varphi(v)](\tau)J_{F,a_1^+}^\alpha[g(\psi(v))\rho(v)](\tau)}{J_{F,a_1^+}^\alpha[\psi(v)](\tau)J_{F,a_1^+}^\lambda[g(\varphi(v))\rho(v)](\tau) + J_{F,a_1^+}^\lambda[\psi(v)](\tau)J_{F,a_1^+}^\alpha[g(\varphi(v))\rho(v)](\tau)} \geq 1, \quad (2.19)$$

*Proof.* From (2.16), we obtain

$$\begin{aligned}
 & J_{F,a_1^+}^\lambda [\varphi(v)](\tau) J_{F,a_1^+}^\alpha \left[ \frac{g(\varphi(v))\rho(v)}{\varphi(v)} \psi(v) \right](\tau) \\
 & + J_{F,a_1^+}^\lambda \left[ \frac{g(\varphi(v))\rho(v)}{\varphi(v)} \psi(v) \right](\tau) J_{F,a_1^+}^\alpha [\varphi(v)](\tau) \\
 & - J_{F,a_1^+}^\lambda \left[ \frac{g(\varphi(v))\rho(v)}{\varphi(v)} \varphi(v) \right](\tau) J_{F,a_1^+}^\alpha [\psi(v)](\tau) \\
 & - J_{F,a_1^+}^\lambda [\psi(v)](\tau) J_{F,a_1^+}^\alpha \left[ \frac{g(\varphi(v))\rho(v)}{\varphi(v)} \varphi(v) \right](\tau) \geq 0.
 \end{aligned} \tag{2.20}$$

Now, using the fact that the function  $\frac{g(w)}{w}$  is increasing, and  $\varphi \leq \psi$  on  $[a_1, a_2]$ , we have

$$J_{F,a_1^+}^k \left[ \frac{g(\varphi(v))\rho(v)}{\varphi(v)} \psi(v) \right] \leq J_{F,a_1^+}^k \left[ \frac{g(\psi(v))\rho(v)}{\psi(v)} \psi(v) \right], \quad k = \alpha, \lambda. \tag{2.21}$$

Thus, from (2.20) and (2.21) we obtain (2.19).  $\square$

**Remark 2.4.** If  $\lambda = \alpha$  then the Theorem 2.4 becomes the Theorem 2.3. As in the previous remarks, we can easily derive from this result Theorem 6 of (Rahman et al., 2020), and Theorem 3.7 of (Dahmani, 2012).

### 3 Conclusions

In this work we have used a generalized integral operator, to obtain integral inequalities, which generalize various interesting results reported in the literature. It is clear that if we use other kernels, for example that of the fractional integral of Hadamard, we can derive new inequalities not yet published.

### References

- Atangana, A. 2016. *Derivative with a New Parameter Theory, Methods and Applications*, Academic Press.
- Baleanu, D. and Fernandez, A. 2019. On fractional operators and their classifications, *Mathematics*. **7**: 830.
- Dahmani, Z. 2012. A note on some new fractional results involving convex functions, *Acta Math. Univ. Comen.* **LXXXI**(2): 241–246.
- Gorenflo, R. and Mainardi, F. 1997. *Fractional Calculus: Integral and Differential Equations of Fractional Order*, Springer, Wien.
- Jarad, F., Abdeljawad, T. and Alzabut, J. 2017. Generalized fractional derivatives generated by a class of local proportional derivatives, *Eur. Phys. J. Spec. Top.* **226**: 3457–3471.
- Liu, W., Ngo, Q. A. and Huy, V. N. 2009. Several interesting integral inequalities, *J. Math. Ineq.* **3**: 201–212.

- Nápoles, J. E., Guzman, P. M., Lugo, L. M. and Kashuri, A. 2020. The local generalized derivative and mittag leffler function., *Sigma J Eng & Nat Sci* **38**(2): 1007–1017.
- Nápoles, J. E., Guzman, P. M., Lugo, L. M. and Vivas, M. 2020. On a new generalized integral properties., *Axioms* **9**(69).
- Ortigueira, M., Martynyuk, V. and Fedula, M. 2019. The failure of certain fractional calculus operators in two physical models, *Fract. Calc. Appl. Anal.* **23**(2).
- Rahman, G., Nisar, K. S., Abdeljawad, T. and Ullah, S. 2020. Certain fractional proportional integral inequalities via convex functions, *Mathematics*. **8**: 222.
- Umarov, S. and Steinberg, S. 2009. Variable order differential equations with piecewise constant order-function and diffusion with changing modes, *Z. Anal. Anwend.* **28**(4): 431–450.
- Zhao, D. and Luo, M. 2017. General conformable fractional derivative and its physical interpretation, *Calcolo*. **54**: 903–917.