

A new result on existence of minimizers for almost convex integrals

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ABSTRACT

We present a new result concerning existence of solutions to some nonconvex autonomous Lagrange problems of the calculus of variations.

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1 Introduction

In this paper we consider the problem of existence of minimizers — let's call them true minimizers — for the nonconvex integral

$$I(x(\cdot)) := \int_a^b L(x(t), x'(t)) dt \quad \text{on } \mathcal{X}_{A,B}^n, \quad (1.1)$$

where the so-called lagrangian function may freely take the ∞ value,

$$L : \mathbb{R}^n \times \mathbb{R}^n \rightarrow (-\infty, \infty];$$

and $\mathcal{X}_{A,B}^n$ is the subclass of absolutely continuous (AC) functions given by

$$\mathcal{X}_{A,B}^n := \{x(\cdot) \in W^{1,1}([a, b], \mathbb{R}^n) : x(a) = A, x(b) = B\}.$$

Since we allow $L(s, \cdot)$ to be nonconvex, we also consider the associated convexified integral

$$I_c(x(\cdot)) := \int_a^b L^{**}(x(t), x'(t)) dt \quad \text{on } \mathcal{X}_{A,B}^n,$$

where $L^{**}(s, \cdot)$ is the bipolar of $L(s, \cdot)$. Any minimizer of $I_c(\cdot)$ will be called a relaxed minimizer.

Assume $L(\cdot)$ is a Borel function bounded from below having $L^{**}(\cdot)$ also Borel, and that there exists a closed set $S \subset \mathbb{R}^n$ such that $L(s, \cdot)$ is almost convex lower semicontinuous (lsc) $\forall s \in S$. Suppose, moreover, that $L(\cdot, 0)$ is lsc on S . It was proved in (Carlota and Ornelas, 2007) that, in this case, if there exists a relaxed minimizer $y_c(\cdot)$ with $y_c([a, b]) \subset S$ then also exists another relaxed minimizer $y(\cdot)$ which is a true minimizer. The proof of this result consists in reducing problem (1.1) into a 1-dimensional problem of the calculus of variations — where the integral is defined on a particular class of reparametrizations of the interval $[a, b]$ — with the aim of finding a good reparametrization $\tau(\cdot)$ in the sense that $y(\cdot) := y_c(\tau(\cdot))$ is a (relaxed and) true minimizer.

In this paper we show that the hypothesis of $L(\cdot, 0)$ to be lsc may be replaced by an extra hypothesis on the relaxed minimizer $y_c(\cdot)$. The proof is done by showing that a special reparametrization of $[a, b]$, defined by the same formula (but using different sets) as the one build in the paper (Cellina and Ornelas, 2003) to obtain existence of solutions to nonconvex inclusions and time optimal control problems, also yields — in this case we are considering calculus of variations general problems with almost convex lagrangians — a solution to the original nonconvex problem.

2 Almost convex functions

The concept of ‘almost convex set’ was introduced in the paper (Cellina and Ornelas, 2003); and was then adapted in (Carlota and Ornelas, 2007) to define ‘almost convex function’.

Definition 2.1. (See (Carlota and Ornelas, 2007, Def. 1).) A function $F : \mathbb{R}^n \rightarrow (-\infty, \infty]$ is said to be almost convex if satisfies the following hypothesis :

$$\text{for each } \xi \text{ where } F^{**}(\xi) < F(\xi) \text{ there must} \\ \exists \lambda = \lambda_F(\xi) \in [0, 1] \quad \exists \Lambda = \Lambda_F(\xi) \in [1, \infty) \quad \exists \alpha = \alpha_F(\xi) \in [0, 1]$$

such that

$$F^{**}(\xi) = (1 - \alpha) F(\lambda \xi) + \alpha F(\Lambda \xi) \\ \xi = (1 - \alpha) (\lambda \xi) + \alpha (\Lambda \xi)$$

(we set $\lambda = 1 = \Lambda = \alpha$ where $F^{**}(\xi) = F(\xi)$ and use $0 \cdot \infty := 0$).

Notice that if $F(\cdot)$ is almost convex, then

(i) $F(\cdot)$ is 0-convex, i.e. $F^{**}(0) = F(0)$, and

(ii) for every $\xi \in \mathbb{R}^n$, $F^{**}(\lambda \xi) = F(\lambda \xi)$ & $F^{**}(\Lambda \xi) = F(\Lambda \xi)$.

In this paper we consider lagrangians $L : \mathbb{R}^n \times \mathbb{R}^n \rightarrow (-\infty, \infty]$ having $L(s, \cdot)$ almost convex for each s on some adequate closed set S . Next we present our basic hypothesis on the lagrangian.

Definition 2.2. Let $S \subset \mathbb{R}^n$ be a closed set. We will say that $L : \mathbb{R}^n \times \mathbb{R}^n \rightarrow (-\infty, \infty]$ is a BHyp-function on S if it is a Borel function bounded from below, having $L^{**}(\cdot)$ Borel and $L(s, \cdot)$ almost convex lsc $\forall s \in S$.

Proposition 2.1. Let $S \subset \mathbb{R}^n$ be a closed set, $L : \mathbb{R}^n \times \mathbb{R}^n \rightarrow (-\infty, \infty]$ a BHyp-function on S and $y_c(\cdot) \in \mathcal{X}_{A,B}^n$ having $y_c([a, b]) \subset S$. Then there exist measurable

$$\lambda = \lambda_L(y_c, y'_c) : [a, b] \rightarrow [0, 1] \quad \text{and} \quad \Lambda = \Lambda_L(y_c, y'_c) : [a, b] \rightarrow [1, \infty)$$

such that

$$\begin{aligned} L^{**}(y_c(t), y'_c(t)) &= \\ &= (1 - \alpha(t)) L(y_c(t), \lambda(t) y'_c(t)) + \alpha(t) L(y_c(t), \Lambda(t) y'_c(t)) \quad \text{a.e. on } [a, b], \end{aligned} \quad (2.1)$$

where $\alpha : [a, b] \rightarrow [0, 1]$ is given by

$$\alpha(t) := \begin{cases} 1 & \text{if } \lambda(t) = \Lambda(t) = 1 \\ \frac{1 - \lambda(t)}{\Lambda(t) - \lambda(t)} & \text{otherwise.} \end{cases}$$

3 Main result

Our main result is the following

Theorem 3.1. Let $S \subset \mathbb{R}^n$ be a closed set and let $L : \mathbb{R}^n \times \mathbb{R}^n \rightarrow (-\infty, \infty]$ be a BHyp-function on S . Assume the following extra hypothesis (EH) :

there exists a relaxed minimizer $y_c(\cdot)$ having $y_c([a, b]) \subset S$ and

either the set $E_0 := \{t \in [a, b] : \lambda(t) = 0\}$ has zero measure

or else : $\overline{E_0} \setminus E_0$ is a null set (with $\overline{E_0}$ the closure of E_0)

and

$\exists t_m \in [a, b]$ such that $L(y_c(t_m), 0) \leq L(y_c(t), 0) \quad \forall t \in [a, b]$.

Then there exists a true minimizer $y(\cdot)$. Moreover,

$$\min_{x(\cdot) \in \mathcal{X}_{A,B}^n} I(x(\cdot)) = \min_{x(\cdot) \in \mathcal{X}_{A,B}^n} I_c(x(\cdot)).$$

Proof. (Generalizing the proof of (Cellina and Ornelas, 2003, Theorem 2).)

In case

$$\int_a^b L^{**}(y_c(t), y'_c(t)) dt = \infty$$

we have $I(x(\cdot)) \equiv \infty$, therefore every $x(\cdot) \in \mathcal{X}_{A,B}^n$ is a minimizer for the nonconvex integral

$$\int_a^b L(x(t), x'(t)) dt \quad \text{on } \mathcal{X}_{A,B}^n.$$

So, in what follows we will assume that

$$\int_a^b L^{**}(y_c(t), y'_c(t)) dt < \infty,$$

hence

$$L^{**}(y_c(t), y'_c(t)) < \infty \quad \text{a.e. on } [a, b].$$

Let $\mu \in \mathbb{R}$ be such that

$$L(s, \xi) \geq \mu \quad \forall (s, \xi) \in \mathbb{R}^n \times \mathbb{R}^n.$$

(a) Suppose, in this part of the proof, that $|E_0| = 0$, where $|\cdot|$ denotes Lebesgue measure, so that $\lambda(t) \neq 0$ a.e. on $[a, b]$.

Define $\varphi : [a, b] \rightarrow [0, \infty)$,

$$\varphi(t) := \left[\frac{1}{\lambda(t)} + \frac{L(y_c(t), \lambda(t) y'_c(t)) - \mu}{\lambda(t)} + \frac{L(y_c(t), \Lambda(t) y'_c(t)) - \mu}{\Lambda(t)} \right] \chi_T(t),$$

where

$$T := \{t \in [a, b] : \exists y'_c(t), L^{**}(y_c(t), y'_c(t)) < \infty, \lambda(t) \neq 0 \text{ and (2.1) holds true}\}$$

and $\chi_T(\cdot)$ is the characteristic function of T :

$$\chi_T(t) := \begin{cases} 1 & \text{for } t \in T \\ 0 & \text{otherwise.} \end{cases}$$

For $n = 1, 2, \dots$, set

$$C^n := \{t \in [a, b] : n-1 \leq \varphi(t) < n\}.$$

Obviously

$$\varphi(\cdot) \in L^1(C^n) \quad \forall n \in \mathbb{N}.$$

Moreover,

$$[a, b] = \bigcup_{n=1}^{\infty} C^n$$

and

$$n_1 \neq n_2 \Rightarrow C^{n_1} \cap C^{n_2} = \emptyset.$$

Let $p_1, p_2 : [a, b] \rightarrow [0, 1]$ be the weight functions

$$p_1(t) := (1 - \alpha(t)) \lambda(t), \quad p_2(t) := \alpha(t) \Lambda(t).$$

By (2.1) and the Liapunov's theorem on the range of vector measures (see e.g. (Cesari, 1983, 16.1.v)), for each $n \in \mathbb{N}$ there exist two disjoint measurable subsets C_1^n, C_2^n of C^n such that $C^n = C_1^n \cup C_2^n$ and

$$\begin{aligned} & \int_{C^n} (1, L^{**}(y_c(t), y_c'(t)) - \mu) dt \\ &= \int_{C^n} \left(p_1(t) \frac{1}{\lambda(t)} + p_2(t) \frac{1}{\Lambda(t)}, p_1(t) \frac{L(y_c(t), \lambda(t) y_c'(t)) - \mu}{\lambda(t)} + p_2(t) \frac{L(y_c(t), \Lambda(t) y_c'(t)) - \mu}{\Lambda(t)} \right) dt \\ &= \int_{C^n} \left[p_1(t) \left(\frac{1}{\lambda(t)}, \frac{L(y_c(t), \lambda(t) y_c'(t)) - \mu}{\lambda(t)} \right) + p_2(t) \left(\frac{1}{\Lambda(t)}, \frac{L(y_c(t), \Lambda(t) y_c'(t)) - \mu}{\Lambda(t)} \right) \right] dt \\ &= \int_{C^n} \left[\chi_{C_1^n}(t) \left(\frac{1}{\lambda(t)}, \frac{L(y_c(t), \lambda(t) y_c'(t)) - \mu}{\lambda(t)} \right) + \chi_{C_2^n}(t) \left(\frac{1}{\Lambda(t)}, \frac{L(y_c(t), \Lambda(t) y_c'(t)) - \mu}{\Lambda(t)} \right) \right] dt. \end{aligned}$$

Hence, in particular,

$$\begin{aligned} & \int_{C^n} [L^{**}(y_c(t), y_c'(t)) - \mu] dt = \\ &= \int_{C^n} \left[\frac{L(y_c(t), \lambda(t) y_c'(t)) - \mu}{\lambda(t)} \chi_{C_1^n}(t) + \frac{L(y_c(t), \Lambda(t) y_c'(t)) - \mu}{\Lambda(t)} \chi_{C_2^n}(t) \right] dt. \end{aligned} \quad (3.1)$$

Let

$$C_1 := \bigcup_{n=1}^{\infty} C_1^n, \quad C_2 := \bigcup_{n=1}^{\infty} C_2^n.$$

Clearly,

$$C_1 \cap C_2 = \emptyset \text{ and } C_1 \cup C_2 = [a, b].$$

Using these sets C_1 and C_2 it is possible to construct — see (Cellina and Ornelas, 2003) for details of such construction — a strictly increasing AC function $\tau : [a, b] \rightarrow [a, b]$ with

$$\tau(a) = a, \quad \tau(b) = b$$

and derivative

$$\tau'(t) = \lambda(\tau(t)) \chi_{C_1}(\tau(t)) + \Lambda(\tau(t)) \chi_{C_2}(\tau(t)) \in (0, \infty) \text{ a.e. on } [a, b].$$

As in (Cellina and Ornelas, 2003), we define a new AC function $y : [a, b] \rightarrow \mathbb{R}^n$ by

$$y(t) := y_c(\tau(t)),$$

so that

$$y(a) = A, \quad y(b) = B$$

and

$$y'(t) = y_c'(\tau(t)) \tau'(t) \text{ a.e. on } [a, b].$$

Notice that the sets

$$\{t \in [a, b] : \tau(t) \in C_1\} \quad \text{and} \quad \{t \in [a, b] : \tau(t) \in C_2\}$$

also constitute a measurable decomposition of $[a, b]$.

We obtain, by (3.1) and the Lebesgue's monotone convergence theorem,

$$\begin{aligned} & \int_a^b [L^{**}(y_c(t), y'_c(t)) - \mu] dt \\ &= \int_{\bigcup_{n=1}^{\infty} C^n} [L^{**}(y_c(t), y'_c(t)) - \mu] dt \\ &= \sum_{n=1}^{\infty} \int_{C^n} [L^{**}(y_c(t), y'_c(t)) - \mu] dt \\ &= \sum_{n=1}^{\infty} \int_{C^n} \left[\frac{L(y_c(t), \lambda(t) y'_c(t)) - \mu}{\lambda(t)} \chi_{C_1^n}(t) + \frac{L(y_c(t), \Lambda(t) y'_c(t)) - \mu}{\Lambda(t)} \chi_{C_2^n}(t) \right] dt \\ &= \int_a^b \sum_{n=1}^{\infty} \left[\frac{L(y_c(t), \lambda(t) y'_c(t)) - \mu}{\lambda(t)} \chi_{C_1^n}(t) + \frac{L(y_c(t), \Lambda(t) y'_c(t)) - \mu}{\Lambda(t)} \chi_{C_2^n}(t) \right] dt \\ &= \int_a^b \left[\frac{L(y_c(t), \lambda(t) y'_c(t)) - \mu}{\lambda(t)} \chi_{C_1}(t) + \frac{L(y_c(t), \Lambda(t) y'_c(t)) - \mu}{\Lambda(t)} \chi_{C_2}(t) \right] dt. \end{aligned}$$

Hence, using the theorem of change of variables in the integral,

$$\begin{aligned} & \int_a^b [L^{**}(y_c(t), y'_c(t)) - \mu] dt \\ &= \int_a^b \left[\frac{L(y_c(\tau(t)), \lambda(\tau(t)) y'_c(\tau(t))) - \mu}{\lambda(\tau(t))} \tau'(t) \chi_{C_1}(\tau(t)) + \right. \\ & \quad \left. + \frac{L(y_c(\tau(t)), \Lambda(\tau(t)) y'_c(\tau(t))) - \mu}{\Lambda(\tau(t))} \tau'(t) \chi_{C_2}(\tau(t)) \right] dt \\ &= \int_a^b \left[\frac{L(y_c(\tau(t)), \lambda(\tau(t)) y'_c(\tau(t))) - \mu}{\lambda(\tau(t))} \lambda(\tau(t)) \chi_{C_1}(\tau(t)) + \right. \\ & \quad \left. + \frac{L(y_c(\tau(t)), \Lambda(\tau(t)) y'_c(\tau(t))) - \mu}{\Lambda(\tau(t))} \Lambda(\tau(t)) \chi_{C_2}(\tau(t)) \right] dt \\ &= \int_a^b [[L(y(t), y'(t)) - \mu] \chi_{C_1}(\tau(t)) + [L(y(t), y'(t)) - \mu] \chi_{C_2}(\tau(t))] dt \\ &= \int_a^b [L(y(t), y'(t)) - \mu] (\chi_{C_1}(\tau(t)) + \chi_{C_2}(\tau(t))) dt \\ &= \int_a^b [L(y(t), y'(t)) - \mu] \chi_{(a,b)}(\tau(t)) dt \\ &= \int_a^b [L(y(t), y'(t)) - \mu] dt. \end{aligned}$$

This proves that $y(\cdot)$ minimizes the nonconvex integral. Indeed, since

$$\int_a^b L^{**}(y_c(t), y'_c(t)) dt = \int_a^b L(y(t), y'(t)) dt \geq \int_a^b L^{**}(y(t), y'(t)) dt$$

and $y_c(\cdot)$ is a relaxed minimizer, we have

$$\int_a^b L^{**}(y_c(t), y'_c(t)) dt = \int_a^b L^{**}(y(t), y'(t)) dt.$$

Thus $y(\cdot)$ is also a relaxed minimizer and

$$\begin{aligned} \int_a^b L(y(t), y'(t)) dt &= \int_a^b L^{**}(y(t), y'(t)) dt \\ &\leq \int_a^b L^{**}(x(t), x'(t)) dt \leq \int_a^b L(x(t), x'(t)) dt \end{aligned}$$

for every $x(\cdot) \in \mathcal{X}_{A,B}^n$.

(b) Along the remaining of this proof, we will consider the case $|E_0| > 0$.

Let

$$C_0 := \overline{E_0}.$$

By hypothesis $|\overline{E_0} \setminus E_0| = 0$, so we certainly have, for a.e. $t \in C_0$,

$$\lambda(t) = 0$$

hence $(y'_c(t) \neq 0, \Lambda(t) > 1$ and)

$$L^{**}(y_c(t), y'_c(t)) = \left(1 - \frac{1}{\Lambda(t)}\right) L(y_c(t), 0) + \frac{1}{\Lambda(t)} L(y_c(t), \Lambda(t) y'_c(t)).$$

Moreover, we may assume, without loss of generality, that

$$L(y_c(t_m), 0) = \min L(y_c([a, b]), 0) = 0.$$

(c) Let us consider the interval $[a, t_m]$.

Since $(a, t_m) \setminus C_0$ is open, there exists a countable family of disjoint open subintervals (a_i^-, b_i^-) of (a, t_m) such that

$$(a, t_m) \setminus C_0 = \bigcup_{i=1}^{\infty} (a_i^-, b_i^-).$$

As in (a), for each i we can obtain a measurable decomposition of (a_i^-, b_i^-) into two subsets C_{1i}^- and C_{2i}^- for which

$$\begin{aligned} & \int_{a_i^-}^{b_i^-} (1, L^{**}(y_c(t), y_c'(t)) - \mu) dt = \\ & = \int_{a_i^-}^{b_i^-} \left[\chi_{C_{1i}^-}(t) \left(\frac{1}{\lambda(t)}, \frac{L(y_c(t), \lambda(t) y_c'(t)) - \mu}{\lambda(t)} \right) + \right. \\ & \quad \left. + \chi_{C_{2i}^-}(t) \left(\frac{1}{\Lambda(t)}, \frac{L(y_c(t), \Lambda(t) y_c'(t)) - \mu}{\Lambda(t)} \right) \right] dt. \end{aligned} \quad (3.2)$$

Let

$$C_0^- := [a, t_m] \cap C_0$$

and

$$t_{m-} := a + \int_{C_0^-} \frac{1}{\Lambda(t)} dt + |[a, t_m] \setminus C_0| \leq t_m.$$

There exists — see (Cellina and Ornelas, 2003) — an AC function $\tau_- : [a, t_m] \rightarrow [a, t_m]$ which is strictly increasing on $[a, t_{m-}]$ with

$$\tau_-(a) = a, \quad \tau_-(t_{m-}) = t_m$$

and derivative

$$\tau_-'(t) = \Lambda(\tau_-(t)) \chi_{C_0^-}(\tau_-(t)) + \sum_{i=1}^{\infty} \left(\lambda(\tau_-(t)) \chi_{C_{1i}^-}(\tau_-(t)) + \Lambda(\tau_-(t)) \chi_{C_{2i}^-}(\tau_-(t)) \right)$$

for a.e. $t \in [a, t_{m-}]$, while

$$\tau_-(t) \equiv t_m \text{ on } [t_{m-}, t_m].$$

Consider — as in (Cellina and Ornelas, 2003) — the AC function $y_- : [a, t_m] \rightarrow \mathbb{R}^n$ defined by

$$y_-(t) := y_c(\tau_-(t)).$$

Therefore

$$y_-(a) = A, \quad y_-(t_m) = y_c(t_m)$$

and

$$L(y_-(t), y_-'(t)) = L(y_c(t_m), 0) = 0 \text{ for a.e. } t \in [t_{m-}, t_m]. \quad (3.3)$$

(d) Let us prove that

$$\int_a^{t_m} L^{**}(y_c(t), y'_c(t)) dt \geq \int_a^{t_m} L(y_-(t), y'_-(t)) dt. \quad (3.4)$$

Due to (3.3), on C_0^- we obtain

$$\begin{aligned} & \int_{C_0^-} L^{**}(y_c(t), y'_c(t)) dt \\ &= \int_a^{t_m} \left[\left(1 - \frac{1}{\Lambda(t)}\right) L(y_c(t), 0) + \frac{L(y_c(t), \Lambda(t)y'_c(t))}{\Lambda(t)} \right] \chi_{C_0^-}(t) dt \\ &= \int_a^{t_m} \left[\left(1 - \frac{1}{\Lambda(\tau_-(t))}\right) L(y_c(\tau_-(t)), 0) + \frac{L(y_c(\tau_-(t)), \Lambda(\tau_-(t))y'_c(\tau_-(t)))}{\Lambda(\tau_-(t))} \right] \tau'_-(t) \chi_{C_0^-}(\tau_-(t)) dt \\ &= \int_a^{t_m} \left[\left(1 - \frac{1}{\Lambda(\tau_-(t))}\right) L(y_c(\tau_-(t)), 0) + \frac{L(y_-(t), y'_-(t))}{\Lambda(\tau_-(t))} \right] \Lambda(\tau_-(t)) \chi_{C_0^-}(\tau_-(t)) dt \\ &= \int_a^{t_m} [(\Lambda(\tau_-(t)) - 1) L(y_c(\tau_-(t)), 0) + L(y_-(t), y'_-(t))] \chi_{C_0^-}(\tau_-(t)) dt \\ &\geq \int_a^{t_m} [(\Lambda(\tau_-(t)) - 1) L(y_c(t_m), 0) + L(y_-(t), y'_-(t))] \chi_{C_0^-}(\tau_-(t)) dt \\ &= \int_a^{t_m} L(y_-(t), y'_-(t)) \chi_{C_0^-}(\tau_-(t)) dt \\ &= \int_a^{t_m} L(y_-(t), y'_-(t)) \chi_{C_0^-}(\tau_-(t)) dt + \int_{t_m}^{t_m} L(y_-(t), y'_-(t)) \chi_{C_0^-}(\tau_-(t)) dt \\ &= \int_a^{t_m} L(y_-(t), y'_-(t)) \chi_{C_0^-}(\tau_-(t)) dt. \end{aligned}$$

On the other hand, using (3.2) and proceeding as in (a), we obtain for each $(a_i^-, b_i^-) \subset (a, t_m) \setminus C_0$:

$$\begin{aligned} & \int_{a_i^-}^{b_i^-} [L^{**}(y_c(t), y'_c(t)) - \mu] dt \\ &= \int_{a_i^-}^{b_i^-} \left[\frac{L(y_c(t), \lambda(t)y'_c(t)) - \mu}{\lambda(t)} \chi_{C_{1i}^-}(t) + \frac{L(y_c(t), \Lambda(t)y'_c(t)) - \mu}{\Lambda(t)} \chi_{C_{2i}^-}(t) \right] dt \\ &= \int_{\tau_-(a_i^-)}^{\tau_-(b_i^-)} [L(y_-(t), y'_-(t)) - \mu] \chi_{(a_i^-, b_i^-)}(\tau_-(t)) dt \\ &= \int_a^{t_m} [L(y_-(t), y'_-(t)) - \mu] \chi_{(a_i^-, b_i^-)}(\tau_-(t)) \chi_{(\tau_-(a_i^-), \tau_-(b_i^-))}(t) dt \\ &= \int_a^{t_m} [L(y_-(t), y'_-(t)) - \mu] \chi_{(a_i^-, b_i^-)}(\tau_-(t)) dt. \end{aligned}$$

Moreover,

$$\begin{aligned} b_i^- - a_i^- &= \int_{\tau_i^-}^{\tau_i^{-1}(b_i^-)} \left[\chi_{C_{1i}^-}(\tau_-(t)) + \chi_{C_{2i}^-}(\tau_-(t)) \right] dt \\ &= \int_a^{t_{m-}} \chi_{(a_i^-, b_i^-)}(\tau_-(t)) dt, \end{aligned}$$

hence

$$\begin{aligned} \int_a^{t_{m-}} \sum_{i=1}^{\infty} \chi_{(a_i^-, b_i^-)}(\tau_-(t)) dt &= \sum_{i=1}^{\infty} \int_a^{t_{m-}} \chi_{(a_i^-, b_i^-)}(\tau_-(t)) dt \\ &= \sum_{i=1}^{\infty} (b_i^- - a_i^-) = |[a, t_m] \setminus C_0|. \end{aligned}$$

Then

$$\begin{aligned} &\int_{[a, t_m] \setminus C_0} [L^{**}(y_c(t), y'_c(t)) - \mu] dt \\ &= \sum_{i=1}^{\infty} \int_{a_i^-}^{b_i^-} [L^{**}(y_c(t), y'_c(t)) - \mu] dt \\ &= \sum_{i=1}^{\infty} \int_a^{t_{m-}} [L(y_-(t), y'_-(t)) - \mu] \chi_{(a_i^-, b_i^-)}(\tau_-(t)) dt \\ &= \int_a^{t_{m-}} [L(y_-(t), y'_-(t)) - \mu] \sum_{i=1}^{\infty} \chi_{(a_i^-, b_i^-)}(\tau_-(t)) dt \\ &= \int_a^{t_{m-}} \left[L(y_-(t), y'_-(t)) \chi_{[a, t_m] \setminus C_0}(\tau_-(t)) - \mu \sum_{i=1}^{\infty} \chi_{(a_i^-, b_i^-)}(\tau_-(t)) \right] dt \\ &= \int_a^{t_{m-}} L(y_-(t), y'_-(t)) \chi_{[a, t_m] \setminus C_0}(\tau_-(t)) dt - \mu |[a, t_m] \setminus C_0|, \end{aligned}$$

which implies that

$$\begin{aligned} &\int_{[a, t_m] \setminus C_0} L^{**}(y_c(t), y'_c(t)) dt \\ &= \int_a^{t_{m-}} L(y_-(t), y'_-(t)) \chi_{[a, t_m] \setminus C_0}(\tau_-(t)) dt \\ &= \int_a^{t_{m-}} L(y_-(t), y'_-(t)) \chi_{[a, t_m] \setminus C_0}(\tau_-(t)) dt + \\ &\quad + \int_{t_{m-}}^{t_m} L(y_-(t), y'_-(t)) \chi_{[a, t_m] \setminus C_0}(\tau_-(t)) dt \\ &= \int_a^{t_m} L(y_-(t), y'_-(t)) \chi_{[a, t_m] \setminus C_0}(\tau_-(t)) dt. \end{aligned}$$

We are now in conditions to prove (3.4) :

$$\begin{aligned}
 & \int_a^{t_m} L^{**}(y_c(t), y'_c(t)) dt \\
 &= \int_{C_0^-} L^{**}(y_c(t), y'_c(t)) dt + \int_{[a, t_m] \setminus C_0} L^{**}(y_c(t), y'_c(t)) dt \\
 &\geq \int_a^{t_m} L(y_-(t), y'_-(t)) \chi_{C_0^-}(\tau_-(t)) dt + \int_a^{t_m} L(y_-(t), y'_-(t)) \chi_{[a, t_m] \setminus C_0}(\tau_-(t)) dt \\
 &= \int_a^{t_m} L(y_-(t), y'_-(t)) \left(\chi_{C_0^-}(\tau_-(t)) + \chi_{[a, t_m] \setminus C_0}(\tau_-(t)) \right) dt \\
 &= \int_a^{t_m} L(y_-(t), y'_-(t)) dt.
 \end{aligned}$$

(e) On the interval $[t_m, b]$ we proceed in a similar way to what we did on $[a, t_m]$.

Let (a_i^+, b_i^+) , $i = 1, 2, \dots$, be the disjoint subintervals of (t_m, b) whose union is $(t_m, b) \setminus C_0$ and, for each i , let C_{1i}^+, C_{2i}^+ be the measurable decomposition of (a_i^+, b_i^+) for which (3.2) holds true with $a_i^-, b_i^-, C_{1i}^-, C_{2i}^-$ replaced by $a_i^+, b_i^+, C_{1i}^+, C_{2i}^+$. Moreover, set

$$C_0^+ := [t_m, b] \cap C_0$$

and

$$t_{m+} := b - \int_{C_0^+} \frac{1}{\Lambda(t)} dt - |[t_m, b] \setminus C_0| \geq t_m.$$

Define $y_+ : [t_m, b] \rightarrow \mathbb{R}^n$ by

$$y_+(t) := y_c(\tau_+(t)),$$

where $\tau_+ : [t_m, b] \rightarrow [t_m, b]$ is the AC reparametrization of $[t_m, b]$ having

$$\tau_+(t) \equiv t_m \text{ on } [t_m, t_{m+}]$$

and is strictly increasing on $[t_{m+}, b]$ with

$$\tau_+(b) = b$$

and derivative

$$\tau'_+(t) = \Lambda(\tau_+(t)) \chi_{C_0^+}(\tau_+(t)) + \sum_{i=1}^{\infty} \left(\lambda(\tau_+(t)) \chi_{C_{1i}^+}(\tau_+(t)) + \Lambda(\tau_+(t)) \chi_{C_{2i}^+}(\tau_+(t)) \right)$$

for a.e. $t \in [t_{m+}, b]$.

Then $y_+(\cdot)$ is AC,

$$y_+(t_m) = y_c(t_m), \quad y_+(b) = B$$

and

$$\int_{t_m}^b L^{**}(y_c(t), y_c'(t)) dt \geq \int_{t_m}^b L(y_+(t), y_+'(t)) dt.$$

(f) Finally, let us joint together $y_-(\cdot)$ and $y_+(\cdot)$ so as to obtain our aimed true minimizer. Let $y : [a, b] \rightarrow \mathbb{R}^n$ be the AC function given by

$$y(t) := \begin{cases} y_-(t) & \text{for } t \in [a, t_m] \\ y_+(t) & \text{for } t \in [t_m, b]. \end{cases} \quad (3.5)$$

Clearly,

$$y(a) = A, \quad y(b) = B \quad (3.6)$$

and, moreover,

$$\int_a^b L^{**}(y_c(t), y_c'(t)) dt \geq \int_a^b L(y(t), y'(t)) dt. \quad (3.7)$$

Therefore $y(\cdot)$ is a minimizer for the convex integral

$$\int_a^b L^{**}(x(t), x'(t)) dt, \quad x(\cdot) \in \mathcal{X}_{A,B}^n,$$

as well as for the nonconvex integral

$$\int_a^b L(x(t), x'(t)) dt, \quad x(\cdot) \in \mathcal{X}_{A,B}^n.$$

This completes the proof of our main result.

□

The next corollary concerns the scalar $n = 1$ case.

Corollary 3.2. Assume $L : \mathbb{R} \times \mathbb{R} \rightarrow (-\infty, \infty]$ is a Borel function bounded from below with $L^{**}(\cdot)$ also Borel, $L(\cdot, 0) = L^{**}(\cdot, 0)$ and $L(s, \cdot)$ lsc $\forall s$.

Assume, in addition, the following extra hypothesis (EH_1) :

there exists a relaxed minimizer $y_c(\cdot)$ for which

the faces of $\text{epi } L^{**}(y_c(t), \cdot)$ are bounded $\forall t \in [a, b]$ and

$$y'_c(t) \notin \text{interior}(\text{co}\{0, \beta(y_c(t))\}) \quad \text{for a.e. } t \in [a, b] \quad (3.8)$$

whenever $L^{**}(s, \cdot)$ is affine on $\text{co}\{0, \beta(s)\} \quad \forall s \in y_c([a, b])$.

(Here $\text{epi } L^{**}(y_c(t), \cdot)$ is the epigraph of $L^{**}(y_c(t), \cdot)$ and $\text{co}\{0, \beta(s)\}$ denotes the convex hull of $\{0, \beta(s)\}$.)

Then there exists a true minimizer $y(\cdot)$.

Proof. Under the hypotheses on $L(\cdot)$ and $L^{**}(\cdot)$, $L(s, \cdot)$ is almost convex lsc $\forall s \in S = y_c([a, b])$. Furthermore,

$$(3.8) \quad \Rightarrow \quad \lambda(t) \neq 0 \quad \text{for a.e. } t \in [a, b].$$

Thus

$$|E_0| = 0.$$

□

Remark 3.1. Notice that when $n = 1$ and we know that some relaxed minimizer $y_c(\cdot)$ is, or can be transformed into a, bimonotone relaxed minimizer $\tilde{y}_c(\cdot)$ — see (Ornelas, 2007) for details on bimonotone minimizers — then $\tilde{y}_c(\cdot)$ will satisfy (3.8) hence the Extra Hypothesis (EH).

Conclusion

We did attain the goal of proving our main result, whose main feature is the complete liberty for the lagrangian to take the ∞ value — provided our Extra Hypothesis on the given relaxed minimizer is satisfied — thus allowing constraints on positions and/or velocities to be automatically incorporated into the lagrangian. Notice also that in some cases, e.g. as in Remark 3.1, our Extra Hypothesis is automatically satisfied. In such cases, absolutely no lower semicontinuity hypothesis on $L(\cdot, \xi)$ is needed.

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