

The nehari manifold for mean curvature equations

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ABSTRACT

In this paper, we study the existence and combination of mean curvature equation with Dirichlet boundary condition by the Nehari manifold method and some variational technique and show that an elliptic equation has at least two positive answers.

Keywords: Nehari manifold; $p(x)$ -Laplacian; Mean curvature; variable exponent Sobolev space; variational methods.

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1 Introduction

In this work, we consider the following elliptic system

$$\begin{cases} -\operatorname{div}\left((1 + \frac{|\nabla v|^{r(x)}}{\sqrt{1+|\nabla v|^{2r(x)}}})|\nabla v|^{r(x)-2}\nabla v\right) = \mu p(x)|v|^{s(x)-2}v + q(x)|v|^{t(x)-2}v & \text{in } D, \\ v = 0 & \text{on } \partial D. \end{cases} \quad (1.1)$$

Where D is a bounded domain with smooth boundary ∂D in $\mathbb{R}^N$. Let $M \geq 2, r, s, t \in C(\overline{D})$ such that $1 < \frac{3}{2}r(x) < t(x) < r^*(x)$, $1 < s(x) < r(x)$, $r^*(x) = \frac{Mr(x)}{M-r(x)}$ if $M > r(x)$, $r^*(x) = \infty$ if $M \leq r(x)$, $1 < r^- := \operatorname{ess\,inf}_{x \in (D)} r(x) \leq r(x) \leq r^+ := \operatorname{ess\,sup}_{x \in (D)} r(x) < \infty$, $0 < \mu \in \mathbb{R}$ and $p, q \in C(\overline{D})$ be the non-negative weight functions with compact support in D . We shall prove the problem (1.1) has at least two positive answers.

An interesting feature of our study, is the use of the method of Nehari manifold, which allows us to obtain existence results of the equation (1.1). Brown and Zhang has used Nehari manifold for a semilinear elliptic system (Brown, 2005). This approach has been used also by several authors, see for example (Fan and Zhao, 2001), (Mashiyev and Avci, 2010), (Shi, 2008), (Wu, 2007).

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2 Notations an preliminaries

In order to discuss problem (1.1), we need some theories and basic properties. Define the variable exponent Sobolev space $W^{1,r(x)}(D)$ by

$$W^{1,r(x)}(D) = \left\{ v \in L^{r(x)}(D); |\nabla v| \in L^{r(x)}(D) \right\}$$

equipped with the norm

$$\|v\| = \inf \left\{ \tau > 0; \int_D \left| \frac{v(x)}{\tau} \right|^{r(x)} + \left| \frac{\nabla v(x)}{\tau} \right|^{r(x)} dx \leq 1 \right\}. \quad (2.1)$$

The space $W_0^{1,r(x)}(D)$ which is the closure of $C_0^\infty(D)$ in $W^{1,r(x)}(D)$, is a separable and reflexive Banach space.

Let us recall the proposition 2.9 in (Mashiyev and Avci, 2010)

Proposition 2.1 ((Mashiyev and Avci, 2010)). *Assume that the conditions of the both theorems 2.7 and 2.8 in (Mashiyev and Avci, 2010) hold. Let $v \in W^{1,r(x)}(D)$, then there are positive constants $b_3, b_4, b_5, b_6 > 0$ such that the following inequalities holds*

$$\int_D q(x)|v|^{t(x)} dx \leq \begin{cases} b_3 \|v\|^{t^+} & \text{if } \|v\| > 1; \\ b_4 \|v\|^{t^-} & \text{if } \|v\| < 1. \end{cases} \quad (2.2)$$

$$\int_D p(x)|v|^{s(x)} dx \leq \begin{cases} b_5 \|v\|^{s^+} & \text{if } \|v\| > 1; \\ b_6 \|v\|^{s^-} & \text{if } \|v\| < 1. \end{cases} \quad (2.3)$$

Proof. It follows as a consequence of 2.7, 2.8 (Mashiyev and Avci, 2010). $\square$

3 The main results

Assume $K \in C^1(X, R)$ be the Euler functional associated with an elliptic problem on Banach space X . In many problems K is not bounded below, but is bounded below on an appropriate subset of X and minimizer on this set (if there are) may give rise to solutions of the dependent elliptic problem. This appropriate subset of X is Nehari manifold. The corresponding Euler functional, as follows

$$K_\mu(v) = \int_D \frac{1}{r(x)} (|\nabla v|^{r(x)} + \sqrt{1 + |\nabla v|^{2r(x)}}) dx - \mu \int_D \frac{p(x)}{s(x)} |v|^{s(x)} dx - \int_D \frac{q(x)}{t(x)} |v|^{t(x)} dx$$

By theorems 2.1 (Fan and Zhao, 2001), 2.7, 2.8 (Mashiyev and Avci, 2010)

$$\begin{aligned} K_\mu(v) &\geq \frac{1}{r^+} \int_D \left(|\nabla v|^{r(x)} + \sqrt{1 + |\nabla v|^{2r(x)}} \right) dx - \frac{\mu}{s^-} \int_D p(x) |v|^{s(x)} dx \\ &\quad - \frac{1}{t^-} \int_D q(x) |v|^{t(x)} dx \\ &\geq \frac{2}{r^+} \int_D |\nabla v|^{r(x)} dx - \frac{\mu}{s^-} \int_D p(x) |v|^{s(x)} dx - \frac{1}{t^-} \int_D q(x) |v|^{t(x)} dx \\ &\geq \frac{2}{r^+} \|v\|^{r^-} - \frac{\mu}{s^-} b_2 (\|v\|^{s^-} + \|v\|^{s^+}) - \frac{b_1}{t^-} (\|v\|^{t^-} + \|v\|^{t^+}). \end{aligned} \quad (3.1)$$

Since $1 < s^- \leq s^+ < r^- < \frac{3}{2}r^- \leq \frac{3}{2}r^+ < t^- \leq t^+$, K_μ is not bounded below on whole $W_0^{1,r(x)}(D)$. The Nehari manifold $H_\mu(D)$, contains every nonzero solution of 1.1.

$$H_\mu(D) = \left\{ v \in W_0^{1,r(x)}(D) \setminus \{0\} : \langle K'_\mu(v), v \rangle = 0 \right\}$$

For $v \in H_\mu(D)$

$$\begin{aligned} \varphi_\mu(v) &:= \langle K'_\mu(v), v \rangle = \int_D (|\nabla v|^{r(x)} + \frac{|\nabla v|^{2r(x)}}{\sqrt{1 + |\nabla v|^{2r(x)}}}) dx - \mu \int_D p(x)|v|^{s(x)} dx \\ &\quad - \int_D q(x)|v|^{t(x)} dx \\ &= 0 \end{aligned} \quad (3.2)$$

$$\begin{aligned} \langle \varphi'_\mu(v), v \rangle &= \int_D r(x)|\nabla v|^{r(x)} dx + \int_D \frac{2r(x)|\nabla v|^{2r(x)}}{\sqrt{1 + |\nabla v|^{2r(x)}}} dx \\ &\quad - \int_D \frac{r(x)|\nabla v|^{4r(x)}}{\sqrt{(1 + |\nabla v|^{2r(x)})^3}} dx - \mu \int_D s(x)p(x)|v|^{s(x)} dx - \int_D t(x)q(x)|v|^{t(x)} dx. \end{aligned}$$

We split $H_\mu(D)$ into three parts:

$$H_\mu^+ = \left\{ v \in H_\mu(D) : \langle \varphi'_\mu(v), v \rangle > 0 \right\}$$

$$H_\mu^0 = \left\{ v \in H_\mu(D) : \langle \varphi'_\mu(v), v \rangle = 0 \right\}$$

$$H_\mu^- = \left\{ v \in H_\mu(D) : \langle \varphi'_\mu(v), v \rangle < 0 \right\}$$

Lemma 3.1. *The energy functional K_μ is coercive and bounded below on $H_\mu(D)$.*

Proof. Let $v \in H_\mu(D)$ and $\|v\| > 1$. Then using propositions 2.1, (Fan and Zhao, 2001)

$$\begin{aligned} K_\mu(v) &= \int_D \frac{1}{r(x)} (|\nabla v|^{r(x)} + \sqrt{1 + |\nabla v|^{2r(x)}}) dx - \mu \int_D \frac{p(x)}{s(x)} |v|^{s(x)} dx \\ &\quad - \int_D \frac{q(x)}{t(x)} |v|^{t(x)} dx \\ &\geq \frac{1}{r^+} \int_D (|\nabla v|^{r(x)} + \sqrt{1 + |\nabla v|^{2r(x)}}) dx - \frac{\mu}{s^-} \int_D p(x)|v|^{s(x)} dx \\ &\quad - \frac{1}{t^-} \left(\int_D |\nabla v|^{r(x)} dx + \int_D \frac{|\nabla v|^{2r(x)}}{\sqrt{1 + |\nabla v|^{2r(x)}}} dx - \mu \int_D p(x)|v|^{s(x)} dx \right) \\ &\geq \left(\frac{2}{r^+} - \frac{1}{t^-} \right) \int_D |\nabla v|^{r(x)} dx - \mu \left(\frac{1}{s^-} - \frac{1}{t^-} \right) \int_D p(x)|v|^{s(x)} dx \\ &\quad - \frac{1}{t^-} \int_D |\nabla v|^{r(x)} dx \\ &= \left(\frac{2}{r^+} - \frac{2}{t^-} \right) \|v\|^{r^-} - \mu \left(\frac{1}{s^-} - \frac{1}{t^-} \right) \int_D p(x)|v|^{s(x)} dx \\ &\geq \left(\frac{2t^- - 2r^+}{r^+t^-} \right) \|v\|^{r^-} - \mu b_5 \left(\frac{t^- - s^-}{s^-t^-} \right) \|v\|^{s^+}. \end{aligned}$$

Therefore K_μ is coercive and bounded below on $H_\mu(D)$. □

Lemma 3.2. *There exists $\mu_1 > 0$ such that for $0 < \mu < \mu_1$, $H_\mu^0(D) = \emptyset$.*

Proof. If $H_\mu^0(D) \neq \emptyset$ for all $\mu \in \mathbb{R} \setminus \{0\}$ and $v \in H_\mu^0(D)$ such that $\|v\| > 1$. By proposition 2.1 and definition of $H_\mu^0(D)$,

$$\begin{aligned}
 0 &= \langle \varphi'_\mu(v), v \rangle \\
 &= \int_D r(x) |\nabla v|^{r(x)} dx + \int_D \frac{2r(x) |\nabla v|^{2r(x)}}{\sqrt{1 + |\nabla v|^{2r(x)}}} dx - \int_D \frac{r(x) |\nabla v|^{4r(x)}}{\sqrt{(1 + |\nabla v|^{2r(x)})^3}} dx \\
 &\quad - \mu \int_D s(x) p(x) |v|^{s(x)} dx - \int_D t(x) q(x) |v|^{t(x)} dx \\
 &\geq r^- \|v\|^{r(x)} + 2r^- \int_D \frac{|\nabla v|^{2r(x)}}{\sqrt{1 + |\nabla v|^{2r(x)}}} dx - r^+ \|v\|^{r(x)} - \mu s^+ \int_D p(x) |v|^{s(x)} dx \\
 &\quad - s^+ \int_D q(x) |v|^{t(x)} dx \\
 &\geq r^- \|v\|^{r(x)} + 2r^- \int_D \frac{|\nabla v|^{2r(x)}}{\sqrt{1 + |\nabla v|^{2r(x)}}} dx - r^+ \|v\|^{r(x)} - s^+ [\|v\|^{r(x)} \\
 &\quad + \int_D \frac{|\nabla v|^{2r(x)}}{\sqrt{1 + |\nabla v|^{2r(x)}}} dx - \int_D q(x) |v|^{t(x)} dx] - t^+ \int_D q(x) |v|^{t(x)} dx \\
 &\geq (r^- - r^+ - s^+) \|v\|^{r(x)} + (2r^- - s^+) \int_D \frac{|\nabla v|^{2r(x)}}{\sqrt{1 + |\nabla v|^{2r(x)}}} dx \\
 &\quad + (s^+ - t^+) \int_D q(x) |v|^{t(x)} dx \\
 &\geq (r^- - r^+ - s^+) \|v\|^{r(x)} + (2r^- - s^+) \int_D \sqrt{1 + |\nabla v|^{2r(x)}} \left(1 - \frac{1}{1 + |\nabla v|^{2r(x)}}\right) dx \\
 &\quad + (s^+ - t^+) \int_D q(x) |v|^{t(x)} dx \\
 &\geq -r^+ \|v\|^{r^+} + (2r^- - s^+) \eta \|v\|^{r^+} - b_3(t^+ - s^+) \|v\|^{t^+} \\
 &\Rightarrow (r^- \eta - r^+) \|v\|^{r^+} - b_3(t^+ - s^+) \|v\|^{t^+} \leq 0
 \end{aligned} \tag{3.3}$$

Since $0 < \eta < 1$ and by proposition 2.1

$$\|v\| \geq b_7 \left(\frac{r^- \eta - r^+}{t^+ - s^+} \right)^{\frac{1}{t^+ - r^+}}. \tag{3.4}$$

Also,

$$\begin{aligned}
 0 &= \langle \varphi'_\mu(v), v \rangle \\
 &= \int_D r(x) |\nabla v|^{r(x)} dx + \int_D \frac{2r(x) |\nabla v|^{2r(x)}}{\sqrt{1 + |\nabla v|^{2r(x)}}} dx - \int_D \frac{r(x) |\nabla v|^{4r(x)}}{\sqrt{(1 + |\nabla v|^{2r(x)})^3}} dx \\
 &\quad - \mu \int_D s(x) p(x) |v|^{s(x)} dx - \int_D q(x) |v|^{t(x)} dx \\
 &\leq r^+ \|v\|^{r(x)} + 2r^+ \|v\|^{r(x)} - r^- \int_D \frac{|\nabla v|^{4r(x)}}{\sqrt{(1 + |\nabla v|^{2r(x)})^3}} dx - \mu s^- \int_D p(x) |v|^{s(x)} dx \\
 &\quad - t^- [\|v\|^{r(x)} + \int_D \frac{|\nabla v|^{2r(x)}}{\sqrt{1 + |\nabla v|^{2r(x)}}} dx - \mu \int_D p(x) |v|^{s(x)} dx]
 \end{aligned} \tag{3.5}$$

By 2.1

$$\begin{aligned} 0 &\leq (3r^+ - 2t^-) \|v\|^{r^-} + \mu b_5 (t^- - s^-) \|v\|^{s^+} \\ \Rightarrow \|v\| &\leq b_8 (\mu \frac{s^- - t^-}{3r^+ - 2t^-})^{\frac{1}{r^- - s^+}} \end{aligned} \quad (3.6)$$

For μ is sufficiently small ($\mu \leq (\frac{3r^+ - 2t^-}{s^- - t^-}) (\frac{\eta r^- - r^+}{t^+ - s^+})^{\frac{r^- - s^+}{t^+ - s^+}}$), from 3.4, 3.6, we get $\|v\| < 1$ which contradicts. Thus $H_\mu^0(D) = \emptyset$. $\square$

By Lemma 3.2, for $0 < \mu < \mu_1$, we can write $H_\mu(D) = H_\mu^+(D) \cup H_\mu^-(D)$,

$$\vartheta_\mu^+ := \inf_{u \in H_\mu^+(D)} K_\mu(v), \quad \vartheta_\mu^- := \inf_{v \in H_\mu^-(D)} K_\mu(v)$$

Theorem 3.3. Let v_0 is a local maximum or minimum for K_μ on $H_\mu(D)$. If $v_0 \notin H_\mu^0(D)$, thus $K'_\mu(v_0) = 0$.

Proof. The proof can be obtained directly from the above lemmas. $\square$

Lemma 3.4. If $0 < \mu < \mu_1$, then for all $v \in H_\mu^+(D)$, $K_\mu(v) < 0$.

Proof. Let $v \in H_\mu^+(D)$

$$\begin{aligned} K_\mu(v) &\leq \frac{1}{r^-} \int_D (|\nabla v|^{r(x)} dx + \sqrt{1 + |\nabla v|^{2r(x)}}) dx - \frac{\mu}{s^+} \int_D p(x) |v|^{s(x)} dx \\ &\quad - \frac{1}{t^+} \int_D q(x) |v|^{t(x)} dx. \end{aligned} \quad (3.7)$$

Since $v \in H_\mu^+(D)$, then $\langle \varphi'_\mu(v), v \rangle > 0$,

$$\begin{aligned} r^+ \int_D |\nabla v|^{r(x)} dx + 2r^+ \int_D \frac{|\nabla v|^{2r(x)}}{\sqrt{1 + |\nabla v|^{2r(x)}}} dx - r^- \int_D \frac{|\nabla v|^{4r(x)}}{\sqrt{1 + |\nabla v|^{2r(x)}}^3} dx \\ - \mu s^- \int_D p(x) |v|^{s(x)} dx - t^- \int_D q(x) |v|^{t(x)} dx > 0. \end{aligned} \quad (3.8)$$

By combination 3.2 by $(-s^-)$ and 3.8

$$\int_D q(x) |v|^{t(x)} dx < (\frac{3r^+ - 2s^-}{t^- - s^-}) \|v\|^{r(x)} \quad (3.9)$$

By substituting 3.9 in 3.2,

$$\begin{aligned} K_\mu(v) &\leq (\frac{1}{r^-} - \frac{1}{s^+}) \|v\|^{r(x)} + \frac{1}{r^-} \int_D \sqrt{1 + |\nabla v|^{2r(x)}} dx - \frac{1}{s^+} \int_D \frac{|\nabla v|^{2r(x)}}{\sqrt{1 + |\nabla v|^{2r(x)}}} dx \\ &\quad + (\frac{1}{s^+} - \frac{1}{t^+}) \int_D q(x) |v|^{t(x)} dx \\ &\leq (\frac{1}{r^-} - \frac{1}{s^+}) \|v\|^{r(x)} + \frac{1}{r^-} \int_D (1 + |\nabla v|^{r(x)}) dx + (\frac{1}{s^+} - \frac{1}{t^+}) \int_D q(x) |v|^{t(x)} dx \\ &\leq (\frac{1}{r^-} - \frac{1}{s^+}) \|v\|^{r(x)} + \frac{|D| + \|v\|^{r(x)}}{r^-} + (\frac{1}{s^+} - \frac{1}{t^+}) \int_D q(x) |v|^{t(x)} dx \end{aligned} \quad (3.10)$$

If $|D| \leq \|v\|^{r^-}$ and assuming that 3.9 is holds in 3.10,

$$K_\mu(v) \leq \left(\frac{3}{r^-} - \frac{1}{s^+}\right)\|v\|^{r(x)} + \left(\frac{1}{s^+} - \frac{1}{t^+}\right)\left(\frac{3r^+ - 2s^-}{t^- - s^-}\right)\|v\|^{r(x)} < 0. \quad (3.11)$$

Therefore

$$\vartheta_\mu^+ = \inf_{v \in H_\mu^+(D)} K_\mu(v) < 0$$

□

Theorem 3.5. *If $0 < \mu < \mu_1$, there exists a minimizer of K_μ on $H_\mu^+(D)$.*

Proof. K_μ is bounded below on $H_\mu(D)$ and so on $H_\mu^+(D)$. Then, there exists a minimizing sequence $\{v_n^+\} \subseteq H_\mu^+(D)$ such that

$$\lim_{n \rightarrow \infty} K_\mu(v_n^+) = \inf_{v \in H_\mu^+(D)} K_\mu(v) = \vartheta_\mu^+ < 0$$

Also K_μ is coercive, v_n^+ is bounded in $W_0^{1,r(x)}(D)$, thus, we may assume that, $v_n^+ \rightharpoonup v_0^+$ and by the compact embedding

$$v_n^+ \rightarrow v_0^+ \quad \text{in} \quad L_{p(x)}^{s(x)}(D)$$

and

$$v_n^+ \rightarrow v_0^+ \quad \text{in} \quad L_{q(x)}^{s(x)}(D)$$

Now, we shall prove $v_n^+ \rightarrow v_0^+$ in $W_0^{1,r(x)}(D)$. Let $v_n^+ \rightarrow v_0^+$ in $W_0^{1,r(x)}(D)$. Then, by Fatou's lemma and compact embedding

$$\begin{aligned} \liminf_{n \rightarrow \infty} \int_D |\nabla v_n^+|^{r(x)} dx &> \int_D |\nabla v_0^+|^{r(x)} dx, \\ \liminf_{n \rightarrow \infty} \int_D \sqrt{1 + |\nabla v_n^+|^{2r(x)}} dx &> \int_D \sqrt{1 + |\nabla v_0^+|^{2r(x)}} dx, \\ \liminf_{n \rightarrow \infty} \int_D p(x) |v_n^+|^{s(x)} dx &= \int_D p(x) |v_0^+|^{s(x)} dx, \\ \liminf_{n \rightarrow \infty} \int_D q(x) |v_n^+|^{s(x)} dx &= \int_D q(x) |v_0^+|^{s(x)} dx. \end{aligned}$$

On the other hand $\langle K'_\mu(v_n^+), v_n^+ \rangle = 0$ and by theorem 2.8 in (Mashiyev and Avci, 2010),

$$\begin{aligned} K_\mu(v_n^+) &\geq \left(\frac{2}{r^+} - \frac{2}{t^-}\right)\|v_n^+\|^{r(x)} - \mu\left(\frac{1}{s^-} - \frac{1}{t^-}\right) \int_D p(x) |v_n^+|^{s(x)} dx \\ \lim_{n \rightarrow \infty} K_\mu(v_n^+) &\geq \left(\frac{2}{r^+} - \frac{2}{t^-}\right) \lim_{n \rightarrow \infty} \int_D |\nabla v_n^+|^{r(x)} dx \\ &\quad - \mu\left(\frac{1}{s^-} - \frac{1}{t^-}\right) \lim_{n \rightarrow \infty} \int_D p(x) |v_n^+|^{s(x)} dx \\ \Rightarrow \vartheta_\mu^+ &= \inf_{v \in H_\mu^+(D)} K_\mu(v) > \left(\frac{2}{r^+} - \frac{2}{t^-}\right)\|v_0^+\|^{r^+} + \mu c_7 \left(\frac{1}{t^-} - \frac{1}{s^-}\right)(\|v_0^+\|^{s^+} + \|v_0^+\|^{s^-}) \end{aligned} \quad (3.12)$$

Since $r^- > s^+$, for $\|v_0^+\| > 1$,

$$\vartheta_\mu^+ = \inf_{v \in H_\mu^+(D)} K_\mu(v) > 0.$$

This is a contradiction. Hence, $v_n^+ \rightarrow v_0^+$ in $W_0^{1,r(x)}(D)$ and

$$K_\mu(v_0^+) = \lim_{n \rightarrow \infty} K_\mu(v_n^+) = \inf_{v \in H_\mu^+(D)} K_\mu(v).$$

v_0^+ is a minimizer for K_μ on $H_\mu^+(D)$. □

Lemma 3.6. *If $0 < \mu < \mu_1$, then for all $v \in H_\mu^-(D)$, $K_\mu(v) > 0$.*

Proof. Let $v \in H_\mu(D)$ and with definition $K_\mu(v)$, making use the equation 3.2, we have

$$\begin{aligned} K_\mu(v) &\geq \frac{1}{r^+} \int_D (|\nabla v|^{r(x)} + \sqrt{1 + |\nabla v|^{2r(x)}}) dx - \frac{\mu}{s^-} \int_D p(x) |v|^{s(x)} dx \\ &\quad - \frac{1}{t^-} \int_D q(x) |v|^{t(x)} dx \\ &\geq \left(\frac{2}{r^+} - \frac{2}{t^-} \right) \|v\|^{r^-} + \mu \left(\frac{1}{t^-} - \frac{1}{s^-} \right) \int_D p(x) |v|^{s(x)} dx. \end{aligned} \quad (3.13)$$

By 2.1, and the condition $r^- > s^+$, it follows:

$$\begin{aligned} K_\mu(v) &\geq \left(\frac{2}{r^+} - \frac{2}{t^-} \right) \|v\|^{r^-} + \mu b_5 \left(\frac{s^- - t^-}{s^- t^-} \right) \|v\|^{s^+} \\ &\geq \left[\frac{2t^- - 2r^+}{r^+ t^-} + \mu b_5 \frac{s^- - t^-}{s^- t^-} \right] \|v\|^{r^-}. \end{aligned} \quad (3.14)$$

If $\mu < \frac{2s^-(t^- - r^+)}{b_5 r^+(t^- - s^-)}$, then $K_\mu(v) > 0$.

Also $H_\mu(D) = H_\mu^+(D) \cup H_\mu^-(D)$, $H_\mu^+(D) \cap H_\mu^-(D) = \emptyset$ and lemma 3.6, we have $v \in H_\mu^-(D)$. □

Theorem 3.7. *If $0 < \mu < \mu_1$, there exists a minimizer of K_μ on $H_\mu^-(D)$.*

Proof. In a similar way, there exists a minimizing sequence $\{v_n^-\} \subseteq H_\mu^-(D)$ such that

$$\lim_{n \rightarrow \infty} K_\mu(v_n^-) = \inf_{v \in H_\mu^-(D)} K_\mu(v) = \vartheta_\mu^- > 0.$$

Thus, we may assume that $v_n^- \rightharpoonup v_0^-$ in $W_0^{1,r(x)}(D)$

$$v_n^+ \rightarrow v_0^- \quad \text{in} \quad L_{p(x)}^{s(x)}(D),$$

$$v_n^+ \rightarrow v_0^- \quad \text{in} \quad L_{q(x)}^{s(x)}(D).$$

If $v_0^- \in H_\mu^-(D)$, then $c > 0$ exists such that $cv_0^- \in H_\mu^-$ and $K_\mu(v_0^-) \geq K_\mu(cv_0^-)$.

$$\begin{aligned} \varphi'_\mu(v) &= \int_D r(x) |\nabla v|^{r(x)} dx + \int_D \frac{2r(x) |\nabla v|^{2r(x)}}{\sqrt{1 + |\nabla v|^{2r(x)}}} dx - \int_D \frac{r(x) |\nabla v|^{4r(x)}}{\sqrt{(1 + |\nabla v|^{2r(x)})^3}} dx \\ &\quad - \mu \int_D s(x) p(x) |v|^{s(x)} dx - \int_D t(x) q(x) |v|^{t(x)} dx. \end{aligned}$$

Therefore,

$$\begin{aligned}
 \varphi'_\mu(cv_0^-) &= \int_D r(x)|\nabla cv_0^-|^{r(x)} dx + \int_D \frac{2r(x)|\nabla cv_0^-|^{2r(x)}}{\sqrt{1+|\nabla cv_0^-|^{2r(x)}}} dx \\
 &\quad - \int_D \frac{r(x)|\nabla cv_0^-|^{4r(x)}}{\sqrt{(1+|\nabla cv_0^-|^{2r(x)})^3}} dx - \mu \int_D s(x)p(x)|cv_0^-|^{s(x)} dx \\
 &\quad - \int_D t(x)q(x)|cv_0^-|^{t(x)} dx \\
 &\leq c^{r^+} r^+ \int_D |\nabla v_0^-|^{r(x)} dx + 2c^{r^+} r^+ \int_D |\nabla v_0^-|^{r(x)} dx \\
 &\quad - \mu c^{s^-} s^- \int_D p(x)|v_0^-|^{s(x)} dx - c^{t^-} t^- \int_D q(x)|v_0^-|^{t(x)} dx
 \end{aligned} \tag{3.15}$$

According to $s^- < r^+ < t^-$, and conditions on p, q , $\varphi'_\mu(cv_0^-) < 0$, by the definition of $H_\mu^-(D)$, $cv_0^- \in H_\mu^-(D)$. We show that $v_n^- \rightarrow v_0^-$ in $W_0^{1,r(x)}(D)$. Suppose $v_n^- \rightarrow v_0^-$ in $W_0^{1,r(x)}(D)$. Then by the Fatou's Lemma

$$\liminf_{n \rightarrow \infty} \int_D |\nabla v_n^-|^{r(x)} dx > \int_D |\nabla v_0^-|^{r(x)} dx,$$

and

$$\liminf_{n \rightarrow \infty} \int_D \sqrt{1+|\nabla v_n^-|^{2r(x)}} dx > \int_D \sqrt{1+|\nabla v_0^-|^{2r(x)}} dx.$$

Thus,

$$\begin{aligned}
 K_\mu(cv_0^-) &\leq \frac{c^{r^+}}{r^-} \int_D |\nabla v_0^-|^{r(x)} dx + \frac{c^{r^+}}{r^-} \int_D \sqrt{1+|\nabla v_0^-|^{2r(x)}} dx \\
 &\quad - \frac{\mu c^{s^-}}{s^+} \int_D p(x)|v_0^-|^{s(x)} dx - \frac{c^{t^-}}{t^+} \int_D q(x)|v_0^-|^{t(x)} dx \\
 &< \lim_{n \rightarrow \infty} \left[\frac{c^{r^+}}{r^-} \int_D |\nabla v_n^-|^{r(x)} dx + \frac{c^{r^+}}{r^-} \int_D \sqrt{1+|\nabla v_n^-|^{2r(x)}} dx \right. \\
 &\quad \left. - \frac{\mu c^{s^-}}{s^+} \int_D p(x)|v_n^-|^{s(x)} dx - \frac{c^{t^-}}{t^+} \int_D q(x)|v_n^-|^{t(x)} dx \right] \\
 &\leq \lim_{n \rightarrow \infty} K_\mu(v_n^-) < \lim_{n \rightarrow \infty} K_\mu(v_n^-) = \inf_{v \in H_\mu^-(D)} K_\mu(v) = \vartheta_\mu^-.
 \end{aligned}$$

Thus we have,

$$K_\mu(cv_0^-) < \inf_{v \in H_\mu^-(D)} K_\mu(v) = \vartheta_\mu^-.$$

Which is a contradiction. So $v_n^- \rightarrow v_0^-$ in $W_0^{1,r(x)}(D)$ and

$$K_\mu(v_0^-) = \lim_{n \rightarrow \infty} K_\mu(v_n^-) = \inf_{v \in H_\mu^-(D)} K_\mu(v).$$

Then v_0^- is a minimizer for $K_\mu(v)$ on $H_\mu^-(D)$. □

Corollary 3.8. According to theorems 3.5, 3.7, there exists $v_0^+ \in H_\mu^+(D)$, $v_0^- \in H_\mu^-(D)$ such that $K_\mu(v_0^+) = \inf_{v \in H_\mu^+(D)} K_\mu(v)$, $K_\mu(v_0^-) = \inf_{v \in H_\mu^-(D)} K_\mu(v)$. Since $K_\mu(v_0^\pm) = K_\mu(|v_0^\pm|)$ and $|v_0^\pm| \in H_\mu^\pm$, we may assume $v_0^\pm > 0$. By Theorem 3.3, $v_0^\pm$ are critical points of K_μ on $W_0^{1,r(x)}(D)$ and hence are weak solutions (and so by standard regularity results classical solutions) of (1.1). Finally, by the Harnack inequality due to (Trudinger, 1967), (Zhang and Liu, 2007), we obtain that $v_0^\pm$ are positive solutions of (1.1).

4 Conclusions

Using Nehari manifold, we considered the mean curvature equation on a domain in the Euclidean space equipped with a norm induced by a variable exponent Sobolev space (See equations (1.1) and (2.1)). We showed that there are two solution of this problem, which are supported on disjoint subset of the domain D .

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