

k —Almost Cobalancing Numbers

Ahmet Tekcan

Bursa Uludag University
 Faculty of Science
 Department of Mathematics
 Bursa, Turkey
 tekcan@uludag.edu.tr

ABSTRACT

The general terms of k —almost cobalancing numbers and k —almost Lucas—cobalancing numbers are determined for an integer $k \geq 1$.

Keywords: Balancing numbers, cobalancing numbers, Pell equations.

2000 Mathematics Subject Classification: 11B37, 11B39, 11D25.

1 Notations and Preliminaries.

$n \in \mathbb{Z}^+$ is called a balancing number ([2]) and is denoted by B_n if

$$1 + 2 + \cdots + (n - 1) = (n + 1) + (n + 2) + \cdots + (n + r)$$

verified for $r \in \mathbb{Z}^+$ which is called balancer, and $n \in \mathbb{Z}^+$ is called a cobalancing number ([12]) and is denoted by b_n if

$$1 + 2 + \cdots + n = (n + 1) + (n + 2) + \cdots + (n + r)$$

verified for $r \in \mathbb{Z}^+$ which is called cobalancer (see [4, 10, 11, 14, 15]).

Recently many authors studied on several type of balancing numbers (see [6, 7, 8, 9, 16, 17, 18]). In [13], the author defined that $n \in \mathbb{Z}^+$ is called an almost cobalancing number (shortly acn) if

$$|[(n + 1) + (n + 2) + \cdots + (n + r)] - (1 + 2 + \cdots + n)| = 1 \quad (1.1)$$

verified for $r \in \mathbb{Z}^+$ which is called almost cobalancer. If (1.1) equals to 1, then n is called an acn of first type and is denoted by b_n^* , and equals to -1 , then n is called an acn of second type and is denoted by b_n^{**} . Thus

$$c_n^* = \sqrt{8(b_n^*)^2 + 8b_n^* + 9} \quad \text{and} \quad c_n^{**} = \sqrt{8(b_n^{**})^2 + 8b_n^{**} - 7}$$

are called the almost Lucas—cobalancing numbers (shortly aLcn) of first type and second type, respectively.

2 Results.

Let $k \in \mathbb{Z}^+$. By considering (1.1), $n \in \mathbb{Z}^+$ is called a k -acn if

$$|(n+1) + (n+2) + \cdots + (n+r)| - [1 + 2 + \cdots + n] = k \quad (2.1)$$

verified for $r \in \mathbb{Z}^+$ which is called k -almost cobalancer (Note that 1-acm are acm). If (2.1) equals to k , then n is called a k -acn of first type and is denoted by b_n^{k*} and

$$c_n^{k*} = \sqrt{8(b_n^{k*})^2 + 8b_n^{k*} + 8k + 1} \quad (2.2)$$

is called a k -aLcn of first type, and equals to $-k$, then n is called a k -acn of second type and is denoted by b_n^{k**} and

$$c_n^{k**} = \sqrt{8(b_n^{k**})^2 + 8b_n^{k**} - 8k + 1} \quad (2.3)$$

is called a k -aLcb of second type.

We note from (2.2) that b_n^{k*} is a k -acn of first type iff $\sqrt{8(b_n^{k*})^2 + 8b_n^{k*} + 8k + 1} \in \mathbb{Z}^+$. So we set

$$8(b_n^{k*})^2 + 8b_n^{k*} + 8k + 1 = y^2 \quad (2.4)$$

for $y \in \mathbb{Z}^+$. Then $2(2b_n^{k*} + 1)^2 - y^2 = -8k + 1$. Let

$$x = 2b_n^{k*} + 1. \quad (2.5)$$

Then we get the Pell equation (see [1, 5])

$$2x^2 - y^2 = -8k + 1.$$

Let $\Omega^* = \{(x, y) \in \mathbb{Z}^+ \times \mathbb{Z}^+ : 2x^2 - y^2 = -8k + 1\}$. Then we can give the following theorem.

Theorem 2.1. *If $\#Rep = 2q$ for $q \in \mathbb{Z}^+$, then for $\forall k \geq 1$,*

$$\Omega^* = \{(x_{2qn+j}, y_{2qn+j}) : n \geq 0\} \cup \{(x_{2qn-j+1}, y_{2qn-j+1}) : n \geq 1\},$$

where

$$\begin{aligned} (x_{2qn+j}, y_{2qn+j}) &= (2k_{2j}B_n + k_{2j-1}C_n, 4k_{2j-1}B_n + k_{2j}C_n) \\ (x_{2qn-j+1}, y_{2qn-j+1}) &= (2k_{2j}B_n - k_{2j-1}C_n, -4k_{2j-1}B_n + k_{2j}C_n) \end{aligned}$$

and k_{2j-1}, k_{2j} are positive integers such that $2k_{2j-1}^2 - k_{2j}^2 = -8k + 1$ for $1 \leq j \leq q, k_1 < k_3 < \cdots < k_{2q-1}, k_2 < k_4 < \cdots < k_{2q}$.

Proof. For the Pell equation $2x^2 - y^2 = -8k + 1$, let $\#Rep = 2q$ for $q \in \mathbb{Z}^+$. Then the set of representatives (see [3]) is

$$Rep = \{[\pm k_{2j-1} \quad k_{2j}]\},$$

where k_{2j-1}, k_{2j} are positive integers such that $2k_{2j-1}^2 - k_{2j}^2 = -8k + 1$ for $1 \leq j \leq q, k_1 < k_3 < \cdots < k_{2q-1}$ and $k_2 < k_4 < \cdots < k_{2q}$. Here (x_{2qn+j}, y_{2qn+j}) comes from $[k_{2j-1} \quad k_{2j}]M^n$ for $n \geq 0$ and $(x_{2qn-j+1}, y_{2qn-j+1})$ comes from $[-k_{2j-1} \quad k_{2j}]M^n$ for $n \geq 1$, where

$$M = \begin{bmatrix} 3 & 4 \\ 2 & 3 \end{bmatrix}.$$

It can be easily seen that the n^{th} power of M is

$$M^n = \begin{bmatrix} C_n & 4B_n \\ 2B_n & C_n \end{bmatrix}$$

for $n \geq 1$, where C_n is the Lucas-balancing number. So we get $\Omega^* = \{(2k_{2j}B_n + k_{2j-1}C_n, 4k_{2j-1}B_n + k_{2j}C_n) : n \geq 0\} \cup \{(2k_{2j}B_n - k_{2j-1}C_n, -4k_{2j-1}B_n + k_{2j}C_n) : n \geq 1\}$. $\square$

In view of Theorem 2.1, we deduce that

Theorem 2.2. *If $\#Rep = 2q$ for $q \in \mathbb{Z}^+$, then for $\forall k \geq 1$,*

$$b_{2qn+j}^{k*} = \frac{2k_{2j}B_n + k_{2j-1}C_n - 1}{2}$$

$$c_{2qn+j}^{k*} = 4k_{2j-1}B_n + k_{2j}C_n$$

for $n \geq 0$ and

$$b_{2qn-j+1}^{k*} = \frac{2k_{2j}B_n - k_{2j-1}C_n - 1}{2}$$

$$c_{2qn-j+1}^{k*} = -4k_{2j-1}B_n + k_{2j}C_n,$$

for $n \geq 1$, where k_{2j-1}, k_{2j} are positive integers such that $2k_{2j-1}^2 - k_{2j}^2 = -8k + 1$ for $1 \leq j \leq q$, $k_1 < k_3 < \dots < k_{2q-1}$ and $k_2 < k_4 < \dots < k_{2q}$.

Proof. If $\#Rep = 2q$ for $q \in \mathbb{Z}^+$, then $\Omega^* = \{(x_{2qn+j}, y_{2qn+j}) : n \geq 0\} \cup \{(x_{2qn-j+1}, y_{2qn-j+1}) : n \geq 1\}$, where

$$(x_{2qn+j}, y_{2qn+j}) = (2k_{2j}B_n + k_{2j-1}C_n, 4k_{2j-1}B_n + k_{2j}C_n)$$

$$(x_{2qn-j+1}, y_{2qn-j+1}) = (2k_{2j}B_n - k_{2j-1}C_n, -4k_{2j-1}B_n + k_{2j}C_n)$$

and k_{2j-1}, k_{2j} are positive integers such that $2k_{2j-1}^2 - k_{2j}^2 = -8k + 1$ for $1 \leq j \leq q$, $k_1 < k_3 < \dots < k_{2q-1}$, $k_2 < k_4 < \dots < k_{2q}$. Since $x = 2b_n^{k*} + 1$ by (2.5), we get

$$b_{2qn+j}^{k*} = \frac{2k_{2j}B_n + k_{2j-1}C_n - 1}{2}$$

and since $8(b_n^{k*})^2 + 8b_n^{k*} + 8k + 1 = y^2$ by (2.4), we get

$$c_{2qn+j}^{k*} = 4k_{2j-1}B_n + k_{2j}C_n.$$

The others are similar. $\square$

Similarly from (2.3), b_n^{k**} is a k -acn of second type iff $\sqrt{8(b_n^{k**})^2 + 8b_n^{k**} - 8k + 1} \in \mathbb{Z}^+$. So we set

$$8(b_n^{k**})^2 + 8b_n^{k**} - 8k + 1 = v^2 \quad (2.6)$$

for $v \in \mathbb{Z}^+$. Then $2(2b_n^{k**} + 1)^2 - v^2 = 8k + 1$. Taking

$$u = 2b_n^{k**} + 1. \quad (2.7)$$

we get the Pell equation

$$2u^2 - v^2 = 8k + 1.$$

Let $\Omega^{**} = \{(u, v) \in \mathbb{Z}^+ \times \mathbb{Z}^+ : 2u^2 - v^2 = 8k + 1\}$. Then we can give the following theorem.

Theorem 2.3. 1. If $8k + 1 = h^2$ and $\#Rep = 2$, then

$$\Omega^{**} = \{(-2hB_n + hC_n, 4hB_n - hC_n) : n \geq 1\}.$$

2. If $8k + 1 = h^2$ and $\#Rep = 2q + 2$ for $q \in \mathbb{Z}^+$, then

$$\Omega^{**} = \{(u_{(2q+1)n+j}, v_{(2q+1)n+j}), (u_{(2q+1)n+q+1}, v_{(2q+1)n+q+1}) : n \geq 0\} \cup \{(u_{(2q+1)n-j+1}, v_{(2q+1)n-j+1}) : n \geq 1\},$$

where

$$\begin{aligned} (u_{(2q+1)n+j}, v_{(2q+1)n+j}) &= (2k_{2j}B_n + k_{2j-1}C_n, 4k_{2j-1}B_n + k_{2j}C_n) \\ (u_{(2q+1)n+q+1}, v_{(2q+1)n+q+1}) &= (2hB_n + hC_n, 4hB_n + hC_n) \\ (u_{(2q+1)n-j+1}, v_{(2q+1)n-j+1}) &= (-2k_{2j}B_n + k_{2j-1}C_n, 4k_{2j-1}B_n - k_{2j}C_n) \end{aligned}$$

and k_{2j-1}, k_{2j} are positive integers such that $2k_{2j-1}^2 - k_{2j}^2 = 8k + 1$ for $1 \leq j \leq q, k_1 < k_3 < \dots < k_{2q-1} < h, k_2 < k_4 < \dots < k_{2q} < h$.

3. If $8k + 1 \neq h^2$ and $\#Rep = 2q$ for $q \in \mathbb{Z}^+$, then

$$\Omega^{**} = \{(u_{2qn+j}, v_{2qn+j}) : n \geq 0\} \cup \{(u_{2qn-j+1}, v_{2qn-j+1}) : n \geq 1\},$$

where

$$\begin{aligned} (u_{2qn+j}, v_{2qn+j}) &= (2k_{2j}B_n + k_{2j-1}C_n, 4k_{2j-1}B_n + k_{2j}C_n) \\ (u_{2qn-j+1}, v_{2qn-j+1}) &= (-2k_{2j}B_n + k_{2j-1}C_n, 4k_{2j-1}B_n - k_{2j}C_n) \end{aligned}$$

and k_{2j-1}, k_{2j} are positive integers such that $2k_{2j-1}^2 - k_{2j}^2 = 8k + 1$ for $1 \leq j \leq q, k_1 < k_3 < \dots < k_{2q-1}, k_2 < k_4 < \dots < k_{2q}$.

Proof. (1) Let $8k + 1$ is a perfect square, say $h = \sqrt{8k + 1}$ and let $\#Rep = 2$. Then the set of representatives is

$$Rep = \{[\pm h \ h]\}$$

and (u_n, v_n) comes from $[h \ -h]M^n$ for $n \geq 1$, where $M = \begin{bmatrix} 3 & 4 \\ 2 & 3 \end{bmatrix}$. So $\Omega^{**} = \{(-2hB_n + hC_n, 4hB_n - hC_n) : n \geq 1\}$.

(2) Let $8k + 1 = h^2$ and let $\#Rep = 2q + 2$ for $q \in \mathbb{Z}^+$. Then the set of representatives is

$$Rep = \{[\pm k_{2j-1} \ k_{2j}], [\pm h \ h]\},$$

where k_{2j-1}, k_{2j} are positive integers such that $2k_{2j-1}^2 - k_{2j}^2 = 8k + 1$ for $1 \leq j \leq q, k_1 < k_3 < \dots < k_{2q-1} < h, k_2 < k_4 < \dots < k_{2q} < h$. Here $(u_{(2q+1)n+j}, v_{(2q+1)n+j})$ comes from $[k_{2j-1} \ k_{2j}]M^n$ for $n \geq 0$, $(u_{(2q+1)n-j+1}, v_{(2q+1)n-j+1})$ comes from $[k_{2j-1} \ -k_{2j}]M^n$ for $n \geq 1$ and $(u_{(2q+1)n+q+1}, v_{(2q+1)n+q+1})$ comes from $[h \ h]M^n$ for $n \geq 0$, where $M = \begin{bmatrix} 3 & 4 \\ 2 & 3 \end{bmatrix}$. So $\Omega^{**} = \{(2k_{2j}B_n + k_{2j-1}C_n, 4k_{2j-1}B_n + k_{2j}C_n), (2hB_n + hC_n, 4hB_n + hC_n) : n \geq 0\} \cup \{(-2k_{2j}B_n + k_{2j-1}C_n, 4k_{2j-1}B_n - k_{2j}C_n) : n \geq 1\}$.

(3) Let $8k + 1$ is not a perfect square and let $\#Rep = 2q$ for $q \in \mathbb{Z}^+$. Then the set of representatives is

$$Rep = \{[\pm k_{2j-1} \ k_{2j}]\},$$

where k_{2j-1}, k_{2j} are positive integers such that $2k_{2j-1}^2 - k_{2j}^2 = 8k + 1$ for $1 \leq j \leq q, k_1 < k_3 < \dots < k_{2q-1}, k_2 < k_4 < \dots < k_{2q}$. Here (u_{2qn+j}, v_{2qn+j}) comes from $[k_{2j-1} \ k_{2j}]M^n$ for $n \geq 0$ and $(u_{2qn-j+1}, v_{2qn-j+1})$ comes from $[k_{2j-1} \ -k_{2j}]M^n$ for $n \geq 1$, where $M = \begin{bmatrix} 3 & 4 \\ 2 & 3 \end{bmatrix}$. Thus $\Omega^{**} = \{(2k_{2j}B_n + k_{2j-1}C_n, 4k_{2j-1}B_n + k_{2j}C_n) : n \geq 0\} \cup \{(-2k_{2j}B_n + k_{2j-1}C_n, 4k_{2j-1}B_n - k_{2j}C_n) : n \geq 1\}$. $\square$

Finally from Theorem 2.3, we can give the following theorem.

Theorem 2.4. 1. If $8k + 1 = h^2$ and $\#Rep = 2$, then

$$b_n^{k**} = \frac{-2hB_n + hC_n - 1}{2} \text{ and } c_n^{k**} = 4hB_n - hC_n$$

for $n \geq 1$.

2. If $8k + 1 = h^2$ and $\#Rep = 2q + 2$ for $q \in \mathbb{Z}^+$, then

$$\begin{aligned} b_{(2q+1)n+j}^{k**} &= \frac{2k_{2j}B_n + k_{2j-1}C_n - 1}{2} \\ b_{(2q+1)n+q+1}^{k**} &= \frac{2hB_n + hC_n - 1}{2} \\ c_{(2q+1)n+j}^{k**} &= 4k_{2j-1}B_n + k_{2j}C_n \\ c_{(2q+1)n+q+1}^{k**} &= 4hB_n + hC_n \end{aligned}$$

for $n \geq 0$ and

$$\begin{aligned} b_{(2q+1)n-j+1}^{k**} &= \frac{-2k_{2j}B_n + k_{2j-1}C_n - 1}{2} \\ c_{(2q+1)n-j+1}^{k**} &= 4k_{2j-1}B_n - k_{2j}C_n, \end{aligned}$$

for $n \geq 1$, where k_{2j-1}, k_{2j} are positive integers such that $2k_{2j-1}^2 - k_{2j}^2 = 8k + 1$ for $1 \leq j \leq q, k_1 < k_3 < \dots < k_{2q-1} < h, k_2 < k_4 < \dots < k_{2q} < h$.

3. If $8k + 1 \neq h^2$ and $\#Rep = 2q$ for $q \in \mathbb{Z}^+$, then

$$\begin{aligned} b_{2qn+j}^{k**} &= \frac{2k_{2j}B_n + k_{2j-1}C_n - 1}{2} \\ c_{2qn+j}^{k**} &= 4k_{2j-1}B_n + k_{2j}C_n \end{aligned}$$

for $n \geq 0$ and

$$\begin{aligned} b_{2qn-j+1}^{k**} &= \frac{-2k_{2j}B_n + k_{2j-1}C_n - 1}{2} \\ c_{2qn-j+1}^{k**} &= 4k_{2j-1}B_n - k_{2j}C_n \end{aligned}$$

for $n \geq 1$, where k_{2j-1}, k_{2j} are positive integers such that $2k_{2j-1}^2 - k_{2j}^2 = 8k + 1$ for $1 \leq j \leq q, k_1 < k_3 < \dots < k_{2q-1}, k_2 < k_4 < \dots < k_{2q}$.

Proof. (1) If $8k+1 = h^2$ and $\#Rep = 2$, then $\Omega^{**} = \{(-2hB_n + hC_n, 4hB_n - hC_n) : n \geq 1\}$. Since $x = 2b_n^{k**} + 1$ by (2.7), we get

$$b_n^{k**} = \frac{-2hB_n + hC_n - 1}{2}$$

and from (2.3), we observe that

$$\begin{aligned} c_n^{k**} &= \sqrt{8(b_n^{k**})^2 + 8b_n^{k**} - 8k + 1} \\ &= \sqrt{8\left(\frac{-2hB_n + hC_n - 1}{2}\right)^2 + 8\left(\frac{-2hB_n + hC_n - 1}{2}\right) - 8k + 1} \\ &= \sqrt{8h^2B_n^2 - 8h^2B_nC_n + 2h^2C_n^2 - 8k - 1} \\ &= \sqrt{h^2(C_n^2 - 1) - 8h^2B_nC_n + 2h^2(1 + 8B_n^2) - h^2} \\ &= \sqrt{h^2(4B_n - C_n)^2} \\ &= 4hB_n - hC_n. \end{aligned}$$

(2) If $8k+1 = h^2$ and $\#Rep = 2q+2$ for $q \in \mathbb{Z}^+$, then $\Omega^{**} = \{(u_{(2q+1)n+j}, v_{(2q+1)n+j}), (u_{(2q+1)n+q+1}, v_{(2q+1)n+q+1}) : n \geq 0\} \cup \{(u_{(2q+1)n-j+1}, v_{(2q+1)n-j+1}) : n \geq 1\}$, where

$$\begin{aligned} (u_{(2q+1)n+j}, v_{(2q+1)n+j}) &= (2k_{2j}B_n + k_{2j-1}C_n, 4k_{2j-1}B_n + k_{2j}C_n) \\ (u_{(2q+1)n+q+1}, v_{(2q+1)n+q+1}) &= (2hB_n + hC_n, 4hB_n + hC_n) \\ (u_{(2q+1)n-j+1}, v_{(2q+1)n-j+1}) &= (-2k_{2j}B_n + k_{2j-1}C_n, 4k_{2j-1}B_n - k_{2j}C_n) \end{aligned}$$

and k_{2j-1}, k_{2j} are positive integers such that $2k_{2j-1}^2 - k_{2j}^2 = 8k+1$ for $1 \leq j \leq q, k_1 < k_3 < \dots < k_{2q-1} < h, k_2 < k_4 < \dots < k_{2q} < h$. So we get

$$\begin{aligned} b_{(2q+1)n+j}^{k**} &= \frac{2k_{2j}B_n + k_{2j-1}C_n - 1}{2} \\ b_{(2q+1)n+q+1}^{k**} &= \frac{2hB_n + hC_n - 1}{2} \\ b_{(2q+1)n-j+1}^{k**} &= \frac{-2k_{2j}B_n + k_{2j-1}C_n - 1}{2} \end{aligned}$$

and from (2.6), we get

$$\begin{aligned} c_{(2q+1)n+j}^{k**} &= 4k_{2j-1}B_n + k_{2j}C_n \\ c_{(2q+1)n+q+1}^{k**} &= 4hB_n + hC_n \\ c_{(2q+1)n-j+1}^{k**} &= 4k_{2j-1}B_n - k_{2j}C_n. \end{aligned}$$

(3) If $8k+1 \neq h^2$ and $\#Rep = 2q$ for $q \in \mathbb{Z}^+$, then $\Omega^{**} = \{(u_{2qn+j}, v_{2qn+j}) : n \geq 0\} \cup \{(u_{2qn-j+1}, v_{2qn-j+1}) : n \geq 1\}$, where

$$\begin{aligned} (u_{2qn+j}, v_{2qn+j}) &= (2k_{2j}B_n + k_{2j-1}C_n, 4k_{2j-1}B_n + k_{2j}C_n) \\ (u_{2qn-j+1}, v_{2qn-j+1}) &= (-2k_{2j}B_n + k_{2j-1}C_n, 4k_{2j-1}B_n - k_{2j}C_n) \end{aligned}$$

and k_{2j-1}, k_{2j} are positive integers such that $2k_{2j-1}^2 - k_{2j}^2 = 8k+1$ for $1 \leq j \leq q, k_1 < k_3 < \dots < k_{2q-1}, k_2 < k_4 < \dots < k_{2q}$. Hence the result is obvious. $\square$

References

- [1] Barbeau, E. J., 2003, *Pell's Equation*, Springer–Verlag New York, Inc.
- [2] Behera, A. and Panda, G.K., 1999, *On the Square Roots of Triangular Numbers*, The Fibonacci Quart. 37(2), 98–105.
- [3] Flath, D. E., 1989, *Introduction to Number Theory*, Wiley.
- [4] Gözeri, G.K., Özkoç, A. and Tekcan, A., 2017, *Some Algebraic Relations on Balancing Numbers*, Utilitas Mathematica 103, 217–236.
- [5] Jacobson, M. Williams, H.C., 2009, *Solving the Pell Equation*, CMS Books in Mathematics, Springer Science, Business Media, LLC.
- [6] Kovacs, T., Liptai, K. and Olajos, P., 2010, *On (a, b) –Balancing Numbers*, Publ. Math. Deb. 77(3-4), 485–498.
- [7] Liptai, K., Luca, F., Pinter, A. and Szalay, L., 2009, *Generalized Balancing Numbers*, Indagationes Math. N.S. 20(1), 87–100.
- [8] Liptai, K., 2004, *Fibonacci Balancing Numbers*, The Fibonacci Quarterly 42(4), 330–340.
- [9] Liptai, K., 2006, *Lucas Balancing Numbers*, Acta Math. Univ. Ostrav. 14, 43–47.
- [10] Olajos, P., 2010, *Properties of Balancing, Cobalancing and Generalized Balancing Numbers*, Annales Mathematicae et Informaticae 37, 125–138.
- [11] Panda, G.K. and Ray, P.K., 2011, *Some Links of Balancing and Cobalancing Numbers with Pell and Associated Pell Numbers*, Bul. of Inst. of Math. Acad. Sinica 6(1), 41–72.
- [12] Panda, G.K. and Ray, P.K., 2005, *Cobalancing Numbers and Cobalancers*, Int. J. Math. Math. Sci. 8, 1189–1200.
- [13] Panda, G.K. and Panda, A.K., 2015, *Almost Balancing Numbers*, Jour. of the Indian Math. Soc. 82(3-4), 147–156.
- [14] Panda, A.K., 2017, *Some Variants of the Balancing Sequences*, Ph.D. dissertation, National Institute of Technology Rourkela, India.
- [15] Ray, P.K., 2009, *Balancing and Cobalancing Numbers*, Ph.D. dissertation, Department of Mathematics, National Institute of Technology Rourkela, India.
- [16] Ray, P.K., 2015, *Balancing and Lucas–balancing Sums by Matrix Methods*, Math. Reports 17(67), 225–233.
- [17] Szalay, L., 2007, *On the Resolution of Simultaneous Pell Equations*, Ann. Math. Inform. 34, 77–87.
- [18] Tengely, S., 2013, *Balancing Numbers which are Products of Consecutive Integers*, Publ. Math. Deb. 83(1–2), 197–205.