

Cauchy Problem for the Rotational-Modified Benjamin Equation

Farida Bousbia¹ and Fatma Zohra Nouri²

^{1,2}Mathematical Modeling and Numerical Simulation Research Laboratory
 Badji Mokhtar University
 PO. Box 12, Annaba 23000, Algeria

¹farida.bousbia@univ-annaba.org and ²fz_nouri@yahoo.fr

ABSTRACT

In this research paper, we are mainly interested in the Cauchy problem for the so-called Rotational-Modified Benjamin equation. This equation was introduced to describe the propagation of long internal waves in two rotating fluids. By using a sharp bilinear estimate, we prove that the problem is well posed in a Sobolev space of a specific order.

Keywords: Rotational-Modified Benjamin, Cauchy problem, Sharp bilinear estimate.

2000 Mathematics Subject Classification: 35Q53, 78A60, 35Q70.

1 Introduction

Throughout this work, we are concerned about the Rotational-Modified Benjamin equation (RMB). Physically, this equation was proposed by (Durán, 2018) as a rotating two-fluid model of internal waves's propagation and characterizes the vertical displacement of the interface between two layers, where the lower and upper layers are respectively bounded below and above by a rigid horizontal plane. The RMB equation takes the following form

$$(u_t + \alpha u_x + uu_x - \beta \mathcal{H}u_{xx} - \delta u_{xxx})_x = \gamma u, \quad t \geq 0, \quad x \in \mathbb{R}, \quad (1.1)$$

with $u = u(x, t), x \in \mathbb{R}, t \geq 0$ and γ, α and δ are positive while $\beta \neq 0$. The parameter γ is a positive real number that measures the effect of rotation, β and δ are used to control the general dispersive effects whereas α depends on the fluids's densities. Here $\mathcal{H}$ denotes the Hilbert transform that is defined on $\mathbb{R}$ by

$$\mathcal{H}f(x) := \frac{1}{\pi} P.V. \int_{-\infty}^{\infty} \frac{f(y)}{x-y} dy,$$

where $P.V.$ stands for the principal value. Related to the RMB equation, the well known Ostrovsky equation

$$(u_t + uu_x - \delta u_{xxx})_x = \gamma u, \quad (1.2)$$

describes the propagation of nonlinear long shallow water waves in a rotating fluid. The Cauchy problem associated to (1.2) has been studied by several authors, for instance (Isaza and Mejia,

2006) and (Isaza and Mejia, 2007) showed its sharp local well-posedness in the Sobolev space H^s if $s > -3/4$ for $\delta < 0$ (negative dispersion). The application of the Fourier technique allowed (Tsugawa, 2009) to show that the Cauchy problem of (1.2) is well-posed locally in the space $H^{a,s}$ for $s > -\frac{a}{2} - \frac{3}{4}$ and $0 \leq a \leq 1$, for more details on this space see (Maouni and Nouri, 2013). Recently, instead of the Cauchy Schwarz inequalities, (Wang and Yan, 2018) used the Strichartz estimates to establish the well-posedness result for negative dispersion while (Yan, Li, Huang and Duan, 2018) proved it for the positive dispersion case in $H^{-3/4}$.

2 Rotational-modified Benjamin-Ono equation

By setting $\delta = 0$ and $\alpha = 0$ in equation (1.1), one obtains the particular case, namely the rotational-modified Benjamin-Ono (RMBO) equation

$$(u_t + uu_x - \beta \mathcal{H}u_{xx})_x = \gamma u, \quad (2.1)$$

which models the internal long wave propagation in a deep fluid subject to rotation effects. The well-posedness of its associated Cauchy problem has been investigated by (Linares and Milanes, 2004) in the space

$$X_s = \{u \in H^s(\mathbb{R}); \partial_x^{-1}u \in H^s(\mathbb{R})\},$$

with the associated norm

$$\|u\|_{X_s} = \|u\|_{H^s(\mathbb{R})} + \|\partial_x^{-1}u\|_{H^s(\mathbb{R})}, \text{ for } s > 3/2.$$

In the absence of a Coriolis forcing, equation (1.1) reduces to the original Benjamin equation (see (Benjamin, 1967)) written in the form

$$u_t + \alpha u_x + uu_x - \beta \mathcal{H}u_{xx} - \delta u_{xxx} = 0, \quad (2.2)$$

which was introduced to describe the propagation of intermediate wave in the stratified fluid with a considerable capillarity on the surface. A sharp well posedness result of the Benjamin equation was established in (Chen and Xiao, 2010) and (Chen et al. 2011), they proved that the associated Cauchy problem is local and global well-posed when $s \geq -3/4$, whereas it is ill-posed for $s < -3/4$. On the case $\alpha = 0$, see (Guo and Huo, 2004) and (Kozono et al. 2001) for the study of Cauchy problem, see also (Angulo, 1999), (Linares and Scialom, 2005) and (Chen and Bona, 1998) for more informations on the topics dealing with existence, nonexistence, stability of solitary waves and their asymptotics. In fact (1.1) is exactly a perturbation of (2.2) by a rotational term. Two possibilities are also worth noting in this case ($\alpha = 0$) :

- 1) The case $\beta = 0$ induces the well known Korteweg-de Vries equation (KdV) for which its associated Cauchy problem has been widely studied by many authors, see for example (Colliander et al. 2001), (Kenig et al. 1996) and (Kishimoto, 2009).
- 2) The case $\delta = 0$ and $\beta \neq 0$ generates the Benjamin-Ono equation, see (Fokas and Ablowitz, 1983), (Ionescu and Kenig, 2007), (Kenig and Koenig, 2003) and (Kenig and Martel, 2009), for mathematical aspects concerning complete integrability, conserved quantities, well-posedness, solitary wave solutions and their asymptotics.

This paper is focused on the general case, i.e. the rotational-modified Benjamin equation.

3 New results for the rotationnal-modified Benjamin equation

Notations: we set $\langle \cdot \rangle = (1 + |\cdot|^2)^{1/2}$; denote by $A \sim B$ to stand for $B \leq A \leq CB$, where C is a varying positive constant; and we denote $b - \epsilon$ and $b + \epsilon$ by b_- and b_+ , respectively for a small $\epsilon > 0$. Let φ be a C_0^∞ function supported in $[-2, +2]$ and equals to 1 over the interval $[-1, +1]$, where $\varphi_\rho = \varphi(\frac{\cdot}{\rho})$ and χ is the usual characteristic function. Finally, $\mathcal{F}f = \hat{f}$ is the Fourier transform of f . Let

$$m(\eta) = \frac{\gamma}{\eta} + \alpha\eta - \beta\eta|\eta| + \delta\eta^3, \quad m(\eta_j) = \frac{\gamma}{\eta_j} + \alpha\eta_j - \beta\eta_j|\eta_j| + \delta\eta_j^3, \quad (3.1)$$

$$\lambda = \tau - m(\eta), \quad \lambda_j = \tau_j - m(\eta_j), \quad \text{for } j = 1, 2. \quad (3.2)$$

Throughout the rest of this article, we will assume that $\gamma, \delta > 0, \beta < 0$ or $0 < \beta < 4\delta^{3/4}\gamma^{1/4}$.

Let $a = \max\left\{1, \frac{2|\beta|}{3|\delta|}\right\}$, $\tau = \tau_1 + \tau_2$, $\eta = \eta_1 + \eta_2$, $\eta_{\max} = \max\{\eta, \eta_1, \eta_2\}$ and $\eta_{\text{med}} = \text{mean}\{\eta, \eta_1, \eta_2\}$ as well as $\eta_{\min} = \min\{\eta, \eta_1, \eta_2\}$.

The Bourgain space is defined by

$$X_{s,b} = \left\{ u \text{ S.T. } \|u\|_{X_{s,b}} = \|\eta^s \lambda^p \hat{u}\|_{L^2_{\tau\eta}(\mathbb{R}^2)} < \infty \right\},$$

where u is a distribution.

Its restriction onto the time interval $[0, T]$ is denoted by $X_{s,p}^T$, and defined by

$$\|u\|_{X_{s,p}^T} = \inf \left\{ \|v\|_{X_{s,b}} : v \in X_{s,b}, u(t) = v(t) \text{ for } 0 \leq t \leq T \right\}$$

Let us now formulate our main results.

Theorem 3.1. *Let $s \geq -\frac{3}{4} + 3\epsilon$, then $\exists b \in (\frac{1}{2}, 1)$ S.T. we have*

$$\|\partial_x(uv)\|_{X_{s,b-1}} \leq C\|u\|_{X_{s,b}}\|v\|_{X_{s,b}} \quad (3.3)$$

Theorem 3.2. *Let $s \geq -\frac{3}{4} + 3\epsilon$, $b \in (\frac{1}{2}, 1)$, then problem (1.1) is well-posed locally in $H^s(\mathbb{R})$, i.e. for $u_0 \in H^s(\mathbb{R})$, $\exists T = T(\|u\|_{H^s} > 0)$, such that (1.1) admits $u \in C([0, T], H^s) \cap X_{s,b}$ as a unique solution. Moreover the solution map $T : u_0 \rightarrow u$ is an analytic map from $H^s(\mathbb{R})$ to $C([0, T]; H^s(\mathbb{R}))$.*

To prove these results, the following preliminaries are needed.

3.1 Preliminary Estimates

For convenience, let us write

$$P^N F = \int_{|\eta| \geq N} e^{ix\eta} \hat{F} d\eta, \quad P_N F = \int_{|\eta| \leq N} e^{ix\eta} \hat{F} d\eta, \quad \forall N > 0$$

and

$$f = \mathcal{F}^{-1}F, \quad \mathcal{F}f_\sigma(\eta, \tau) = \frac{F(\eta, \tau)}{\langle \lambda \rangle^\sigma}.$$

The solution of (1.1) can be formally written as follows:

$$u(x, t) = S(t)u_0(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{i(\eta x - m(\eta))t} u_0 d\eta. \quad (3.4)$$

Let us now state some useful lemmas that will be used later

Lemma 3.3. *Let $m(\cdot)$ be given by (3.1) with $\alpha = 0$, and assume that $m(0) = 0$, then*

$$|m''(\eta)| \geq -2\beta + 8\gamma^{\frac{1}{4}}\delta^{\frac{3}{4}}, \quad \eta \neq 0.$$

Lemma 3.4. *Let $f \in L^2$, we have*

$$\|S(t)f\|_{L_t^q L_x^p} \leq \|f\|_{L^2}, \quad (3.5)$$

for $q = \frac{4}{\vartheta}$, $p = \frac{2}{1-\vartheta}$, $\vartheta \in [0, 1]$.

Proof. For the proof of Lemmas 3.3 and 3.4, see Lemma 1 and Theorem 2 in (Durán, 2018). □

Lemma 3.5. *For $\rho \in [0, 1]$, $-\frac{1}{2} < b' \leq 0 \leq b \leq b' + 1$ and $s \in \mathbb{R}$, the following inequalities hold*

$$\|\varphi(\frac{t}{\rho})S(t)u_0\|_{X_{s,b}(\mathbb{R}^2)} \leq C\rho^{\frac{1}{2}-b}\|u_0\|_{H^s(\mathbb{R})}, \quad (3.6)$$

$$\left\| \varphi(\frac{t}{\rho}) \int_0^t S(t-\tau)F(\tau)d\tau \right\|_{X_{s,b}(\mathbb{R}^2)} \leq C\rho^{1+b'-b}\|F\|_{X_{s,b'}(\mathbb{R}^2)}. \quad (3.7)$$

Proof. For the proof of this Lemma, see (kenig et al. 1993). □

Lemma 3.6. *Let $\theta \in [0, 1]$, $N \geq 2$, if $r > \frac{\theta}{2}$, then*

$$\|D_x^\theta P^N f_r\|_{L_x^{\frac{2}{1-\theta}} L_t^2} \leq C\|F\|_{L_\eta^2 L_\tau^2}.$$

Lemma 3.7. *For $r > \frac{1}{3}$, $N \geq 2$ and $0 < c < +\infty$, it holds that*

$$\|f_r\|_{L_x^4 L_t^4} \leq C\|F\|_{L_\eta^2 L_\tau^2}. \quad (3.8)$$

If $r > \frac{1}{2}$, then we have

$$\|P_c f_r\|_{L_x^2 L_t^\infty} \leq C\|F\|_{L_\eta^2 L_\tau^2}. \quad (3.9)$$

Lemma 3.8. *For $b > \frac{1}{2}$, $N \geq 2$, then*

$$\|f\|_{L_x^2 L_t^2} \leq C\|F\|_{X_{0,0}}, \quad (3.10)$$

$$\|f\|_{L_x^2 L_t^4} \leq C\|F\|_{X_{0,\frac{1}{2}b}}, \quad (3.11)$$

$$\|D_x^{\frac{1}{4}} P^N f\|_{L_x^\infty L_t^4} \leq C\|F\|_{X_{0,b}}, \quad (3.12)$$

$$\|D_x^{\frac{1}{8}} P^N f\|_{L_x^4 L_t^4} \leq C\|F\|_{X_{0,\frac{3}{4}b}}. \quad (3.13)$$

Proof. For the proof of Lemmas 3.6, 3.7 and 3.8, the reader can be referred to (Wang and Cui, 2007) and (Guo and Huo, 2004). $\square$

Lemma 3.9. *Let $\alpha, \beta, \gamma, \delta$ and $m(\cdot)$ be given by (3.1). If $|\eta_{\min}| \geq 1$ or $|\eta_{\min}|^2 |\eta_{\max}| |\eta_{\text{med}}| \geq 1$ and $|\eta_{\max}| \sim |\eta_{\text{med}}| \geq 2a$, then one of the following cases must hold*

$$|\lambda| := \max \{|\lambda|, |\lambda_1|, |\lambda_2|\} \geq |\eta_1 \eta_2 \eta|, \quad (3.14)$$

$$|\lambda_2| := \max \{|\lambda|, |\lambda_1|, |\lambda_2|\} \geq |\eta_1 \eta_2 \eta|, \quad (3.15)$$

$$|\lambda_1| := \max \{|\lambda|, |\lambda_1|, |\lambda_2|\} \geq |\eta_1 \eta_2 \eta|. \quad (3.16)$$

Proof. From equation (3.2), we have

$$\begin{aligned} L(\eta) &= -\lambda + \lambda_1 + \lambda_2 \\ &= 3\delta\eta_1\eta_2\eta - \beta(|\eta_1|\eta_1 + |\eta_2|\eta_2 + |\eta|\eta) + \gamma\left(\frac{\eta_1^2 + \eta_2^2 + \eta_1\eta_2}{\eta_1\eta_2\eta}\right), \end{aligned} \quad (3.17)$$

using the symmetry, we may assume that $|\eta_1| < |\eta_2|$, so we can simplify (3.17) with respect to the four regions:

$$D_1 = \{(\tau_1, \eta_1, \tau_2, \eta_2) \in \mathbb{R}^4; \eta_1 > 0, \eta_2 > 0\},$$

$$D_2 = \{(\tau_1, \eta_1, \tau_2, \eta_2) \in \mathbb{R}^4; \eta_1 < 0, \eta_2 < 0\},$$

$$D_3 = \{(\tau_1, \eta_1, \tau_2, \eta_2) \in \mathbb{R}^4; \eta_1 > 0, \eta_2 < 0\},$$

$$D_4 = \{(\tau_1, \eta_1, \tau_2, \eta_2) \in \mathbb{R}^4; \eta_1 < 0, \eta_2 > 0\}.$$

Then, we have

$$\begin{aligned} L_{D_1}(\eta) &= 3\delta\eta_1\eta_2\eta - \beta(\eta_1^2 + \eta_2^2 - (\eta_1 + \eta_2)^2) + \gamma\frac{\eta_1^2 + \eta_2^2 + \eta_1\eta_2}{\eta_1\eta_2\eta} \\ &= 3\delta\eta_1\eta_2(\eta + \frac{2\beta}{3\delta}) + \gamma\frac{\eta_1^2 + \eta_2^2 + \eta_1\eta_2}{\eta_1\eta_2\eta}. \\ L_{D_2}(\eta) &= 3\delta\eta_1\eta_2\eta - \beta(-\eta_1^2 - \eta_2^2 + (\eta_1 + \eta_2)^2) + \gamma\frac{\eta_1^2 + \eta_2^2 + \eta_1\eta_2}{\eta_1\eta_2\eta} \\ &= 3\delta\eta_1\eta_2(\eta - \frac{2\beta}{3\delta}) + \gamma\frac{\eta_1^2 + \eta_2^2 + \eta_1\eta_2}{\eta_1\eta_2\eta}. \\ L_{D_3}(\eta) &= 3\delta\eta_1\eta_2\eta - \beta(\eta_1^2 - \eta_2^2 - (\eta_1 + \eta_2)^2) + \gamma\frac{\eta_1^2 + \eta_2^2 + \eta_1\eta_2}{\eta_1\eta_2\eta} \\ &= 3\delta\eta_1\eta_2(\eta + \frac{2\beta}{3\delta}) + \gamma\frac{\eta_1^2 + \eta_2^2 + \eta_1\eta_2}{\eta_1\eta_2\eta}. \\ L_{D_4}(\eta) &= 3\delta\eta_1\eta_2\eta - \beta(-\eta_1^2 + \eta_2^2 - (\eta_1 + \eta_2)^2) + \gamma\frac{\eta_1^2 + \eta_2^2 + \eta_1\eta_2}{\eta_1\eta_2\eta} \\ &= 3\delta\eta_1\eta_2(\eta - \frac{2\beta}{3\delta}) + \gamma\frac{\eta_1^2 + \eta_2^2 + \eta_1\eta_2}{\eta_1\eta_2\eta}. \end{aligned}$$

We will only treat with the restriction of L on D_1 . From (3.17), we have

$$\begin{aligned} |L| &= |-\lambda + \lambda_1 + \lambda_2| \\ &\leq |\lambda| + |\lambda_1| + |\lambda_2| \\ &\leq 3 \max\{|\lambda|, |\lambda_1|, |\lambda_2|\}. \end{aligned}$$

If we denote by λ_{max} the maximum of (λ, λ_i) for $i = 1, 2$, we have

$$|\lambda_{max}| \geq \frac{1}{3}|L|,$$

with

$$|L| \geq 3\delta|\eta_1||\eta_2| \left| \left(\eta + \frac{2\beta}{3\delta} \right) \right| - \gamma|\eta_1||\eta_2||\eta| \left(\frac{|\eta_1^2 + \eta_2^2 + \eta_1\eta_2|}{(\eta_1\eta_2\eta)^2} \right), \quad (3.18)$$

$$\left| \left(\eta + \frac{2\beta}{3\delta} \right) \right| \geq |\eta| - a \geq |\eta| - \frac{1}{2}|\eta| \geq \frac{1}{2}|\eta|, \quad (3.19)$$

and

$$\frac{|\eta_1^2 + \eta_2^2 + \eta_1\eta_2|}{(\eta_1\eta_2\eta)^2} \leq \frac{1}{(\eta_2\eta)^2} + \frac{1}{(\eta_1\eta)^2} + \frac{1}{\eta_1\eta_2\eta^2} \leq \frac{3}{\eta_{min}^2\eta_{max}\eta_{med}} \leq \frac{3}{\eta_{min}^4} \leq 3. \quad (3.20)$$

□

From (3.18), (3.19) and (3.20), we conclude that

$$|L| \geq C|\eta_1||\eta_2||\eta|.$$

For the others three cases $(D_i)_{i=2}^4$ results are shown by similar arguments.

Lemma 3.10. For $\eta \neq 0$, let $m(\eta) = \frac{1}{\eta} + \eta|\eta| + \eta^3$ and $0 \leq s \leq \frac{1}{2}$ as well as

$$\mathcal{F}(K^s(u, v)(\eta, \tau)) = \int_{\eta\tau} |m'(\eta - \eta_1) - m'(\eta_1)|^s \mathcal{F}u(\eta, \tau) \mathcal{F}v(\eta_1, \tau_1) d\eta_1 d\tau_1.$$

Then, we find that

$$\|K^s(u, v)\|_{L_{xt}^2} \leq C\|u\|_{X_{0, \frac{2+2s}{3}(b+)}} \|v\|_{X_{0, \frac{2+2s}{3}(b+)}}.$$

Proof. For the proof of this Lemma, we can use the same arguments as Lemma 2.9 in (Yan et al. 2018). □

3.2 Proof of the Bilinear Estimate

In this section, we discuss an estimate for the nonlinear term in (1.1), which is the crucial step to prove the well-posedness by the Picard iteration.

Proof. of Theorem 3.1 By the use of duality and the Plancherel identity, (3.3) follows if we show

$$\int_{\mathbb{R}^2} \overline{w} \partial_x(uv) dx dt \leq C[\|u\|_{X_{s, b+}} \|v\|_{X_{s, b+}}] \|w\|_{X_{-s, (1-b)-}}, \quad (3.21)$$

where $b = \frac{1}{2}$. If we define

$$F(\eta, \tau) = \langle \eta \rangle^{-s} \langle \lambda \rangle^{(1-b)-} \widehat{w}(\eta, \tau),$$

$$F_1(\eta_1, \tau_1) = \langle \eta_1 \rangle^s \langle \lambda_1 \rangle^{b+} \widehat{u}(\eta_1, \tau_1),$$

$$F_2(\eta_2, \tau_2) = \langle \eta_2 \rangle^s \langle \lambda_2 \rangle^{b+} \widehat{v}(\eta_2, \tau_2),$$

then (3.21) is equivalent to

$$\int_{\mathbb{R}^2} \int_{\eta\tau} \frac{|\eta_1 + \eta_2| \langle \eta \rangle^s F(\eta, \tau) \prod_{j=1}^2 F_j(\eta_j, \tau_j)}{\langle \lambda_j \rangle^{(1-b)-} \prod_{j=1}^2 \langle \eta_j \rangle^s \langle \lambda_j \rangle^{b+}} d\eta_1 d\tau_1 d\eta d\tau \leq C \|F\|_{L^2_{\eta\tau}} \left(\prod_{j=1}^2 \|F_j\|_{L^2_{\eta\tau}} \right). \quad (3.22)$$

We may assume that $|\eta_1| \leq |\eta_2|$ and $F(\eta, \tau) \geq 0$, $F_j(\eta_j, \tau_j)_{j=1,2} \geq 0$ and define

$$M(\eta_1, \tau_1, \eta, \tau) := \frac{|\eta| \langle \eta \rangle^s}{\langle \lambda \rangle^{(1-b)-} \prod_{j=1}^2 \langle \eta_j \rangle^s \langle \lambda_j \rangle^{b+}}, \quad (3.23)$$

and

$$I = \int_{\mathbb{R}^2} \int_{\eta\tau} M(\eta_1, \tau_1, \eta, \tau) F(\eta, \tau) \prod_{j=1}^2 F_j(\eta_j, \tau_j) d\eta_1 d\tau_1 d\eta d\tau. \quad (3.24)$$

If we set

$$\Lambda := \{(\eta_1, \tau_1, \eta, \tau) \in \mathbb{R}^4 : \eta = \eta_1 + \eta_2, \tau = \tau_1 + \tau_2, |\eta_1| \leq |\eta - \eta_1|\},$$

then we can define

$$\begin{aligned} \Lambda_1 &:= \{(\eta_1, \tau_1, \eta, \tau) \in \Lambda : |\eta - \eta_1| \leq 2a\}, \\ \Lambda_2 &:= \{(\eta_1, \tau_1, \eta, \tau) \in \Lambda : |\eta_1| \geq 2a, 2|\eta_1| \leq |\eta - \eta_1|\}, \\ \Lambda_3 &:= \{(\eta_1, \tau_1, \eta, \tau) \in \Lambda : |\eta_1| \leq 2a, |\eta - \eta_1| \geq 4a, 2|\eta_1| \leq |\eta - \eta_1|\}, \\ \Lambda_4 &:= \{(\eta_1, \tau_1, \eta, \tau) \in \Lambda : |\eta| \geq \frac{1}{2}|\eta - \eta_1|, |\eta - \eta_1| \geq 4a, |\eta - \eta_1| \leq 2|\eta_1|\}, \\ \Lambda_5 &:= \{(\eta_1, \tau_1, \eta, \tau) \in \Lambda : |\eta| \leq 2a, |\eta - \eta_1| \geq 4a, |\eta - \eta_1| \leq 2|\eta_1|\}. \end{aligned}$$

Let I_j be the restriction of I on the regions $(\Lambda_j)_{j=1}^5$.

- For region Λ_1 , we have $|\eta_1| \leq |\eta - \eta_1| \leq 2a$ and $|\eta| \leq 2|\eta - \eta_1| \leq 4a$, then we obtain

$$M(\eta, \tau, \eta_1, \tau_1) \leq \frac{C}{\langle \lambda \rangle^{(1-b)-} \prod_{j=1}^2 \langle \lambda_j \rangle^{b+}}. \quad (3.25)$$

In view of the Cauchy–Schwarz and Hölder inequalities, and Lemma 3.7 as well as the Plancherel identity, we find that

$$\begin{aligned} I_1 &\leq C \int_{\mathbb{R}^2} \int_{\eta\tau} \frac{F(\eta, \tau) F_1(\eta_1, \tau_1) F_2(\eta - \eta_1, \tau - \tau_1)}{\langle \lambda \rangle^{(1-b)-} \langle \lambda_1 \rangle^{b+} \langle \lambda_2 \rangle^{b+}} d\eta_1 d\tau_1 d\eta d\tau \\ &\leq \left\| \frac{F(\eta, \tau)}{\langle \lambda \rangle^{(1-b)-}} \right\|_{L^2_{\eta\tau}} \left\| \int_{\eta\tau} \frac{F_1(\eta_1, \tau_1) F_2(\eta - \eta_1, \tau - \tau_1)}{\langle \lambda_1 \rangle^{b+} \langle \lambda_2 \rangle^{b+}} d\eta_1 d\tau_1 \right\|_{L^2_{\eta\tau}} \\ &\leq \|F(\eta, \tau)\|_{L^2_{\eta\tau}} \left\| \mathcal{F}^{-1} \frac{F_1(\eta_1, \tau_1)}{\langle \lambda_1 \rangle^{b+}} \right\|_{L^4_{xt}} \left\| \mathcal{F}^{-1} \frac{F_2(\eta - \eta_1, \tau - \tau_1)}{\langle \lambda_2 \rangle^{b+}} \right\|_{L^4_{xt}} \\ &\leq \|F\|_{L^2_{\eta\tau}} \|F_1\|_{L^2_{\eta\tau}} \|F_2\|_{L^2_{\eta\tau}}. \end{aligned}$$

- For region Λ_2 , since $2|\eta_1| \leq |\eta - \eta_1|$, we obtain $\frac{1}{2}|\eta - \eta_1| \leq |\eta| \leq \frac{3}{2}|\eta - \eta_1|$, which means that $|\eta| \sim |\eta - \eta_1|$. Next, we study (3.14)-(3.16), respectively.

1- If (3.14) is valid then $|\lambda| \geq |\eta - \eta_1|^2 |\eta_1|$. Since $s \geq -\frac{3}{4} + 3\epsilon$, we get

$$\begin{aligned} M(\eta, \tau, \eta_1, \tau_1) &\leq C \frac{|\eta||\eta_1|^{-s}}{\langle \lambda \rangle^{(1-b)-} \prod_{j=1}^2 \langle \lambda_j \rangle^{b+}} \leq C \frac{|\eta|^{2\epsilon} |\eta_1|^{-s+(b-1)+}}{\prod_{j=1}^2 \langle \lambda_j \rangle^{b+}} \\ &\leq C \frac{|\eta|^{2\epsilon} |\eta_1|^{-s+(b-1)+}}{\prod_{j=1}^2 \langle \lambda_j \rangle^{b+}} \leq C \frac{|\eta|^{\frac{1}{4}}}{\prod_{j=1}^2 \langle \lambda_j \rangle^{b+}} \leq C \frac{|\eta - \eta_1|^{\frac{1}{4}}}{\prod_{j=1}^2 \langle \lambda_j \rangle^{b+}}. \end{aligned} \quad (3.26)$$

By using Similar arguments as before (Plancherel identity, Cauchy–Schwarz and Hölder inequalities) together with Lemma 3.8, we find that

$$\begin{aligned} I_2 &\leq C \int_{\mathbb{R}^2} \int_{\eta\tau} \frac{F_1(\eta_1, \tau_1) \mathcal{X}_{|\eta - \eta_1| \geq 4a} |\eta - \eta_1|^{\frac{1}{4}} F_2(\eta - \eta_1, \tau - \tau_1)}{\langle \lambda_1 \rangle^{b+} \langle \lambda_2 \rangle^{b+}} F(\eta, \tau) d\eta_1 d\tau_1 d\eta d\tau \\ &\leq \|F(\eta, \tau)\|_{L_{\eta\tau}^2} \left\| \mathcal{F}^{-1} \frac{F_1(\eta_1, \tau_1)}{\langle \lambda_1 \rangle^{b+}} \right\|_{L_x^2 L_t^4} \left\| P^{4a} D_x^{\frac{1}{4}} \mathcal{F}^{-1} \frac{F_2(\eta - \eta_1, \tau - \tau_1)}{\langle \lambda_2 \rangle^{b+}} \right\|_{L_t^4 L_x^\infty} \\ &\leq \|F\|_{L_{\eta\tau}^2} \|F_1\|_{L_{\eta\tau}^2} \|F_2\|_{L_{\eta\tau}^2}. \end{aligned}$$

2- If (3.15) is valid, since $s \geq -\frac{3}{4} + 3\epsilon$ and $\langle \lambda \rangle^{(b-1)+} \langle \lambda_2 \rangle^{(-b)-} \leq \langle \lambda_2 \rangle^{(b-1)+} \langle \lambda \rangle^{(-b)-}$ as well as

$$\begin{aligned} |m'(\eta) - m'(\eta_1)| &= \left| -\frac{1}{\eta^2} + \frac{1}{\eta_1^2} + |\eta| - |\eta_1| + 3(\eta^2 - \eta_1^2) \right| \\ &= 3|\eta^2 - \eta_1^2| \left| 1 + \frac{1}{3\eta^2\eta_1^2} + \frac{1}{3(|\eta| + |\eta_1|)} \right| \\ &\geq C|\eta^2 - \eta_1^2| \\ &\geq C|\eta|^2, \end{aligned}$$

as a result, we infer that

$$\begin{aligned} M(\eta, \tau, \eta_1, \tau_1) &\leq C \frac{|\eta||\eta_1|^{-s}}{\langle \lambda_2 \rangle^{(1-b)-} \langle \lambda \rangle^{b+} \langle \lambda_1 \rangle^{b+}} \leq C \frac{|\eta|^{2\epsilon} |\eta_1|^{-s+(b-1)+}}{\langle \lambda \rangle^{b+} \langle \lambda_1 \rangle^{b+}} \\ &\leq C \frac{|\eta|^{\frac{1}{4}}}{\langle \lambda \rangle^{b+} \langle \lambda_1 \rangle^{b+}} \leq C \frac{|m'(\eta) - m'(\eta_1)|^{\frac{1}{2}}}{\langle \lambda \rangle^{b+} \langle \lambda_1 \rangle^{b+}}. \end{aligned} \quad (3.27)$$

By using the following transformed variables

$$\begin{aligned} \eta' &= \eta - \eta_1, \quad \tau' = \tau - \tau_1, \quad \eta'_1 = -\eta_1, \quad \tau'_1 = -\tau_1, \quad \eta' - \eta'_1 = \eta, \quad \tau - \tau'_1 = \tau, \\ \lambda' &= \tau' - m(\eta') = \lambda_2, \quad \lambda'_1 = \tau'_1 - m(\eta'_1) = -\lambda_1, \quad \lambda'_2 = \tau' - \tau'_1 - m(\eta - \eta'_1) = \lambda, \\ F'(\eta', \tau') &= F_2(\eta - \eta_1, \tau - \tau_1), \quad F'_1(\eta'_1, \tau'_1) = F_1(-\eta_1, -\tau_1), \quad F'_2(\eta' - \eta'_1, \tau' - \tau'_1) = F(\eta, \tau), \end{aligned}$$

we get

$$M(\eta, \tau, \eta_1, \tau_1) \leq C \frac{|m'(\eta' - \eta'_1) - m'(\eta'_1)|^{\frac{1}{2}}}{\langle \lambda'_2 \rangle^{b+} \langle \lambda'_1 \rangle^{b+}}.$$

Again, the use of Plancherel identity, Cauchy–Schwarz and Hölder inequalities with (3.27) as well as Lemma 3.10, we deduce that

$$\begin{aligned} I_2 &\leq C \int_{\mathbb{R}^2} \int_{\eta\tau} \frac{|m'(\eta' - \eta'_1) - m'(\eta'_1)|^{\frac{1}{2}} F'(\eta', \tau') F'_1(\eta'_1, \tau'_1) F_2(\eta' - \eta'_1, \tau' - \tau'_1)}{\langle \lambda'_2 \rangle^{b+} \langle \lambda'_1 \rangle^{b+}} d\eta_1 d\tau_1 d\eta d\tau \\ &\leq \| \mathcal{F}^{-1} F(\eta', \tau') \|_{L_{xt}^2} \left\| K^{\frac{1}{2}} \left(\mathcal{F}^{-1} \left(\frac{F'_2(\eta' - \eta'_1, \tau' - \tau'_1)}{\langle \lambda'_2 \rangle^{b+}} \right), \mathcal{F}^{-1} \left(\frac{F'_1(\eta'_1, \tau'_1)}{\langle \lambda'_1 \rangle^{b+}} \right) \right) \right\|_{L_{xt}^2} \\ &\leq C \|F'\|_{L_{\eta, \tau}^2} \|F'_1\|_{L_{\eta, \tau}^2} \|F'_2\|_{L_{\eta, \tau}^2}. \end{aligned}$$

3- If (3.16) is valid, we can prove this case similarly to the previous one.

- In region Λ_3 , we also have $|\eta| \sim |\eta - \eta_1|$ so that we consider the most interesting case $-\frac{3}{4} + 3\epsilon \leq s < 0$, and conclude that

$$M(\eta, \tau, \eta_1, \tau_1) \leq \frac{C|\eta|}{\langle \lambda \rangle^{(1-b)-} \prod_{j=1}^2 \langle \lambda_j \rangle^{b+}}. \quad (3.28)$$

Let us consider the two cases $|\eta_1| \geq 1$ and $|\eta_1| < 1$.

The case $|\eta_1| \geq 1$ is treated as in the previous region. If $|\eta_1| < 1$, by the use of Lemmas 3.6 and 3.7 it follows that

$$\begin{aligned} I_3 &\leq C \int_{\mathbb{R}^2} \int_{\eta\tau} \frac{F(\eta, \tau) F_1(\eta_1, \tau_1) |\eta - \eta_1| F_2(\eta - \eta_1, \tau - \tau_1)}{\langle \lambda \rangle^{(1-b)-} \langle \lambda_1 \rangle^{b+} \langle \lambda_2 \rangle^{b+}} d\eta_1 d\tau_1 d\eta d\tau \\ &\leq \left\| \frac{F(\eta, \tau)}{\langle \lambda \rangle^{(1-b)-}} \right\|_{L_{\eta\tau}^2} \left\| \int_{\eta\tau} \frac{F_1(\eta_1, \tau_1) F_2(\eta - \eta_1, \tau - \tau_1)}{\langle \lambda_1 \rangle^{b+} \langle \lambda_2 \rangle^{b+}} d\eta_1 d\tau_1 \right\|_{L_{\eta\tau}^2} \\ &\leq \left\| \mathcal{F}^{-1} \frac{F(\eta, \tau)}{\langle \lambda \rangle^{(1-b)-}} \right\|_{L_{xt}^2} \left\| P_1 \mathcal{F}^{-1} \frac{F_1(\eta_1, \tau_1)}{\langle \lambda_1 \rangle^{b+}} \right\|_{L_x^2 L_t^\infty} \left\| P^{4a} D_x \mathcal{F}^{-1} \frac{F_2(\eta - \eta_1, \tau - \tau_1)}{\langle \lambda_2 \rangle^{b+}} \right\|_{L_t^2 L_x^\infty} \\ &\leq \|F\|_{L_{\eta\tau}^2} \|F_1\|_{L_{\eta\tau}^2} \|F_2\|_{L_{\eta\tau}^2}. \end{aligned}$$

- For region Λ_4 , since $\frac{1}{2}|\eta - \eta_1| \leq |\eta| \leq 2|\eta - \eta_1|$ and $|\eta_1| \leq |\eta - \eta_1| \leq 2|\eta_1|$, it implies that $|\eta| \sim |\eta_1| \sim |\eta - \eta_1|$. We have to consider the three scenarios (3.14), (3.15) and (3.16)
- 1- If (3.14) is valid, then $|\lambda| \geq |\eta|^3$, and we have

$$\begin{aligned} M(\eta, \tau, \eta_1, \tau_1) &\leq C \frac{|\eta - \eta_1|^{-s} |\eta|}{\langle \lambda \rangle^{(1-b)-} \langle \lambda_1 \rangle^{b+} \langle \lambda_2 \rangle^{b+}} \leq C \frac{|\eta - \eta_1|^{-s+3(b-1)+3\epsilon+1}}{\langle \lambda_1 \rangle^{b+} \langle \lambda_2 \rangle^{b+}} \\ &\leq C \frac{|\eta_1|^{\frac{1}{8}} |\eta - \eta_1|^{\frac{1}{8}}}{\langle \lambda_1 \rangle^{b+} \langle \lambda_2 \rangle^{b+}}. \end{aligned} \quad (3.29)$$

By Cauchy, Plancherel and Hölder inequalities and Lemma 3.8, we get

$$\begin{aligned}
 I_4 &\leq C \int_{\mathbb{R}^2} \int_{\eta\tau} \frac{F(\eta, \tau) \mathcal{X}_{|\eta_1| \geq 2a} |\eta_1|^{\frac{1}{8}} F_1(\eta_1, \tau_1) \mathcal{X}_{|\eta - \eta_1| \geq 4a} |\eta - \eta_1|^{\frac{1}{8}} F_2(\eta - \eta_1, \tau - \tau_1)}{\langle \lambda_1 \rangle^{b+} \langle \lambda_2 \rangle^{b+}} d\eta_1 d\tau_1 d\eta d\tau \\
 &\leq \|F(\eta, \tau)\|_{L^2_{\eta\tau}} \left\| \int_{\eta\tau} \frac{\mathcal{X}_{|\eta| \geq 2a} |\eta|^{\frac{1}{8}} F_1(\eta_1, \tau_1) \mathcal{X}_{|\eta - \eta_1| \geq 4a} |\eta - \eta_1|^{\frac{1}{8}} F_2(\eta - \eta_1, \tau - \tau_1)}{\langle \lambda_1 \rangle^{b+} \langle \lambda_2 \rangle^{b+}} d\eta_1 d\tau_1 \right\|_{L^2_{\eta\tau}} \\
 &\leq \|F(\eta, \tau)\|_{L^2_{\eta\tau}} \left\| D_x^{\frac{1}{8}} P^{2a} \mathcal{F}^{-1} \frac{F_1(\eta_1, \tau_1)}{\langle \lambda_1 \rangle^{b+}} \right\|_{L_t^4 L_x^4} \left\| D_x^{\frac{1}{8}} P^{4a} \mathcal{F}^{-1} \frac{F_2(\eta - \eta_1, \tau - \tau_1)}{\langle \lambda_2 \rangle^{b+}} \right\|_{L_t^4 L_x^4} \\
 &\leq \|F\|_{L^2_{\eta\tau}} \|F_1\|_{L^2_{\eta\tau}} \|F_2\|_{L^2_{\eta\tau}}.
 \end{aligned}$$

2- If (3.15) is valid, then $\langle \lambda_2 \rangle^{(-b)-} \leq |\eta|^{-3b-3\epsilon}$ which implies

$$\begin{aligned}
 M(\eta, \tau, \eta_1, \tau_1) &\leq C \frac{|\eta|^{-s+1-3b-3\epsilon}}{\langle \lambda \rangle^{(1-b)-} \langle \lambda_1 \rangle^{b+}} \leq C \frac{|\eta|^{\frac{1}{4}}}{\langle \lambda_1 \rangle^{b+} \langle \lambda \rangle^{(1-b)-}} \\
 &\leq C \frac{|\eta|^{\frac{1}{8}} |\eta_1|^{\frac{1}{8}}}{\langle \lambda_1 \rangle^{b+} \langle \lambda \rangle^{(1-b)-}}.
 \end{aligned} \tag{3.30}$$

By similar arguments as before and because $(1-b) > \frac{3}{8}$, we can use (3.13) in Lemma 3.8 and conclude that

$$\begin{aligned}
 I_4 &\leq C \int_{\mathbb{R}^2} \int_{\eta\tau} \frac{\mathcal{X}_{|\eta_1| \geq 2a} |\eta_1|^{\frac{1}{8}} F_1(\eta_1, \tau_1) \mathcal{X}_{|\eta| \geq 2a} |\eta|^{\frac{1}{8}} F(\eta, \tau) F_2(\eta - \eta_1, \tau - \tau_1)}{\langle \lambda_1 \rangle^{b+} \langle \lambda \rangle^{(1-b)-}} d\eta_1 d\tau_1 d\eta d\tau \\
 &\leq \|F_2(\eta - \eta_1, \tau - \tau_1)\|_{L^2_{\eta\tau}} \left\| \int_{\eta\tau} \frac{\mathcal{X}_{|\eta_1| \geq 2a} |\eta_1|^{\frac{1}{8}} F_1(\eta_1, \tau_1) \mathcal{X}_{|\eta| \geq 2a} |\eta|^{\frac{1}{8}} F(\eta, \tau)}{\langle \lambda_1 \rangle^{b+} \langle \lambda \rangle^{(1-b)-}} d\eta_1 d\tau_1 \right\|_{L^2_{\eta\tau}} \\
 &\leq \|F_2(\eta - \eta_1, \tau - \tau_1)\|_{L^2_{\eta\tau}} \left\| D_x^{\frac{1}{8}} P^{2a} \mathcal{F}^{-1} \frac{F_1(\eta_1, \tau_1)}{\langle \lambda_1 \rangle^{b+}} \right\|_{L_t^4 L_x^4} \left\| D_x^{\frac{1}{8}} P^{2a} \mathcal{F}^{-1} \frac{F(\eta, \tau)}{\langle \lambda \rangle^{(1-b)-}} \right\|_{L_t^4 L_x^4} \\
 &\leq \|F\|_{L^2_{\eta\tau}} \|F_1\|_{L^2_{\eta\tau}} \|F_2\|_{L^2_{\eta\tau}}.
 \end{aligned}$$

3- If (3.16) is valid, then we can proceed as the previous case.

- For region Λ_5 , we consider: $s \geq 0$ and $-\frac{3}{4} \leq s < 0$, respectively.

1- If $s \geq 0$, we find that

$$\begin{aligned}
 M(\eta, \tau, \eta_1, \tau_1) &\leq C \frac{|\eta|^{1+s} |\eta_1|^{-2s}}{\langle \lambda \rangle^{(1-b)-} \langle \lambda_1 \rangle^{b+} \langle \lambda_2 \rangle^{b+}} \\
 &\leq \frac{C}{\langle \lambda \rangle^{(1-b)-} \langle \lambda_1 \rangle^{b+} \langle \lambda_2 \rangle^{b+}}.
 \end{aligned} \tag{3.31}$$

This is established in a similar way as for (3.25).

2- If $-\frac{3}{4} + 3\epsilon \leq s < 0$, let us consider $|\eta| \geq 1$ or $|\eta| |\eta_1|^{\frac{1}{2}} |\eta - \eta_1|^{\frac{1}{2}} \geq 1$ and $|\eta| < 1$ or $|\eta| |\eta_1|^{\frac{1}{2}} < 1 |\eta - \eta_1|^{\frac{1}{2}} < 1$, respectively.

– When $|\eta| \geq 1$, we need to deal with (3.14), (3.15) and (3.16).

1- If (3.14) is valid, then $|\lambda| \geq |\eta||\eta - \eta_1|^2$. Since $0 > s \geq -\frac{3}{4} + 3\epsilon$, we obtain that

$$\begin{aligned} M(\eta, \tau, \eta_1, \tau_1) &\leq C \frac{|\eta||\eta - \eta_1|^{\frac{3}{2} - 6\epsilon}}{\langle \lambda \rangle^{(1-b) - \langle \lambda_1 \rangle^{b+} \langle \lambda_2 \rangle^{b+}}} \\ &\leq C \frac{|\eta|^{\frac{1}{2} + \epsilon} |\eta - \eta_1|^{\frac{1}{2} - 4\epsilon}}{\langle \lambda_1 \rangle^{b+} \langle \lambda_2 \rangle^{b+}} \\ &\leq C \frac{|\eta|^{\frac{1}{2}} |\eta - \eta_1|^{\frac{1}{2}}}{\langle \lambda_1 \rangle^{b+} \langle \lambda_2 \rangle^{b+}}, \end{aligned} \quad (3.32)$$

yielding to

$$\begin{aligned} |m'(\eta - \eta_1) - m'(\eta_1)| &= \left| -\frac{1}{(\eta - \eta_1)^2} + \frac{1}{\eta_1^2} + |\eta - \eta_1| - |\eta_1| + 3((\eta - \eta_1)^2 - \eta_1^2) \right| \\ &= 3|(\eta - \eta_1)^2 - \eta_1^2| \left| 1 + \frac{1}{3(\eta - \eta_1)^2 \eta_1^2} + \frac{1}{3(|\eta - \eta_1| + |\eta_1|)} \right| \\ &\geq C|(\eta - \eta_1)^2 - \eta_1^2| \\ &\geq C|\eta - \eta_1||\eta|. \end{aligned} \quad (3.33)$$

From (3.32), (3.33) and by using the same arguments as before (Plancherel identity, Cauchy Schwarz and Hölder inequalities) together with Lemma 3.10, we get

$$\begin{aligned} I_5 &\leq C \int_{\mathbb{R}^2} \int_{\eta\tau} \frac{F(\eta, \tau) |m'(\eta_1) - m'(\eta - \eta_1)|^{\frac{1}{2}} F_1(\eta_1, \tau_1) F_2(\eta - \eta_1, \tau - \tau_1)}{\langle \lambda \rangle^{(1-b) - \langle \lambda_1 \rangle^{b+} \langle \lambda_2 \rangle^{b+}}} d\eta_1 d\tau_1 d\eta d\tau \\ &\leq \|F(\eta, \tau)\|_{L_{\eta\tau}^2} \left\| \int_{\eta\tau} \frac{|m'(\eta_1) - m'(\eta - \eta_1)|^{\frac{1}{2}} F_1(\eta_1, \tau_1) F_2(\eta - \eta_1, \tau - \tau_1)}{\langle \lambda_1 \rangle^{b+} \langle \lambda_2 \rangle^{b+}} d\eta_1 d\tau_1 \right\|_{L_{\eta\tau}^2} \\ &\leq \|F(\eta, \tau)\|_{L_{\eta\tau}^2} \left\| K^{\frac{1}{2}} \left(\mathcal{F}^{-1} \left(\frac{F_1(\eta_1, \tau_1)}{\langle \lambda_1 \rangle^{b+}} \right), \mathcal{F}^{-1} \left(\frac{F_2(\eta - \eta_1, \tau - \tau_1)}{\langle \lambda_2 \rangle^{b+}} \right) \right) \right\|_{L_{xt}^2} \\ &\leq C \|F\|_{L_{\eta, \tau}^2} \|F_1\|_{L_{\eta, \tau}^2} \|F_2\|_{L_{\eta, \tau}^2}. \end{aligned}$$

2- If (3.15) is valid, we have $|\lambda_2| \geq |\eta||\eta_1|^2$, hence

$$\begin{aligned} M(\eta, \tau, \eta_1, \tau_1) &\leq C \frac{|\eta|^{\frac{1}{2} + \epsilon} |\eta_1|^{\frac{1}{2} - 4\epsilon}}{\langle \lambda \rangle^{b+} \langle \lambda_1 \rangle^{b+}} \\ &\leq C \frac{|\eta_1|}{\langle \lambda \rangle^{b+} \langle \lambda_1 \rangle^{b+}}. \end{aligned} \quad (3.34)$$

Furthermore

$$\begin{aligned} |m'(\eta) - m'(\eta_1)| &= \left| -\frac{1}{\eta^2} + \frac{1}{\eta_1^2} + |\eta| - |\eta_1| + 3(\eta^2 - \eta_1^2) \right| \\ &= 3|\eta^2 - \eta_1^2| \left| 1 + \frac{1}{3\eta^2 \eta_1^2} + \frac{1}{3(|\eta| + |\eta_1|)} \right| \\ &\geq C|\eta^2 - \eta_1^2| \\ &\geq C|\eta_1|^2. \end{aligned} \quad (3.35)$$

Thus from (3.34) and (3.35), we find that

$$M(\eta, \tau, \eta_1, \tau_1) \leq C \frac{|m'(\eta) - m'(\eta_1)|^{\frac{1}{2}}}{\langle \lambda \rangle^{b+} \langle \lambda_1 \rangle^{b+}}. \quad (3.36)$$

3- If (3.16) is valid, we can prove this case as the previous one.

– When $|\eta||\eta_1|^{\frac{1}{2}}|\eta - \eta_1|^{\frac{1}{2}} < 1$, with a treatment similar to (3.32) we have

$$\begin{aligned} M(\eta, \tau, \eta_1, \tau_1) &\leq C \frac{|\eta|^{1+s} |\eta_1|^{-s} |\eta - \eta_1|^{-s}}{\langle \lambda \rangle^{(1-b)-} \langle \lambda_1 \rangle^{b+} \langle \lambda_2 \rangle^{b+}} \\ &\leq C \frac{|\eta - \eta_1|^{-1-2s}}{\langle \lambda \rangle^{(1-b)-} \langle \lambda_1 \rangle^{b+} \langle \lambda_2 \rangle^{b+}} \\ &\leq C \frac{|\eta - \eta_1|^{\frac{1}{2}}}{\langle \lambda \rangle^{(1-b)-} \langle \lambda_1 \rangle^{b+} \langle \lambda_2 \rangle^{b+}} \\ &\leq C \frac{|\eta - \eta_1|}{\langle \lambda \rangle^{(1-b)-} \langle \lambda_1 \rangle^{b+} \langle \lambda_2 \rangle^{b+}} \\ &\leq C \frac{|m'(\eta - \eta_1) - m'(\eta_1)|^{\frac{1}{2}}}{\langle \lambda \rangle^{(1-b)-} \langle \lambda_1 \rangle^{b+} \langle \lambda_2 \rangle^{b+}}, \end{aligned} \quad (3.37)$$

and

$$\begin{aligned} I_5 &\leq C \int_{\mathbb{R}^2} \int_{\eta\tau} \frac{F(\eta, \tau) |m'(\eta_1) - m'(\eta - \eta_1)|^{\frac{1}{2}} F_1(\eta_1, \tau_1) F_2(\eta - \eta_1, \tau - \tau_1)}{\langle \lambda \rangle^{(1-b)-} \langle \lambda_1 \rangle^{b+} \langle \lambda_2 \rangle^{b+}} d\eta_1 d\tau_1 d\eta d\tau \\ &\leq \left\| \frac{F(\eta, \tau)}{\langle \lambda \rangle^{(1-b)-}} \right\|_{L^2_{\eta\tau}} \left\| \int_{\eta\tau} \frac{|m'(\eta_1) - m'(\eta - \eta_1)|^{\frac{1}{2}} F_1(\eta_1, \tau_1) F_2(\eta - \eta_1, \tau - \tau_1)}{\langle \lambda_1 \rangle^{b+} \langle \lambda_2 \rangle^{b+}} d\eta_1 d\tau_1 \right\|_{L^2_{\eta\tau}} \\ &\leq \left\| \frac{F(\eta, \tau)}{\langle \lambda \rangle^{(1-b)-}} \right\|_{L^2_{\eta\tau}} \left\| K^{\frac{1}{2}} \left(\mathcal{F}^{-1} \left(\frac{F_1(\eta_1, \tau_1)}{\langle \lambda_1 \rangle^{b+}} \right), \mathcal{F}^{-1} \left(\frac{F_2(\eta - \eta_1, \tau - \tau_1)}{\langle \lambda_2 \rangle^{b+}} \right) \right) \right\|_{L^2_{xt}} \\ &\leq C \|F\|_{L^2_{\eta, \tau}} \|F_1\|_{L^2_{\eta, \tau}} \|F_2\|_{L^2_{\eta, \tau}}. \end{aligned}$$

□

This ends the proof of our first result stated in Theorem 3.1.

3.3 Proof of the second result

In this section we demonstrate the result stated in Theorem 3.2.

Proof. For $u_0 \in H^s$ ($s > -\frac{3}{4}$), we define the set

$$\mathcal{S} = \{u \in X_{s,b} : \|u\|_{X_{s,b}} \leq 2C \|u_0\|_{H^s}\},$$

and the operator

$$\Theta_{u_0}(u) = \Theta(u) = \varphi(t)S(t)u_0 - \frac{1}{2}\varphi_\rho(t) \int_0^t S(t-t')(\partial_x u^2)(t')dt.$$

For the fixed point theorem, we need to prove that Θ is a contraction mapping on $\mathcal{S}$. First, we will show that $\Theta(\mathcal{S}) \subset \mathcal{S}$. Using Theorem 3.1, Lemma 3.4 and Lemma 3.5, we conclude that

$$\|\Theta(u)\|_{X_{s,b}} \leq C\|u_0\|_{H^s} + C\rho^{b'-b}\|u\|_{X_{s,b}}^2 \leq C\|u_0\|_{H^s} + C\rho^{b'-b}\|u\|_{H^s}\|u\|_{X_{s,b}}.$$

Thus, if we fix ρ for which $C\rho^{b'-b}\|u\|_{H^s} \leq \frac{1}{2}$, then we obtain $\Theta(\mathcal{S}) \subset \mathcal{S}$.

For $u, v \in \mathcal{S}$, a similar argument as above, we have that

$$\|\Theta(u) - \Theta(v)\|_{X_{s,b}} \leq C\rho^{b'-b}(\|u\|_{H^s} + \|v\|_{H^s})\|u - v\|_{X_{s,b}} \leq \frac{1}{2}\|u - v\|_{X_{s,b}}.$$

Therefore, Θ defines a contraction on $\mathcal{S}$ with $C = \frac{1}{2}$. Hence we can apply the fixed point theorem for $T < \frac{\rho}{2}$ to show that the Cauchy problem derived from (1.1) has a unique solution. $\square$

4 Conclusion

In this research paper, we were principally interested in checking over the well-posedness of the derived Cauchy problem that is related to the Rotational-Modified Benjamin equation. As shown in Theorem 3.2 that the RMB equation is locally well-posed in H^s for $s > -\frac{3}{4}$ which means that we get the same qualitative results for the local well-posedness of the standard Benjamin equation, i.e. RMB without the rotation term. Therefore it is essential to point out that the addition of rotational forces does not effectively affect the resulting Cauchy problem of the RMB. Intuitively we expect similar results for $s \leq -\frac{3}{4}$ and this is under consideration. Furthermore it is important and crucial to reveal that the global results can also be proved in L^2 due to the fact that RMB equation has energy as conserved quantity, see (Durán, 2018) and (Nouri, 2019).

Acknowledgment

Miss Farida Bousbia is grateful to the MESRS-Algeria for her Phd financial support.

References

- Angulo, J. 1999. Existence and stability of solitary wave solution of the benjamin equation, *J. Differential Equations* **152**: 136–159.
- Benjamin, T. B. 1967. Internal waves of permanent form in fluids of great depth, *J. Fluid. Mech* **29**: 559–592.
- Chen, H. and Bona, J. L. 1998. Existence and asymptotic properties of solitary-wave solutions of the benjamin-type equations, *Adv. Diff. Eqns* **3**: 51–84.
- Chen, W. and Xiao, J. 2010. A sharp bilinear estimate for the bourgain-type space with application to the benjamin equation, *Comm. in Part. Diff. Eqns* **13**: 1739–1762.

- Durán, A. 2018. On a model for internal waves in rotating fluids, *Appl Math Nonlinear Sci* **3**: 627–648.
- Fokas, A. S. and Ablowitz, M. J. 1983. The inverse scattering transform for the benjamin ono equation: a pivot to multidimensional problems, *Stud. Appl. Math* **68**: 1–10.
- Guo, B. and Huo, Z. 2004. The well-posedness of the korteweg-de vries-benjamin-ono equation, *J. Math. Anal. Appl* **295**: 444–458.
- Ionescu, A. D. and Kenig, C. E. 2007. Global well-posedness of the benjamin-ono equation in low regularity spaces, *J. Amer. Math. Soc* **20**: 753–798.
- Isaza, P. and Mejia, J. 2006. Cauchy problem for the ostrovsky equation in spaces of low regularity, *J. Differential Equations* **230**: 661–681.
- Isaza, P. and Mejia, J. 2007. Global cauchy problem for the ostrovsky equation, *Nonlinear Anal. TMA* **67**: 1482–1503.
- Kenig, C. E. and Koenig, K. D. 2003. On the local well-posedness of the benjamin ono and modified benjamin-ono equations, *Math. Res. Lett* **10**: 879–895.
- Kenig, C. E. and Martel, Y. 2009. Asymptotic stability of solitons for the benjamin-ono equation, *Revista Matematica Iberoamericana* **25**: 909–970.
- Kishimoto, N. 2009. Well-posedness of the cauchy problem for the korteweg-de vries equation at the critical regularity, *Diff. Int. Eqns* **22**: 447–464.
- Linares, F. and Milanes, A. 2004. A note on solutions to a model for long internal waves in a rotating, *Mat. Contemp* **27**.
- Linares, F. and Scialom, M. 2005. On generalized benjamin type equations, *Discrete and Continuous Dynamical Systems* **12**: 161–174.
- Maouni, M. and Nouri, F. Z. 2013. Image restoration based on p-gradient mode, *Int. J. of App. Maths and Statistics* **41**: 48–57.
- Nouri, F. Z. 2019. A new approach for a mutiphase flow problem, *Int. J. of Maths and Computation* **30**: 321–42.
- Tsugawa, K. 2009. Well-posedness and weak rotation limit for the ostrovsky equation, *J. Differential Equations* **247**: 3163–3180.
- Wang, H. and Cui, S. 2007. Well-posedness of the cauchy problem of ostrovsky equation in anisotropic sobolev spaces, *J. Math. Anal. Appl* **327**: 88–100.
- Wang, J. and Yan, W. 2018. The cauchy problem for quadratic and cubic ostrovsky equation with negative dispersion, *Nonlinear Anal. Real World Applications* **43**: 283–307.
- Yan, W., Li, Y. S., Huang, J. H. and Duan, J. 2018. The cauchy problem for the ostrovsky equation with positive dispersion in $h^{-3/4}(r)$, *Nonlinear Differ. Equ. Appl* **25**: 1–37.