

On the inviscid limit of the diffusive 3D periodic Burgers equations in Sobolev spaces

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ABSTRACT

We prove local in time well-posedness of solution to the viscous and non-viscous three dimensional periodic Burgers equations. We establish convergence of the viscous solution to the non-viscous one and give its rate, as the viscosity vanishes. Such convergence is uniform in time with respect to viscosity. Moreover, we present some ideas that may help interested researchers to ameliorate this convergence result. We use energy methods in large indexes Sobolev spaces.

Keywords: Burgers equations, viscous and non-viscous solution, existence and uniqueness, vanishing viscosity, convergence and rate of convergence.

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1 Introduction

In this article, we study the asymptotic behavior at small viscosity of the solutions to the three-dimensional periodic diffusive Burgers equations

$$\begin{cases} \frac{\partial u^\nu}{\partial t} - \nu \Delta u^\nu + (u^\nu \cdot \nabla) u^\nu = 0, & (t, x) \in (0, T) \times \mathbb{T}^3 \\ u^\nu(t=0) = u_0, \end{cases} \quad (1.1)$$

where $\nu > 0$ is the kinematic viscosity, u_0 the initial data, $\mathbb{T}^3 = \mathbb{R}^3/2\pi\mathbb{Z}^3$ is the torus and $u^\nu(t, x) = (u_1^\nu, u_2^\nu, u_3^\nu)(t, x)$ is the unknown velocity. The non-viscous system corresponding to (1.1) is

$$\begin{cases} \frac{\partial u}{\partial t} + (u \cdot \nabla) u = 0, & (t, x) \in (0, T) \times \mathbb{T}^3 \\ u(0) = u_0. \end{cases} \quad (1.2)$$

We start by giving existence of a unique regular solution to viscous and non-viscous Burgers equations in Sobolev spaces. The time of existence is uniform with respect to the viscosity.

As for Euler equation, to obtain existence and uniqueness result to the non-viscous Burgers equations, we are constraint to use large Sobolev indexes. Then, we prove convergence of the viscous solution to the non-viscous one, and give the corresponding rate of convergence which is uniform in time. These are new results dealing with 3D Burgers equations. Moreover, we present some ideas that may help interested researchers to ameliorate our results, using a parallel mathematical analysis to the one done for Navier-Stokes and Euler equations.

As a physical motivation, three-dimensional Burgers equations are known in cosmology as the Zeldovich approximation (Zeldovich, 1970) and as an adhesion model (Gurbatov and Saichev, 1984). The non-viscous Burgers equations are perhaps the simplest equations that model the nonlinear phenomena in a force free mass transfer, as said in (Alverio and Rozanova, 2009).

Mathematically, Burgers equations are the incompressible Navier-Stokes equations, without the divergence free condition and pressure. For this fact, we perform some parallel mathematical analysis as in (Pooley and Robinson, 2016, Selmi and Chaabani, 2019), where authors proved existence and uniqueness of global in time solution to the viscous Burgers equations, respectively in Sobolev spaces and in critical Gevrey-Sobolev spaces.

About the three-dimensional incompressible Euler equations, the standard energy method gives local wellposedness for initial data in $H^s(\mathbb{T}^3)$, $s > s_c := \frac{3}{2} + 1$. The borderline case s_c is still an open question, see (Bourgain and Li, 2015) for a complete survey. For the non-viscous Burgers equations, we prove the same in Theorem 1. About convergence, the inviscid limit to the Navier-Stokes equations in the whole space was investigated by many authors. Among them, we cite (Constantin, 1986), (Kato, 1972), (Kato, 1975) and (Swann, 1971). In all these references, only convergence in $H^{s'}$, $s' < s$ was proved. Except in (Kato, 1975), the case $s = s'$ was investigated, but on short existence time only. Later, in (Masmoudi, 2007), Masmoudi improved the above results by proving the convergence as long as the solution of the Euler equations exists, that is $H^{s'}$ for $s' = s$. In this paper, because of the condition $\nabla u \in L^\infty([0, T]; H^s(\mathbb{T}^3))$ in Theorem 2, our convergence result belongs to those in $H^{s'}$, $s' < s$. However, we guess that convergence in the space of existence itself can be performed. To do so, one has to manage an extra term that appears while taking the scalar product in H^s . This term was cancelled out in the case of Navier-Stokes equations thanks to the divergence free condition. About convergence rate of viscous solution to the non-viscous one, we established a rate of convergence which is uniform in time.

We define for $s > 0$ the real Sobolev spaces $H^s(\mathbb{T}^3)$ as follows. Let $u \in H^s(\mathbb{T}^3)$ has the Fourier series

$$u = \sum_{k \in \mathbb{Z}^3} \hat{u}_k e^{ikx} \in H^s(\mathbb{T}^3),$$

then,

$$\|u\|_{H^s(\mathbb{T}^3)} = \left(\sum_{k \in \mathbb{Z}^3} (1 + |k|^2)^s |\hat{u}_k|^2 \right)^{1/2} < \infty.$$

The following preliminary results will be useful throughout this paper.

Lemma 1.1. *For $s > d/2$, and $f \in H^s(\mathbb{T}^d)$, there exists $C_s > 0$ such that*

$$\|f\|_{L^\infty(\mathbb{T}^d)} \leq C_s \|f\|_{H^s(\mathbb{T}^d)}.$$

In other words, $H^s \subseteq L^\infty$ for $s > d/2$, and there is a bound on the inclusion operator $i : H^s \rightarrow L^\infty$.

Proof. Let $f \in H^s(\mathbb{T}^d)$, we have

$$f(x) = \sum_{k \in \mathbb{Z}^d} \widehat{f}_k e^{ikx}.$$

Then, $\forall x \in \mathbb{T}^d$

$$\begin{aligned} |f(x)| &\leq \sum_{k \in \mathbb{Z}^d} |\widehat{f}_k| |e^{ikx}| \\ &\leq \sum_{k \in \mathbb{Z}^d} \frac{1}{(1 + |k|^2)^{s/2}} (1 + |k|^2)^{s/2} |\widehat{f}_k|. \end{aligned}$$

Using the Cauchy-Schwarz inequality, we obtain

$$|f(x)| \leq \left(\sum_{k \in \mathbb{Z}^d} \frac{1}{(1 + |k|^2)^s} \right)^{1/2} \left(\sum_{k \in \mathbb{Z}^d} (1 + |k|^2)^s |\widehat{f}_k|^2 \right)^{1/2}. \quad (1.3)$$

By one hand, by definition of the H^s norm, for any reel number s , we have

$$\left(\sum_{k \in \mathbb{Z}^d} (1 + |k|^2)^s |\widehat{f}_k|^2 \right)^{1/2} = \|f\|_{H^s(\mathbb{T}^d)}. \quad (1.4)$$

By the other hand, the series

$$\sum_{k \in \mathbb{Z}^d} \frac{1}{(1 + |k|^2)^s} \quad (1.5)$$

converges. In fact, as k approaches ∞ ,

$$\frac{1}{(1 + |k|^2)^s} \simeq \frac{1}{|k|^{2s}},$$

which is a Riemann series that converges when $k \rightarrow \infty$ if $2s > d$, that is $s > d/2$, where d is the dimension. The same holds for $-\infty$ because we have $|k|$. Hence, if $s > d/2$, the quantity $\left(\sum_{k \in \mathbb{Z}^d} \frac{1}{(1 + |k|^2)^s} \right)^{1/2}$ is finite and depends on the sobolev indexes s ; it is equal to C_s .

Thus, $\forall x \in \mathbb{T}^d$

$$|f(x)| \leq C_s \|f\|_{H^s(\mathbb{T}^d)}.$$

In particular, this holds for the essential supremum that is

$$\|f\|_{L^\infty(\mathbb{T}^d)} \leq C_s \|f\|_{H^s(\mathbb{T}^d)}.$$

Note that

$$\|f\|_\infty = \sup_k |f_k|.$$

□

Lemma 1.2. If $s > d/2$, $H^s(\mathbb{T}^d)$ is a Banach algebra space:

$$\exists C > 0, \forall u, v \in H^s(\mathbb{T}^d), uv \in H^s(\mathbb{T}^d)$$

and

$$\|uv\|_{H^s(\mathbb{T}^d)} \leq C \|u\|_{H^s(\mathbb{T}^d)} \|v\|_{H^s(\mathbb{T}^d)}.$$

Using the result of Lemma 1.1 and Lemma 1.2, we deduce the following inequality which will be useful later.

Lemma 1.3. *For $s > 3/2 + 1$, and $u \in H^s(\mathbb{T}^3)$, there exists $\kappa > 0$ such that*

$$|(u \nabla u, u)_{H^s}| \leq \kappa \|u\|_{H^s}^3.$$

Proof. By Cauchy-Schwarz inequality, we have

$$|(u \nabla u, u)_{H^s}| \leq \|u \nabla u\|_{H^s} \|u\|_{H^s}.$$

Since

$$\begin{aligned} \|u \nabla u\|_{H^s} &= \left(\sum_{k=-\infty}^{k=\infty} |\widehat{u \nabla u}(k)|^2 (1 + |k|^2)^s \right)^{1/2} \\ &\leq \sup_{k \in \mathbb{Z}^3} |\widehat{\nabla u}(k)| \left(\sum_{k=-\infty}^{k=\infty} |\hat{u}(k)|^2 (1 + |k|^2)^s \right)^{1/2} \\ &= \|\nabla u\|_{L^\infty} \|u\|_{H^s}, \end{aligned}$$

then we have $u \in H^s(\mathbb{T}^3)$, for $s > 3/2 + 1$. Thus, $\nabla u \in H^{s-1}(\mathbb{T}^3)$, where $s - 1 > 3/2$. By using Lemma 1.1 above, we obtain

$$\|\nabla u^\nu\|_{L^\infty} \leq C \|\nabla u^\nu\|_{H^{s-1}}.$$

Finally, we use

$$\|\nabla u^\nu\|_{H^{s-1}} \leq \|u^\nu\|_{H^s}.$$

□

2 Uniform well-posedness

Following, we prove existence and uniqueness of solution to (1.1) and (1.2).

Theorem 2.1. *Let $u_0 \in H^s$, $s > 3/2 + 1$. There exists $T^* = T^*(\|u_0\|_{H^s}) > 0$, such that for any $T < T^*$, there exists a unique solution u^ν to (1.1) and a unique solution u to (1.2) such that $u^\nu, u \in C^1([0, T], H^s(\mathbb{T}^3))$.*

Proof. To show the existence and the regularity of strong solutions u^ν , we first approximate equation (1.1) by an approximating system using any approximation procedure such as Friedrichs method for example, as in (Benamur and Selmi, 2008) and (Selmi, 2012). Then, we apply the classical theory of ordinary differential equations to find the approximate solutions. Next, by conservation laws, we get uniform estimates on the approximate solutions. Finally, we pass to the limit in the approximate equation.

Let $T_n^*(\nu)$ be the maximal existence time to the approximate system. taking the scalar product in H^s , we obtain, for all $t \in [0, T_n^*(\nu)]$,

$$\begin{aligned} \|u_n^\nu(t)\|_{H^s(\mathbb{T}^3)}^2 &+ 2\nu \int_0^t \|\nabla u_n^\nu(\tau)\|_{H^s(\mathbb{T}^3)}^2 d\tau \leq \|u_0\|_{H^s(\mathbb{T}^3)}^2 + \int_0^t |(u_n^\nu \nabla u_n^\nu, u_n^\nu)_{H^s(\mathbb{T}^3)}| d\tau \\ &\leq \|u_0\|_{H^s(\mathbb{T}^3)}^2 + C \int_0^t \|\nabla u_n^\nu(\tau)\|_{L^\infty(\mathbb{T}^3)} \|u_n^\nu(\tau)\|_{H^s(\mathbb{T}^3)}^2 d\tau \\ &\leq \|u_0\|_{H^s(\mathbb{T}^3)}^2 + C \int_0^t \|\nabla u_n^\nu(\tau)\|_{H^{s-1}(\mathbb{T}^3)} \|u_n^\nu(\tau)\|_{H^s(\mathbb{T}^3)}^2 d\tau. \end{aligned}$$

It holds that

$$\|u_n^\nu(t)\|_{H^s(\mathbb{T}^3)}^2 + 2\nu \int_0^t \|\nabla u_n^\nu(\tau)\|_{H^s(\mathbb{T}^3)}^2 d\tau \leq \|u_0\|_{H^s(\mathbb{T}^3)}^2 + C \int_0^t \|u_n^\nu(\tau)\|_{H^s(\mathbb{T}^3)}^3 d\tau. \quad (2.1)$$

Here, as $s - 1 > 3/2$, we have $\|\nabla u^\nu\|_{L^\infty} \leq \|\nabla u^\nu\|_{H^{s-1}} \leq c \|u^\nu\|_{H^s}$. Set

$$T(n, \nu) := \sup \left\{ 0 \leq t < T_n^*(\nu); \forall \tau \in [0, t], \|u_n^\nu(\tau)\|_{H^s(\mathbb{T}^3)} \leq 2\|u_0\|_{H^s(\mathbb{T}^3)} \right\}.$$

Using (2.1) and Gronwall's lemma, we obtain for all $t \in [0, T(n, \nu))$,

$$\|u_n^\nu(t)\|_{H^s(\mathbb{T}^3)}^2 \leq \|u_0\|_{H^s(\mathbb{T}^3)}^2 \exp(2Ct\|u_0\|_{H^s(\mathbb{T}^3)}). \quad (2.2)$$

Thus,

$$T(n, \nu) > T^* := \frac{1}{C\|u_0\|_{H^s(\mathbb{T}^3)}} > 0. \quad (2.3)$$

Moreover, for all $t \in [0, T]$,

$$\|u_n^\nu(t)\|_{H^s(\mathbb{T}^3)}^2 + 2\nu \int_0^t \|\nabla u_n^\nu(\tau)\|_{H^s(\mathbb{T}^3)}^2 d\tau \leq 2\|u_0\|_{H^s(\mathbb{T}^3)}^2. \quad (2.4)$$

Using Ascoli's theorem, the Cantor diagonal process, and the estimate (2.4), for n tends to infinity, we obtain a solution that satisfies the following estimate:

$$\|u^\nu(t)\|_{H^s(\mathbb{T}^3)}^2 + 2\nu \int_0^t \|\nabla u^\nu(\tau)\|_{H^s(\mathbb{T}^3)}^2 d\tau \leq 2\|u_0\|_{H^s(\mathbb{T}^3)}^2, \quad \forall t \in [0, T]. \quad (2.5)$$

This regularity implies in a standard way the uniqueness. By (2.3), it is clear that T^* is independent of the viscosity ν . Moreover, all of the above estimates are independent of ν , so we let $\nu = 0$ in the approximating system, and proceed exactly as above to obtain similar results for the a priori estimate about solution u to equation (1.2). $\square$

Remark 2.1. The uniform time T is actually depending on s and $\|u_0\|_{H^s}$. This is because C is a constant depending on s and created by Leibniz formula and Sobolev embedding inequality, when estimating the nonlinear term. We have no idea if this time possibly vanishes as s goes to infinity, (this can be investigated). However, for the index s less then a given s_0 , we are certain that $T = T(s)$ is positive.

3 Convergence results

In this section, we prove the convergence result. We determine the convergence rate as a function of the viscosity ν .

Theorem 3.1. For $s > 3/2 + 1$, there exist a time $T^* > 0$, and a constant $C > 0$ depending on the initial data u_0 and T^* but not on ν , such that if $\nabla u \in L^\infty([0, T]; H^s(\mathbb{T}^3))$, the solutions u^ν and u satisfy for all $0 < t < T^*$,

$$\|u^\nu(t, \cdot) - u(t, \cdot)\|_{H^s(\mathbb{T}^3)} \leq C\sqrt{\nu}. \quad (3.1)$$

Proof. The difference $w^\nu = u^\nu - u$ satisfies, for all $(t, x) \in (0, T) \times \mathbb{T}^3$,

$$\begin{cases} \frac{\partial w^\nu}{\partial t} - \nu \Delta w^\nu + (u^\nu \cdot \nabla) w^\nu + (w^\nu \cdot \nabla) u = \nu \Delta u \\ w^\nu(0, x) = 0. \end{cases} \quad (3.2)$$

Taking the scalar product in H^s , we obtain

$$\frac{1}{2} \frac{d}{dt} \|w^\nu\|_{H^s(\mathbb{T}^3)}^2 + \nu \|\nabla w^\nu\|_{H^s(\mathbb{T}^3)}^2 = J_0 + J_1 + J_2, \quad (3.3)$$

where

$$\begin{aligned} J_0 &= -((u^\nu \cdot \nabla) w^\nu, w^\nu)_{H^s(\mathbb{T}^3)} \\ J_1 &= -((w^\nu \cdot \nabla) u, w^\nu)_{H^s(\mathbb{T}^3)} \\ J_2 &= \nu (\Delta u, w^\nu)_{H^s(\mathbb{T}^3)}. \end{aligned}$$

To estimate J_0 , we note that

$$\begin{aligned} |J_0| &\leq \|u^\nu(t)\|_{H^s(\mathbb{T}^3)} \|\nabla w^\nu(t)\|_{L^\infty(\mathbb{T}^3)} \|w^\nu(t)\|_{H^s(\mathbb{T}^3)} \\ &\leq \|u^\nu(t)\|_{H^s(\mathbb{T}^3)} \|\nabla w^\nu(t)\|_{H^{s-1}(\mathbb{T}^3)} \|w^\nu(t)\|_{H^s(\mathbb{T}^3)} \\ &\leq \|u^\nu(t)\|_{H^s(\mathbb{T}^3)} \|w^\nu(t)\|_{H^s(\mathbb{T}^3)}^2. \end{aligned}$$

Since $u^\nu \in C^1([0, T], H^s(\mathbb{T}^3))$, we have $\|u^\nu(t)\|_{H^s(\mathbb{T}^3)} \leq \kappa_1$, where κ_1 is a constant that may depend on u_0 . Thus,

$$|J_0| \leq \kappa_1 \|w^\nu(t)\|_{H^s(\mathbb{T}^3)}^2. \quad (3.4)$$

For J_1 , we have

$$\begin{aligned} |J_1| &\leq \|\nabla u(t)\|_{L^\infty(\mathbb{T}^3)} \|w^\nu(t)\|_{H^s(\mathbb{T}^3)}^2 \\ &\leq \|\nabla u(t)\|_{H^{s-1}(\mathbb{T}^3)} \|w^\nu(t)\|_{H^s(\mathbb{T}^3)}^2 \\ &\leq \|u(t)\|_{H^s(\mathbb{T}^3)} \|w^\nu(t)\|_{H^s(\mathbb{T}^3)}^2. \end{aligned}$$

As above, there exists κ_2 , such that $\|u(t)\|_{H^s(\mathbb{T}^3)} \leq \kappa_2$. Hence,

$$|J_1| \leq \kappa_2 \|w^\nu(t)\|_{H^s(\mathbb{T}^3)}^2. \quad (3.5)$$

Now, we turn to J_2 . Since $s > 3/2 + 1$, we have

$$\begin{aligned} |J_2| &= \nu |(\nabla u(t, x), \nabla w^\nu(t, x))_{H^s(\mathbb{T}^3)}| \\ &\leq \nu \|\nabla u(t)\|_{H^s(\mathbb{T}^3)} \|\nabla w^\nu(t)\|_{H^s(\mathbb{T}^3)} \\ &\leq \frac{\nu}{2} \|\nabla u(t)\|_{H^s(\mathbb{T}^3)}^2 + \frac{\nu}{2} \|\nabla w^\nu(t)\|_{H^s(\mathbb{T}^3)}^2. \end{aligned}$$

Since, by hypothesis of Theorem 3.1, $\nabla u \in L^\infty([0, T]; H^s(\mathbb{T}^3))$, there exists a constant κ_3 , such that $\|\nabla u(t)\|_{H^s(\mathbb{T}^3)}^2 \leq \kappa_3$, $\forall t \in [0, T]$. Then, we deduce

$$|J_2| \leq \kappa_3 \frac{\nu}{2} + \frac{\nu}{2} \|\nabla w^\nu(t)\|_{H^s(\mathbb{T}^3)}^2. \quad (3.6)$$

Let $\kappa = \sup(\kappa_1, \kappa_2, \kappa_3)$, by (3.4), (3.5) and (3.6), we obtain

$$\frac{1}{2} \frac{d}{dt} \|w^\nu\|_{H^s(\mathbb{T}^3)}^2 + \frac{\nu}{2} \|\nabla w^\nu\|_{H^s(\mathbb{T}^3)}^2 \leq \kappa \|w^\nu(t)\|_{H^s(\mathbb{T}^3)}^2 + \kappa \frac{\nu}{2}.$$

It follows that

$$\frac{d}{dt} \|w^\nu(t)\|_{H^s(\mathbb{T}^3)}^2 \leq 2\kappa \|w^\nu(t)\|_{H^s(\mathbb{T}^3)}^2 + \kappa \nu. \quad (3.7)$$

Since $w^\nu(0) = 0$, applying Gronwall's inequality to (3.7), we obtain

$$\|w^\nu(t)\|_{H^s(\mathbb{T}^3)} \leq C\sqrt{\nu}. \quad (3.8)$$

□

4 Conclusion

In this paper, we proved the existence of a unique three-dimensional regular solution to both of the viscous and non-viscous Burgers equations, in the case of periodic boundary conditions and initial data in $H^s(\mathbb{T}^3)$ with $s > s_c : 3/2 + 1$. Also, we proved that the viscous solution converges to the non-viscous solution as the viscosity vanishes and we gave the rate of convergence. Our result can be improved by proving that convergence occurs as long as the solution of the non-viscous Burger equation exists, as (Masmoudi, 2007) did for Navier-Stokes and Euler equations. This is still an open problem for the Burgers equations.

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