

A New Distribution of Combined Parallel System Which Contains Components with Bivariate Geometric Distribution

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ABSTRACT

This paper introduces a new distribution of combined parallel system which contains components with geometric distribution. Let the structure of a system includes two components which combined parallel and there are random component additions to these two core components. Suppose that component additions are distributed with bivariate geometric distribution. This situation gives a special system structure and distribution of this system is studied.

Keywords: Bivariate geometric distribution; exponentially distributed components; reliability analyses; survival function.

Mathematics Subject Classification: 62E10, 62N05

1. INTRODUCTION

In engineering, especially in system engineering, reliability and lifetimes of complicated systems have vital importance. The aim of this study is screening reliability characteristics of a combined parallel system which contains components with geometric distribution.

There are many different researches for gaining statistical distributions (Mahmoud et.al 2017). Especially in lifetime data sets new models with big efficiency suggested by researchers (Shanker and Shukla 2018). While gaining new distributions some different base distributions have been used (Hemeda and Hamoda 2018). Besides these valuable studies some other techniques have also been used in gaining new distributions (Shah et.al 2020).

1.1 Structure of System

The structure of our system includes two components which combined parallel and there are random m and n component additions to these two core components. Component additions are binary

geometrically distributed and components adding to upper part with probability p_1 , components adding to lower part with probability p_2 , and no components adding to upper or lower part with probability p_0 .

Suppose, random number of components parallel combined to upper part is M , random number of components parallel combined to second part is N , then common probability density function is,

$$Pr(N = n, M = m) = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \binom{m+n}{n} p_1^n p_2^m p_0, 0 < p_0, p_1, p_2 < 1, p_0 + p_1 + p_2 = 1$$

and $n, m = 0, 1, 2, \dots$

We assume that core components and adding components have independent lifetimes and with $1/\beta$ mean exponential distribution. X and Y are random variables which show core components lifetimes, $V_1, V_2, \dots, V_N$ and $Z_1, Z_2, \dots, Z_M$ show lifetimes of adding components with order that first and second core components, the probability density function of these components is,

$$f(t) = \beta e^{-t\beta}, t > 0, \beta > 0.$$

2. METARIAL AND METHODS

2.1. Binary Geometric Distribution

In a study binary geometric distribution examined and the formulation of binary geometric distribution is as follow (Jayakumar and Mundassery 2007).

$$Pr(N = n, M = m) = \sum \binom{m+n}{n} p_1^n p_2^m p_0 \quad p_0 + p_1 + p_2 = 1 \quad n, m = 0, 1, 2, \dots$$

p_0 : probability of no component adding,

p_1 : probability of coming component in upper part,

p_2 : probability of coming component in lower part.

For deciding the number of adding components in system we need marginal distribution functions. We reach these marginals as follow.

$$Pr(N = n) = \sum_{m=0}^{\infty} \frac{(m+n)!}{m!n!} p_1^n p_2^m p_0 = \frac{p_1^n p_0}{(1-p_2)^{n+1}} \sum_{m=0}^{\infty} \frac{(m+n)!}{m!n!} p_2^m (1-p_2)^{n+1}$$

With using negative binomial distribution the sum is 1. So marginal distribution for n is in equation (1).

$$Pr(N = n) = \frac{p_1^n p_0}{(1-p_2)^{n+1}} \quad (1)$$

We gain geometric distribution with success rate $\frac{p_0}{p_0+p_1}$. We can obtain marginal for m with the same way like equation (2).

$$Pr(M = m) = \frac{p_2^m p_0}{(1 - p_1)^{m+1}} \quad (2)$$

We emerged with geometric distribution with $\frac{p_0}{p_0 + p_2}$ success rate. As coming of components dependant, we need conditional probability function. At first we decide n for the coming component number for the upper part. After this we decide m for the coming component number for the lower part, with n is given condition (Yılmaz and Ünözkan 2015).

$$Pr(M = m | N = n) = \frac{(m + n)!}{m! n!} p_2^m (1 - p_2)^{n+1}$$

Now we meet negative binomial distribution with unsuccessful try number is m , success number is $n + 1$.

After now firstly the number is generated from $Pr(N = n)$, later the other number is generated from $Pr(M = m | N = n)$, so (n, m) number group gained.

We claimed that every component had exponential distribution with $1/\beta$ mean before. So we generate $m + n$ number exponential distribution lifetime with $1/\beta$ mean. After that we choose maksimum of maksimums for system lifetime.

$$T_{SYSTEM} = \max\{\max\{X, T_1, T_2\}, \max\{Y, Z_1\}\}$$

3. RESULTS

3.1. Lifetime of System

After M and N number of components adding to system, if lifetime of this system shows with T_{SYSTEM}

than lifetime can be defined as,

$$T_{SYSTEM} = \max\{\max\{X, V_1, V_2, \dots, V_N\}, \max\{Y, Z_1, Z_2, \dots, Z_M\}\}$$

Once the reliability of every components in t time figured out as $R(t)$, and $N=n$, $M=m$ observed, the reliability of our system is as below,

$$Pr(T_{SYSTEM} > t | N = n, M = m) = 1 - (1 - R(t))^{n+1} (1 - R(t))^{m+1}$$

For more simple form, we assume $1 - R(t) = Q(t)$. When $S(t)$ shows survival function, we gain this function as follows.

$$S(t) = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} Pr(T_{SYSTEM} > t | N = n, M = m) Pr(N = n, M = m)$$

$$= \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \binom{m+n}{n} p_1^n p_2^m p_0 (1 - Q(t)^{m+n+2})$$

Now we continue with some specialties of geometric distribution.

$$\begin{aligned} S(t) &= \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \binom{m+n}{n} p_1^n p_2^m p_0 - \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \binom{m+n}{n} p_1^n p_2^m p_0 Q(t)^{m+n+2} \\ &= 1 - \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \binom{m+n}{n} (Q(t)p_1)^n (Q(t)p_2)^m p_0 Q(t)^2 \\ &= 1 - Q(t)^2 p_0 \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \binom{m+n}{n} (Q(t)p_1)^n (Q(t)p_2)^m \frac{(1 - Q(t)(p_1 + p_2))}{(1 - Q(t)(p_1 + p_2))} \\ &= 1 - \frac{Q(t)^2 p_0}{(1 - Q(t)(p_1 + p_2))} \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \binom{m+n}{n} (Q(t)p_1)^n (Q(t)p_2)^m (1 - Q(t)(p_1 + p_2)) \\ &= 1 - \frac{Q(t)^2 p_0}{(1 - Q(t)(p_1 + p_2))} \end{aligned} \quad (3)$$

If we assume that $Q(t) = 1 - e^{-t\beta}$ as exponential distribution function, we obtain graphs of survival

function as below. In Figure 1:

$\beta = 0.5, p_1 = p_2 = 0.49$ (blue), $p_1 = p_2 = 0.09$ (red), $p_1 = p_2 = 0.009$ (black), $p_1 = 0.49$ and

$p_2 = 0.009$ (green)

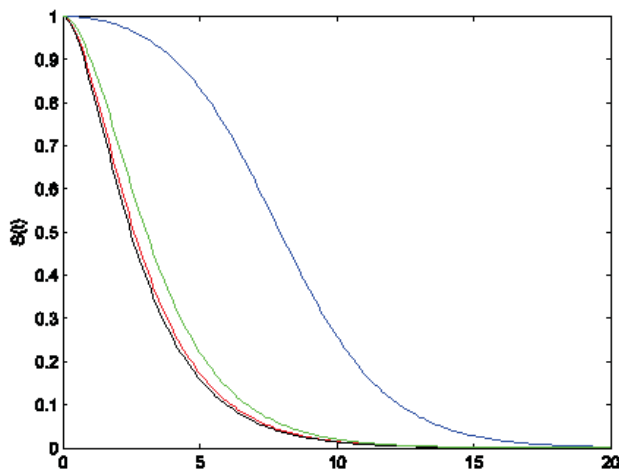


Figure 1 Graph of survival function of system lifetime

When we examine figure 1, once the value of p_0 decreases, which means the probability of adding new components increases, the value of survival function increases, so the graph moves to right side and the lifetime of the system becomes longer.

3.2. The Lifetime Distribution Function of System

Distribution function can be gained easily by using equality (1)

$$F(t) = 1 - S(t) = \frac{Q(t)^2 p_0}{(1 - Q(t)(p_1 + p_2))} \quad (4)$$

3.3. Probability Density Function of System Lifetime

We obtain probability density function (pdf) as below by using equality (4).

$$f(t) = \frac{d}{dt} F(t) = \frac{d}{dt} \left(\frac{Q(t)^2 p_0}{(1 - Q(t)(p_1 + p_2))} \right) = \frac{p_0 Q(t) Q'(t) (2 - Q(t)(p_1 + p_2))}{(1 - Q(t)(p_1 + p_2))^2} \quad (5)$$

The graph of pdf is as below. In Figure 2:

$\beta = 0.005, p_1 = p_2 = 0.49$ (blue), $p_1 = p_2 = 0.09$ (red), $p_1 = p_2 = 0.009$ (black), $p_1 = 0.49$

$p_2 = 0.009$ (green)

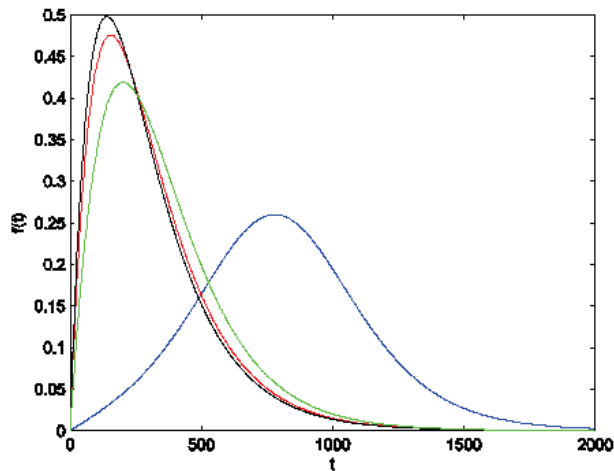


Figure 2 Graph of Probability Density Function of System Lifetime

Once we examine figure 2, we reach that when the value of p_0 decreases, the shape of the graph seems like normal distribution curve. The other parameter β is scale parameter.

3.4. Hazard Rate of System Lifetime

Hazard rate of system lifetime can be obtained via using equalities (1) and (3).

$$r(t) = \frac{f(t)}{S(t)} = \left(\frac{\frac{p_0 Q(t) Q'(t) (2 - Q(t)(p_1 + p_2))}{(1 - Q(t)(p_1 + p_2))^2}}{1 - \frac{Q(t)^2 p_0}{(1 - Q(t)(p_1 + p_2))}} \right) = \frac{p_0 Q(t) Q'(t) (2 - Q(t)(p_1 + p_2))}{(1 - Q(t)(p_1 + p_2))(1 - Q(t)(p_1 + p_2) - Q(t)^2 p_0)}$$

The graph of this hazard rate is as follows. In Figure 3:

$\beta = 0.5, p_1 = p_2 = 0.49$ (blue), $p_1 = p_2 = 0.09$ (red), $p_1 = p_2 = 0.009$ (black), $p_1 = 0.49$

$p_2 = 0.009$ (green)

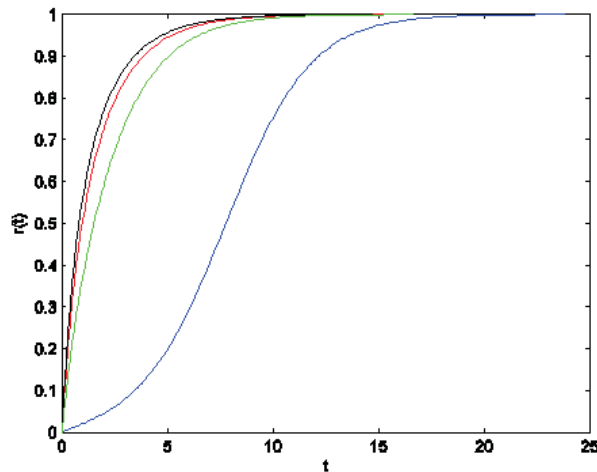


Figure 3 Graph of Hazard Rate Function of System Lifetime

When the value of p_0 is low (blue line) in the beginning the acceleration rate of hazard is increasing

and after a while the increasing of hazard rate is going on with a decreasing acceleration rate. In other values of p_0 hazard rate function is increasing with a decreasing acceleration rate and when p_0 value

becomes higher, the graph moves to right side and so the value of hazard rate becomes higher. We can show that hazard rate function has increasing specialty as below either.

$$r(t) = \frac{f(t)}{1 - F(t)}, \quad r_{\text{component}}(t) = \frac{Q(t)'}{1 - Q(t)}$$

$$r(t) = \frac{p_0 Q(t)(1 - Q(t))(2 - Q(t)(p_1 + p_2))}{(1 - Q(t)(p_1 + p_2))(1 - Q(t)(p_1 + p_2) - Q(t)^2 p_0)} r_{\text{component}}(t)$$

$$r(t) = \frac{p_0 Q(t)(2 - Q(t)(1 - p_0))}{(1 - Q(t) + Q(t)p_0)(1 + Q(t)p_0)} r_{\text{component}}(t)$$

$$r(t) = \left(1 - \frac{1}{(1 + Q(t)p_0)}\right) \left(1 + \frac{1}{(1 - Q(t) + Q(t)p_0)}\right) r_{\text{component}}(t)$$

Now, we suppose that every component is exponentially distributed and we know that exponential distribution has a constant hazard rate, and every part of the function is increasing, so the function is increasing.

3.5. Expected Lifetime of System

Expected lifetime of our system can be gained as below.

$$E(T_{SYSTEM}) = \int_0^{\infty} S(t) dt = \int_0^{\infty} \left(\frac{(1 + p_0)e^{-t\beta} - p_0 e^{-2t\beta}}{(1 - p_0)e^{-t\beta} + p_0} \right) dt = \int_0^{\infty} \left(\frac{(1 + p_0)e^{-t\beta}}{(1 - p_0)e^{-t\beta} + p_0} \right) dt - \int_0^{\infty} \left(\frac{p_0 e^{-2t\beta}}{(1 - p_0)e^{-t\beta} + p_0} \right) dt$$

Now with operations as follow we can reach the result with solving the first part of the equation.

$$\int \left(\frac{(1 + p_0)e^{-t\beta}}{(1 - p_0)e^{-t\beta} + p_0} \right) dt$$

Changing variables $e^{-t} = u$, $-e^{-t} dt = du$

$$\begin{aligned} \int \left(\frac{(1 + p_0)u^{\beta-1}}{(1 - p_0)u^{\beta} + p_0} \right) du &= \left(\frac{(1 + p_0)}{(1 - p_0)\beta} \ln((1 - p_0)u^{\beta} + p_0) \right) \Big|_{u=0}^{u=1} \\ &= -\frac{(1 + p_0)}{(1 - p_0)\beta} \ln(p_0) \end{aligned}$$

With solving the second part we can reach the following result.

$$E(T_{SYSTEM}) = \frac{1}{(1 - p_0)^2 \beta} (-\ln(p_0) - p_0(1 - p_0)), \quad p_0 \in (0, 1)$$

The expected lifetime which equal with first moment of distribution has graph as follow. In Figure 4: $\beta = 0.2$ (blue), $\beta = 0.21$ (red), $\beta = 0.22$ (green).

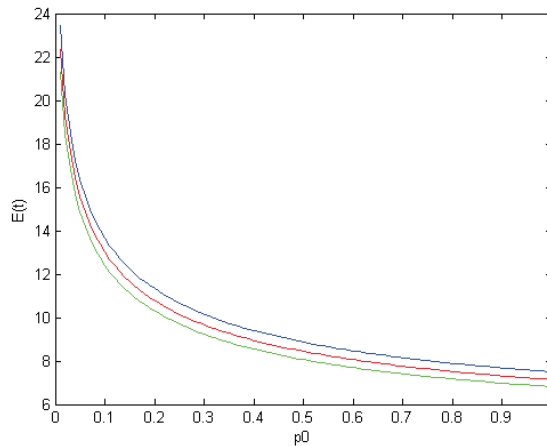


Figure 4 Graph of Function of Expected System Lifetime

Once we examine the graph of expected lifetime, we confront a decreasing line on p_0 . In the first part there is a system which takes more component with probability $1 - p_0$. Adding new component in a parallel system increases lifetime, so the line in graph in figure 4 decreasing according to p_0 . In system each component distributed exponential with mean $1/\beta$.

3.6. Second Moment

We can reach second moment of our system probability distribution with the process below.

$$E(T^2) = \int tS(t)dt = \int t \left(1 - \frac{Q^2 p_0}{(1 - Q(p_1 + p_2))} \right) dt = \int t \left(\frac{(1 + p_0)e^{-t\beta} - p_0 e^{-2t\beta}}{(1 - p_0)e^{-t\beta} + p_0} \right) dt$$

Changing variables $(1 - p_0)e^{-t\beta} + p_0 = a$, $-\beta(1 - p_0)e^{-t\beta} dt = da$

$$\begin{aligned} &= \int_0^1 \frac{1 + p_0}{(1 - p_0)\beta^2} \frac{\ln\left(\frac{a - p_0}{1 - p_0}\right)}{a} da - \int_0^1 \frac{p_0}{(1 - p_0)^2\beta^2} \frac{\ln\left(\frac{a - p_0}{1 - p_0}\right)}{a} (a - p_0) da \\ &= \frac{1}{(1 - p_0)^2\beta^2} \int_0^1 \frac{\ln\left(\frac{a - p_0}{1 - p_0}\right)}{a} da - \frac{p_0}{(1 - p_0)^2\beta^2} \int_0^1 \ln\left(\frac{a - p_0}{1 - p_0}\right) da \end{aligned}$$

In this position second part of integration is as below,

$$\int_0^1 \ln\left(\frac{a - p_0}{1 - p_0}\right) da = p_0 \ln\left(\frac{p_0}{p_0 - 1}\right) - 1$$

For the first part of integration there needs a second variable change.

$$\frac{a-p_0}{1-p_0} = u, \quad da = (1-p_0)du$$

$$\int_0^1 \frac{\ln\left(\frac{a-p_0}{1-p_0}\right)}{a} da = [\ln(1-p_0) - \ln(-p_0)] \ln(p_0) + Li_2(1) - Li_2\left(\frac{1}{p_0}\right)$$

After these calculations we gain second moment as,

$$E(T^2) = \frac{1}{(1-p_0)^2\beta^2} \left[[\ln(1-p_0) - \ln(-p_0)] \ln(p_0) + Li_2(1) - Li_2\left(\frac{1}{p_0}\right) \right] - \frac{p_0}{(1-p_0)^2\beta^2} \left[p_0 \ln\left(\frac{p_0}{p_0-1}\right) - 1 \right]$$

3.7. Variance

$$Var(T) = E(T^2) - E(T)^2$$

$$= \frac{1}{(1-p_0)^2\beta^2} \left[[\ln(1-p_0) - \ln(-p_0)] \ln(p_0) + Li_2(1) - Li_2\left(\frac{1}{p_0}\right) - p_0^2 \ln\left(\frac{p_0}{p_0-1}\right) + p_0 \right]$$

$$- \left(\frac{1}{(1-p_0)^2\beta} (-\ln(p_0) - p_0(1-p_0)) \right)^2$$

3.8. Least Squares Estimation (LSE)

By using equation (4) mean squares of error can be gained as below.

$$F(t) = \frac{Q(t)^2 p_0}{(1 - Q(t)(p_1 + p_2))} = u$$

$$\frac{Q(t)^2 p_0}{(1 - Q(t)(1 - p_0))} = u$$

$$Q(t)^2 p_0 + Q(t)(1 - p_0)u - u = 0$$

$$Q(t)_{1,2} = \frac{-(1-p_0)u \pm \sqrt{(1-p_0)^2 u^2 + 4u p_0}}{2p_0}$$

$$Q(t)_{1,2} = \frac{-(1-p_0)u \pm \sqrt{(1+p_0)^2 u^2 + 4u(1-u)p_0}}{2p_0} \quad (6)$$

In equation (6) there is one root provide non negativity. So we reach the situation below.

$$Q(t) = \frac{-(1-p_0)u + \sqrt{(1+p_0)^2 u^2 + 4u(1-u)p_0}}{2p_0}$$

Define this equation as,

$$\frac{-(1-p_0)u + \sqrt{(1+p_0)^2 u^2 + 4u(1-u)p_0}}{2p_0} = \xi$$

In equation (3) with which we defined system lifetime of every component, lifetime of component is $Q(t) = 1 - e^{-t\beta}$. By using this lifetime we reach some conclusions as follow.

$$1 - e^{-t\beta} = \xi$$

$$t\beta = -\ln(1 - \xi)$$

$$t = -\frac{1}{\beta} \ln(1 - \xi) \quad (7)$$

With this situation, there is an equation for both generating number and parameter estimation. Generalization of equation (7) gives us MSE equation.

$$MSE = \sum_{i=1}^n (t_i \beta + \ln(1 - \xi_i))^2$$

$$\frac{d}{d\beta} \sum_{i=1}^n (t_i \beta + \ln(1 - \xi_i))^2 = 2 \sum_{i=1}^n (t_i \beta + \ln(1 - \xi_i)) t_i = 0$$

$$\widehat{\beta}_{LSE} = \frac{-\sum_{i=1}^n \ln(1 - \xi_i) t_i}{\sum_{i=1}^n t_i^2} \quad (8)$$

After now the same process will be carried out for the other parameter.

$$\frac{Q(t)^2 p_0}{(1 - Q(t)(1 - p_0))} = u$$

$$\frac{u - Q(t)}{(1 - Q(t))Q(t)} = \frac{1}{1 - (1 - p_0)Q(t)}$$

$$\frac{(1 - Q(t))Q(t)}{u - Q(t)} = 1 - (1 - p_0)Q(t)$$

$$MSE = \sum_{i=1}^n \left(\frac{(1 - Q(t_i))Q(t_i)}{u_i - Q(t_i)} - 1 + (1 - p_0)Q(t_i) \right)^2$$

$$\begin{aligned}
 &= \sum_{i=1}^n \left(\frac{(1 - Q(t_i))Q(t_i)}{u_i - Q(t_i)} - (1 - Q(t_i)) - Q(t_i)p_0 \right)^2 \\
 &\frac{d}{dp_0} \sum_{i=1}^n \left(\frac{(1 - Q(t_i))Q(t_i)}{u_i - Q(t_i)} - (1 - Q(t_i)) - Q(t_i)p_0 \right)^2 = 0 \\
 &2 \sum_{i=1}^n \left(\frac{(2Q(t_i) - u)(1 - Q(t_i))Q(t_i)}{u_i - Q(t_i)} - Q(t_i)^2 p_0 \right) = 0 \\
 &\widehat{p_{0_{LSE}}} = \frac{\sum_{i=1}^n \left(\frac{(2Q(t_i) - u)(1 - Q(t_i))Q(t_i)}{u_i - Q(t_i)} \right)}{\sum_{i=1}^n Q(t_i)^2} \quad (9)
 \end{aligned}$$

3.9. Maksimum Likelihood Estimation (MLE)

With equation (5) the probability density function of the system expressed as below.

$$f(t) = \frac{d}{dt} F(t) = \frac{d}{dt} \left(\frac{Q(t)^2 p_0}{(1 - Q(t)(p_1 + p_2))} \right) = \frac{p_0 Q(t) Q'(t) (2 - Q(t)(p_1 + p_2))}{(1 - Q(t)(p_1 + p_2))^2} \quad (5)$$

Therefore, likelihood function can be gained as below.

$$L(f(t); \beta, p_0) = \prod_{i=1}^h \frac{p_0 Q(t) q(t) (2 - Q(t)(1 - p_0))}{(1 - Q(t)(1 - p_0))^2} = p_0^h \prod_{i=1}^h \frac{Q(t) q(t) (2 - Q(t)(1 - p_0))}{(1 - Q(t)(1 - p_0))^2}$$

Natural logarithm of this function is below.

$$\log(L(f(t); \beta, p_0)) = h \ln(p_0) + \sum_{i=1}^h \ln(Q(t) q(t) (2 - Q(t)(1 - p_0))) - \sum_{i=1}^h \ln((1 - Q(t)(1 - p_0))^2)$$

Derivatives of this function is below.

$$\begin{aligned}
 \frac{\partial \log(L(f(t); \beta, p_0))}{\partial p_0} &= \frac{h}{p_0} + \sum_{i=1}^h \frac{Q(t)}{(2 - Q(t)(1 - p_0))} - 2 \sum_{i=1}^h \frac{Q(t)}{(1 - Q(t)(1 - p_0))} = 0 \\
 \frac{\partial \log(L(f(t); \beta, p_0))}{\partial \beta} &= \sum_{i=1}^h \frac{\frac{d}{d\beta} (Q(t) q(t) (2 - Q(t)(1 - p_0)))}{(Q(t) q(t) (2 - Q(t)(1 - p_0)))} + 2(1 - p_0) \sum_{i=1}^h \frac{\frac{d}{d\beta} Q(t)}{(1 - Q(t)(1 - p_0))} = 0
 \end{aligned}$$

3.10. EM Algorithm Estimation (EM)

EM Algorithm contains two step as Expectation step and Maximization step. By the equations below EM Algorithm expectation values can calculate easily.

$$\sum_{i=1}^o \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \frac{\partial \log(f(t_i, m, n))}{\partial \beta} f(n, m | t_i; \beta^0, p_1^0, p_2^0) = 0$$

$$\sum_{i=1}^o \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \frac{\partial \log(f(t_i, m, n))}{\partial p_0} f(n, m | t_i; \beta^0, p_0^0) = 0$$

For gaining these equations, at first natural logarithm of probability density function and conditional function has to be obtained.

$$f(t, m, n) = f(t | n, m) Pr(n, m) = \frac{dF(t | n, m)}{dt} Pr(n, m)$$

$$f(t, m, n) = \frac{dQ^{n+m+2}}{dt} \binom{m+n}{n} \left(\frac{1-p_0}{2}\right)^n \left(\frac{1-p_0}{2}\right)^m p_0$$

$$= (n+m+2)Q^{n+m+1}q \binom{m+n}{n} \left(\frac{1-p_0}{2}\right)^{n+m} p_0$$

With equation (5) the probability density function of the system expressed as below.

$$f(t) = \frac{d}{dt} F(t) = \frac{d}{dt} \left(\frac{Q(t)^2 p_0}{(1-Q(t)(p_1+p_2))} \right) = \frac{p_0 Q(t) Q'(t) (2-Q(t)(p_1+p_2))}{(1-Q(t)(p_1+p_2))^2} \quad (5)$$

Then conditional probability density function gains as below.

$$f(n, m | t) = \frac{(n+m+2)Q^{n+m+1}q \binom{m+n}{n} \left(\frac{1-p_0}{2}\right)^{n+m} p_0}{\frac{p_0 Q q (2-Q(1-p_0))}{(1-Q(1-p_0))^2}}$$

$$= \frac{(n+m+2)(1-Q(1-p_0^{(0)}))}{(2-Q^{(0)}(1-p_0^{(0)}))} (1-Q^{(0)}(1-p_0^{(0)})) \binom{m+n}{n} \left(Q^{(0)} \left(\frac{1-p_0^{(0)}}{2}\right)\right)^n \left(Q^{(0)} \left(\frac{1-p_0^{(0)}}{2}\right)\right)^m$$

Natural logarithm of common probability density function can be obtained as below.

$$\log(f(t, m, n)) = \ln(n+m+2) + (n+m+1) \ln(Q) + \ln(q) + (n+m) \ln\left(\frac{1-p_0}{2}\right) + \ln\binom{n+m}{n} + \ln(p_0)$$

Therefore, the calculations can be acquired as below.

$$\frac{\partial \log(f(t_i, m, n))}{\partial \beta} = \frac{t_i e^{-\beta^{(1)} t_i}}{1 - e^{-\beta^{(1)} t_i}} (n+m+1) + \frac{1}{\beta^{(1)}} - t_i$$

$$\frac{\partial \log(f(t_i, m, n))}{\partial p_0} = \frac{-(n+m)}{1-p_0^{(1)}} + \frac{1}{p_0^{(1)}}$$

By following the steps below, a full equation form which contains every information can be acquired.

$$\sum_{i=1}^o \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \frac{\partial \log(f(t_i, m, n))}{\partial p_0} f(n, m | t_i; \beta^0, p_0^0)$$

$$\begin{aligned}
 &= \sum_{i=1}^o \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \left(\frac{-(n+m)}{1-p_0^{(1)}} + \frac{1}{p_0^{(1)}} \right) \frac{(n+m+2)(1-Q(1-p_0^{(0)}))}{(2-Q^{(0)}(1-p_0^{(0)}))} \\
 &\quad * (1-Q^{(0)}(1-p_0^{(0)})) \binom{m+n}{n} \left(Q^{(0)} \left(\frac{1-p_0^{(0)}}{2} \right) \right)^n \left(Q^{(0)} \left(\frac{1-p_0^{(0)}}{2} \right) \right)^m \\
 \text{For } A &= \binom{m+n}{n} \left(Q^{(0)} \left(\frac{1-p_0^{(0)}}{2} \right) \right)^n \left(Q^{(0)} \left(\frac{1-p_0^{(0)}}{2} \right) \right)^m, \\
 &= \sum_{i=1}^o \left(\frac{-1}{1-p_0^{(1)}} \right) \frac{(1-Q(1-p_0^{(0)}))}{(2-Q^{(0)}(1-p_0^{(0)}))} \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} (n+m)(n+m+2) (1-Q^{(0)}(1-p_0^{(0)})) A \\
 &\quad + \sum_{i=1}^o \left(\frac{1}{p_0^{(1)}} \right) \frac{(1-Q(1-p_0^{(0)}))}{(2-Q^{(0)}(1-p_0^{(0)}))} \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} (n+m+2) (1-Q^{(0)}(1-p_0^{(0)})) A \\
 &= \sum_{i=1}^o \frac{(1-Q(1-p_0^{(0)}))}{(2-Q(1-p_0^{(0)}))} \left(\frac{-1}{1-p_0^{(1)}} \right) E(N^2 + M^2 + 2NM + 2N + 2M) \\
 &\quad + \sum_{i=1}^o \left(\frac{1}{p_0^{(1)}} \right) \frac{(1-Q(1-p_0^{(0)}))}{(2-Q(1-p_0^{(0)}))} E(N + M + 2)
 \end{aligned}$$

Other parameter estimation form can be gained as below.

$$\begin{aligned}
 &\sum_{i=1}^o \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \frac{\partial \log(f(t_i, m, n))}{\partial \beta} f(n, m | t_i; \beta^0, p_0^0) = 0 \\
 &= \sum_{i=1}^o \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \left(\frac{t_i e^{-\beta^{(1)} t_i}}{1 - e^{-\beta^{(1)} t_i}} (n+m+1) + \frac{1}{\beta^{(1)}} - t_i \right) \frac{(n+m+2)(1-Q(1-p_0^{(0)}))}{(2-Q^{(0)}(1-p_0^{(0)}))} (1-Q^{(0)}(1-p_0^{(0)})) A \\
 &= \sum_{i=1}^o \frac{t_i e^{-\beta^{(1)} t_i}}{1 - e^{-\beta^{(1)} t_i}} \frac{(1-Q(1-p_0^{(0)}))}{(2-Q(1-p_0^{(0)}))} E(N^2 + M^2 + 2NM + 3N + 3M + 2) \\
 &\quad + \sum_{i=1}^o \left(\frac{1}{\beta^{(1)}} - t_i \right) \frac{(1-Q(1-p_0^{(0)}))}{(2-Q(1-p_0^{(0)}))} E(N + M + 2)
 \end{aligned}$$

Moment generating function of bivariate geometric distribution can help future steps for estimation.

Bivariate distribution function which defines the system can be expressed as below.

$$f(t, n, m) = \binom{m+n}{n} \left(Q^{(0)} \left(\frac{1-p_0^{(0)}}{2} \right) \right)^n \left(Q^{(0)} \left(\frac{1-p_0^{(0)}}{2} \right) \right)^m p_5, \quad p_5 = 1 - Q^{(0)}(1-p_0^{(0)})$$

For this situation moment generating function gained as below. (Krishna and Singh 2009).

$$E(e^{t_1 n + t_2 m}) = \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} e^{t_1 n + t_2 m} \binom{m+n}{n} \left(Q \left(\frac{1-p_0}{2} \right) \right)^n \left(Q \left(\frac{1-p_0}{2} \right) \right)^m p_5$$

$$= \frac{p_5}{1 - Q\left(\frac{1-p_0}{2}\right)e^{t_1} - Q\left(\frac{1-p_0}{2}\right)e^{t_2}} \quad (10)$$

Expectations can calculate with equation (10).

$$E(N) = \frac{d}{dt_1} \left(\frac{p_5}{1 - Q\left(\frac{1-p_0}{2}\right)e^{t_1} - Q\left(\frac{1-p_0}{2}\right)e^{t_2}} \right) \bigg|_{t_1=0, t_2=0} = \frac{Q\left(\frac{1-p_0}{2}\right)}{p_5}$$

The next calculations are below.

$$E(M) = \frac{d}{dt_2} \left(\frac{p_5}{1 - Q\left(\frac{1-p_0}{2}\right)e^{t_1} - Q\left(\frac{1-p_0}{2}\right)e^{t_2}} \right) \bigg|_{t_1=0, t_2=0} = \frac{Q\left(\frac{1-p_0}{2}\right)}{p_5}$$

$$E(NM) = \frac{d^2}{dt_1 dt_2} \left(\frac{p_5}{1 - Q\left(\frac{1-p_0}{2}\right)e^{t_1} - Q\left(\frac{1-p_0}{2}\right)e^{t_2}} \right) \bigg|_{t_1=0, t_2=0} = \frac{2Q^2\left(\frac{1-p_0}{2}\right)^2}{p_5^2}$$

$$E(N^2) = \frac{d^2}{dt_1^2} \left(\frac{p_5}{1 - Q\left(\frac{1-p_0}{2}\right)e^{t_1} - Q\left(\frac{1-p_0}{2}\right)e^{t_2}} \right) \bigg|_{t_1=0, t_2=0} = \frac{Q\left(\frac{1-p_0}{2}\right)}{p_5} \left(1 + \frac{2Q\left(\frac{1-p_0}{2}\right)}{p_5} \right)$$

$$E(M^2) = \frac{d^2}{dt_2^2} \left(\frac{p_5}{1 - Q\left(\frac{1-p_0}{2}\right)e^{t_1} - Q\left(\frac{1-p_0}{2}\right)e^{t_2}} \right) \bigg|_{t_1=0, t_2=0} = \frac{Q\left(\frac{1-p_0}{2}\right)}{p_5} \left(1 + \frac{2Q\left(\frac{1-p_0}{2}\right)}{p_5} \right)$$

New form of upper equations are below.

$$\begin{aligned} & \sum_{i=1}^o \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \frac{\partial \log(f(t_i, m, n))}{\partial p_0} f(n, m | t_i; \beta^0, p_0^0) = 0 \\ & \sum_{i=1}^o \frac{(1 - Q(1 - p_0^{(0)}))}{(2 - Q(1 - p_0^{(0)}))} \left(\frac{-1}{1 - p_0^{(1)}} \right) E(N^2 + M^2 + 2NM + 2N + 2M) \\ & + \sum_{i=1}^o \left(\frac{1}{p_0^{(1)}} \right) \frac{(1 - Q(1 - p_0^{(0)}))}{(2 - Q(1 - p_0^{(0)}))} E(N + M + 2) = 0 \\ & \sum_{i=1}^o \frac{(1 - Q(1 - p_0^{(0)}))}{(2 - Q(1 - p_0^{(0)}))} \left(\frac{1}{1 - p_0^{(1)}} \right) E(N^2 + M^2 + 2NM + 2N + 2M) \\ & = \sum_{i=1}^o \left(\frac{1}{p_0^{(1)}} \right) \frac{(1 - Q(1 - p_0^{(0)}))}{(2 - Q(1 - p_0^{(0)}))} E(N + M + 2) \\ & (p_0^{(1)}) \sum_{i=1}^o \frac{(1 - Q(1 - p_0^{(0)}))}{(2 - Q(1 - p_0^{(0)}))} E(N^2 + M^2 + 2NM + 2N + 2M) \end{aligned}$$

$$\begin{aligned}
 &= (1 - p_0^{(1)}) \sum_{i=1}^o \frac{(1 - Q(1 - p_0^{(0)}))}{(2 - Q(1 - p_0^{(0)}))} E(N + M + 2) \\
 &(p_0^{(1)}) \sum_{i=1}^o \left(\frac{3Q(1 - p_0^{(0)})}{(2 - Q(1 - p_0^{(0)}))} + \frac{2Q^2(1 - p_0^{(0)})^2}{p_5^{(0)}(2 - Q(1 - p_0^{(0)}))} \right) = o(1 - p_0^{(1)}) \quad (11)
 \end{aligned}$$

Same calculations for other parameter are below.

$$\begin{aligned}
 &\sum_{i=1}^o \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \frac{\partial \log(f(t_i, m, n))}{\partial \beta} f(n, m | t_i; \beta^0, p_0^0) = 0 \\
 &= \sum_{i=1}^o \frac{t_i e^{-\beta^{(1)} t_i}}{1 - e^{-\beta^{(1)} t_i}} \frac{(1 - Q(1 - p_0^{(0)}))}{(2 - Q(1 - p_0^{(0)}))} E(N^2 + M^2 + 2NM + 3N + 3M + 2) \\
 &+ \sum_{i=1}^o \left(\frac{1}{\beta^{(1)}} - t_i \right) \frac{(1 - Q(1 - p_0^{(0)}))}{(2 - Q(1 - p_0^{(0)}))} E(N + M + 2) \\
 &= \sum_{i=1}^o \frac{t_i e^{-\beta^{(1)} t_i}}{1 - e^{-\beta^{(1)} t_i}} \left(\frac{3Q(1 - p_0^{(0)})}{(2 - Q(1 - p_0^{(0)}))} + \frac{2Q^2(1 - p_0^{(0)})^2}{p_5^{(0)}(2 - Q(1 - p_0^{(0)}))} + 1 \right) + \sum_{i=1}^o \left(\frac{1}{\beta^{(1)}} - t_i \right) = 0 \quad (12)
 \end{aligned}$$

By using Equations (11) and (12) estimation for p_0 can be gained as below. For other parameter β using program can be better.

$$\hat{p}_0 = \frac{o}{o + \sum_{i=1}^o \left(\frac{3Q(1 - p_0^{(0)})}{(2 - Q(1 - p_0^{(0)}))} + \frac{2Q^2(1 - p_0^{(0)})^2}{p_5^{(0)}(2 - Q(1 - p_0^{(0)}))} \right)}$$

3.11. Simulation

In Table I there are the results of parameter estimations for generated lifetimes in which parameter values are decided. Repeating number of every iteration is 100, n shows observation number.

Table I Parameter Estimation of the Model with 3 Parameters.

$p_0 = 0.05, p_1 = 0.91, p_2 = 0.04, \beta = 1$									
	n=30			n=50			n=100		
	Mse	Mle	EM	Mse	Mle	EM	Mse	Mle	EM
$\hat{p}_1$	0.52	0.48	0.49	0.53	0.5	0.48	0.53	0.46	0.43
Rmse($\hat{p}_1$)	0.38	0.47	0.48	0.38	0.48	0.49	0.37	0.44	0.54
$\hat{p}_2$	0.42	0.47	0.49	0.41	0.44	0.49	0.41	0.47	0.53
Rmse($\hat{p}_2$)	0.38	0.46	0.48	0.38	0.43	0.48	0.37	0.45	0.43

From Table 1, we identify that best estimation method varies according to the observation number. According to root mean squares of error least squares estimation provides better estimations. Maximum likelihood estimation provides best estimation accuracy for parameter value.

4. APPLICATION

We use Kolmogorov-Smirnov test statistics for looking the availability of our distribution to data sets. In using this statistic, we carry out a hypothesis test and we reach a conclusion that whether our distribution and nonparametric distribution of data set is equal or not (DeGroot and Schervish 2012). In Kolmogorov-Smirnov test statistics p value indicate the rate of distributions explanation success.

After we reach that the two distribution is equal, we have a new problem that which of the distribution is better for this data set. Because there may be many distributions that equal to nonparametric distribution according to hypothesis test. For comparing these distributions, we use Akaike Information Criterion (AIC). While using AIC, we choose better distribution based on minimum AIC value. Because AIC is a penalty value and minimum value shows maximum resemblance to nonparametric distribution of data set and distribution (Snipes and Taylor 2014), (Akaike 1974).

Now we will study with data sets for deciding the usability range of our new distribution (bivariate geometric with exponential). We will study with 14 data sets. While studying these data, we will compare our new distribution with most known and previously used distributions.

4.1 Fit Test for Data Sets

First data group was studied by Singh et al. (2014). Data contains 1000 redundant fiber carbons strengths. They were denominated as Gpa. The p value of Kolmogorov-Smirnov test statistics of new distribution is 0.9986.

Table II Strength data test results.

Model	K-S	AIC	A-D	C-W-M
BG-E	0.0438	101.79	0.143	0.01625
Weibull (2p)	0.0437	102	0.2095	0.02201
Power Lindley	0.0405	101.35	0.1634	0.021676

Once Table II examined we realized that the new distribution has second minimum Akaike penalty value. The suggested model has minimum Anderson-Darling and Cramer von Mises distance value. Histogram and Q-Q graphs about studies for this data set are in Figure 5 and 6 orderly.

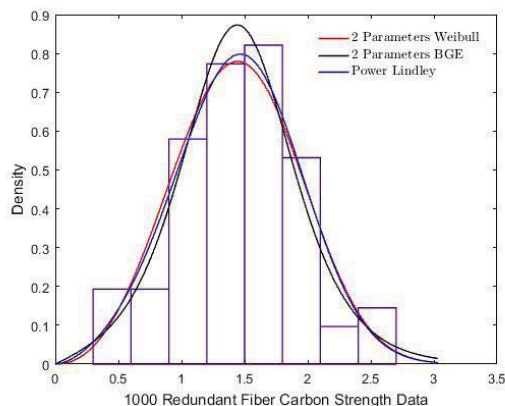


Figure 5 1000 redundant fiber carbons histogram graph

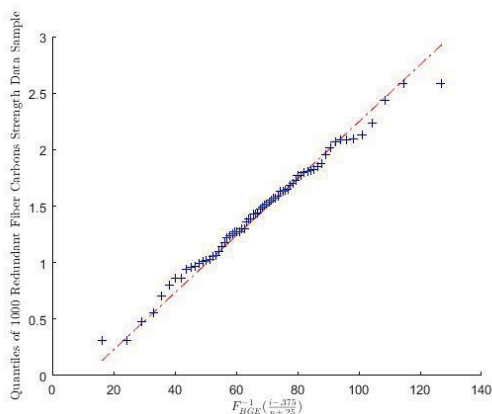


Figure 6 1000 redundant fiber carbons Q-Q graph

Next data contains aluminum sheets strength under maximum stress. Data group was studied by Shanker (2015) for comparing Exponential and Lindley distributions. The p value of Kolmogorov-Smirnov test statistics of new distribution is 0.9762.

Table III Aluminum sheets strength data under maximum stress test results

Model	K-S	AIC	A-D	C-W-M
BG-E	0.0463	905.1631	0.2277	0.02964
Weibull (2p)	0.071	912.529	0.6688	0.09113
Weibull (3p)	0.0642	913.393	0.3806	0.05347
Lindley Distribution	0.242	985.11	11.25	2.2
Exponential Distribution	0.357	1046.87	13.69	2.78

Once Table III examined we realized that the new distribution has least minimum Akaike penalty value. The suggested model has minimum Kolmogorov Smirnov, Anderson-Darling and Cramer von Mises distance value. Histogram and Q-Q graphs about studies for this data set are in Figure 7 and 8 orderly.

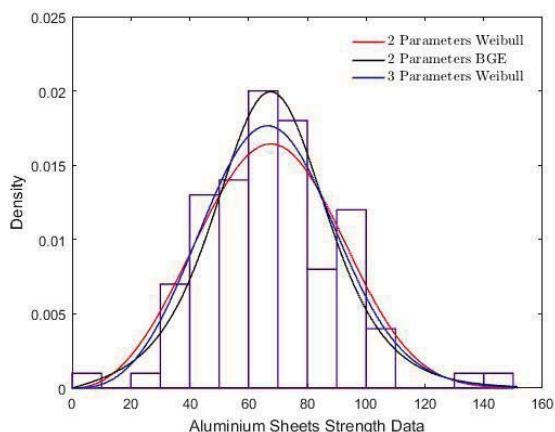


Figure 7 Aluminum sheets histogram graph

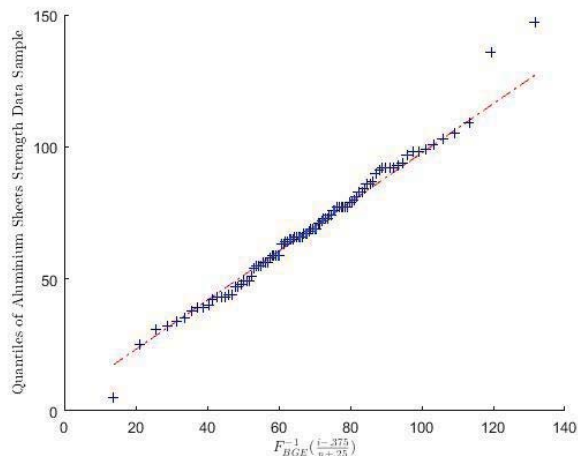


Figure 8 Aluminum sheets Q-Q graph

The next results related to 1000 redundant fiber carbons strengths which were studied Bader and Priest (1982) at first. After this study data group was used in an article in which a new distribution was gained from exponential distribution. In this study Weibull distribution offered the best model (Mahdavi and Kundu 2017). During measuring process, the length of the experiment piece was 1mm. The p value of Kolmogorov-Smirnov test statistics of new distribution is **0.9958**.

Table IV 1000 redundant fiber carbons strength data test results-2

Model	K-S	AIC	A-D	C-W-M
BG-E	0.0523	140.6622	0.157	0.0204
Weibull (2p)	0.0902	141.8672	0.4418	0.068
Weibull (3p)	0.0606	143.7376	0.1659	0.0231

Once Table IV examined we realized that the new distribution has least minimum Akaike penalty value. The suggested model has minimum Kolmogorov Smirnov, Anderson-Darling and Cramer von Mises distance value. Histogram and Q-Q graphs about studies for this data set are in Figure 9 and 10 orderly.

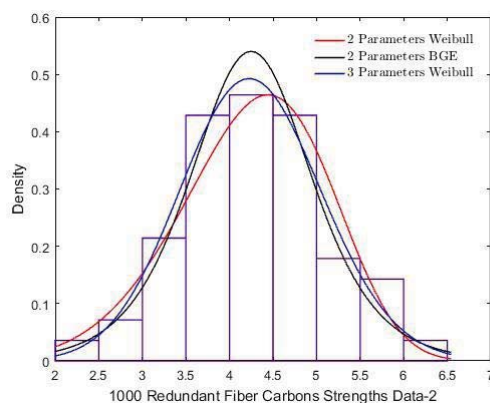


Figure 9 1000 redundant fiber carbons strengths histogram graph-2

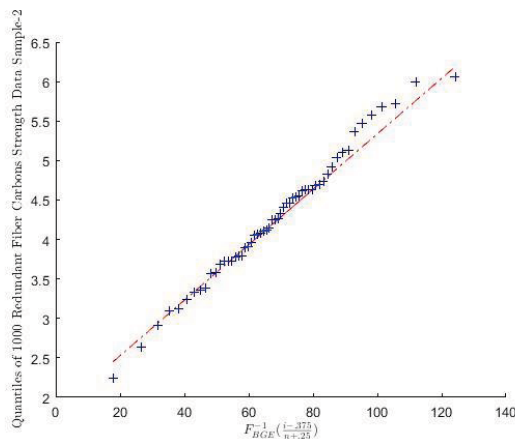


Figure 10 1000 redundant fiber carbons Q-Q graph-2

The next data set contains 66 observations from broken pressure of fiber carbons. The measurement of this data is Gba. Data set was used by Nichols et al. (2006) later was used by Pogany et al. (2015) for four parameters Marshall-Olkin Exponential Weibull distribution. The p value of Kolmogorov-Smirnov test statistics of new distribution is **0.952**.

Table V Broken pressure of fiber carbons data test results.

Model	K-S	AIC	A-D	C-W-M
BG-E	0.06134	174.44	0.3482	0.05451
Weibull (2p)	0.0823	176.13	0.4859	0.08366
Weibull (4p)	0.06105	177.43	0.2534	0.03711

Once Table V examined we realized that the new distribution has least minimum Akaike penalty value. The suggested model has second distance value in Kolmogorov Smirnov, Anderson-Darling and Cramer von Mises. Histogram and Q-Q graphs about studies for this data set are in Figure 11 and 12 orderly.

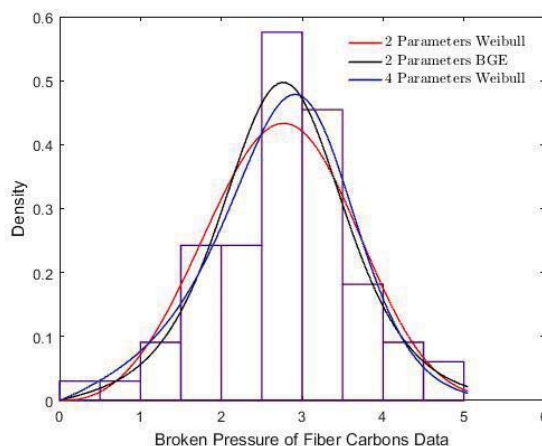


Figure 11 Broken pressure of fiber carbons histogram graph.

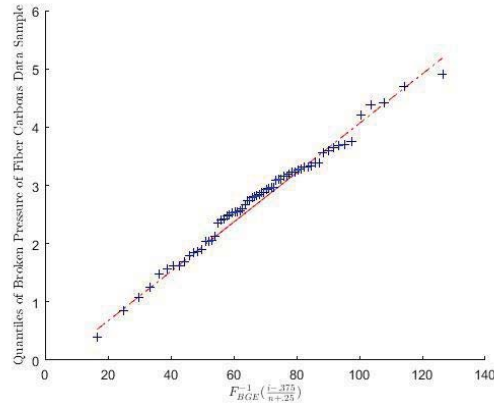


Figure 12 Broken pressure of fiber carbons Q-Q graph

The next results contain test results of samples of a concrete cover for raft foundation. These observations gained from being under 25 Mpa pressure between 28-56 days. The measurement of this data is kg/cm^2 . (site.iugaza.edu.ps). The p value of Kolmogorov-Smirnov test statistics of new distribution is **0.9991**.

Table VI Test results of samples of a concrete cover for raft foundation

Model	K-S	AIC	A-D	C-W-M
BG-E	0.0762	211.05	0.125	0.0151
Weibull (2p)	0.0913	210.23	0.152	0.0222
Weibull (3p)	0.0809	214.09	0.133	0.0177

Once Table VI examined we realized that the new distribution has second minimum Akaike penalty value. The suggested model has least distance value in Kolmogorov Smirnov, Anderson-Darling and Cramer von Mises. Histogram and Q-Q graphs about studies for this data set are in Figure 13 and 14 orderly.

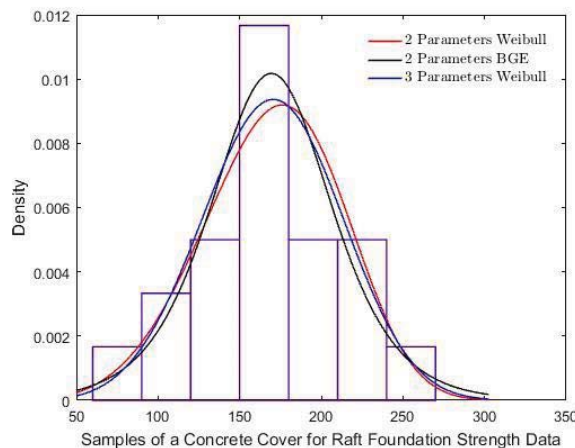


Figure 13 Samples of a concrete cover for raft foundation histogram graph

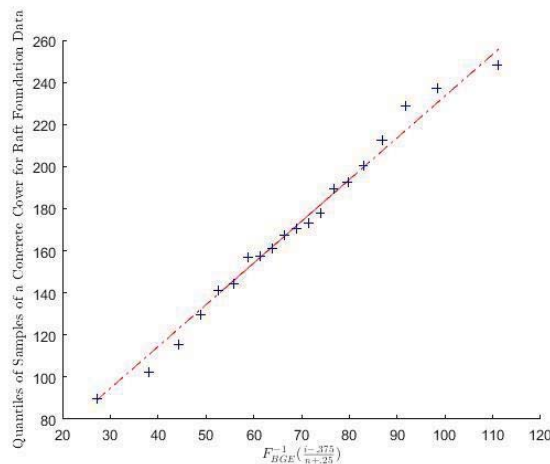


Figure 14 Samples of a concrete cover for raft foundation Q-Q graph

The next data was studied by Barlow et.al (1979) to study a reliability research for a given strength data set from vessel life under pressure. In that research their main target is to reach a different statistical model because exponential distribution is not available according to hazard rate function. The shape of hazard rate function of the new distribution in this paper gives a big hope for being available for the same data set. In table VII there is the data of lifetimes of Kevlar 49/Epoxy cables in %70 pressure level. In table VIII there are results about the modelling of lifetimes of Kevlar 49/Epoxy cables in %70 pressure level. p value of our distribution is **0.8634**.

Table VII Data of lifetimes of Kevlar 49/Epoxy cables in %70 pressure level.

1051	1337	1387	1921	1942	2322	3629	4006	4012	4063
4921	5445	5620	5817	5905	5956	6068	6121	6473	7501
7886	8108	8546	8666	8831	9106	9711	9806	10205	10396
10861	11026	11214	11362	11604	11608	11745	11762	11895	12044
13520	13670	14110	14496	15395	16179	17092	17568	17568	

Table VIII Results about the modelling of lifetimes of Kevlar 49/Epoxy cables in %70 pressure level.

Model	K-S	AIC	A-D	C-W-M
BG-E	0.0827	967.1604	0.6182	0.0752
Weibull (2p)	0.0876	965.6986	0.5503	0.0777
Weibull (3p)	0.0848	969.5231	0.5142	0.0714

This data was studied by Barlow et.al (1979) to generate baseline data on vessel life under pressure and to apply statistical reliability theory to vessel and strand data in order to predict reliability at given stress levels. In their examination their main aim was the observation that the hazard rates for data sets were time-dependent at low stress levels; as a result, the exponential lifetime distribution is not valid. After viewing their work, with transfusable hazard rate function shape the new distribution in this paper may be available for the same data. In table IX there is the data of lifetimes of Kevlar 49/Epoxy

sphere boxes in %74 pressure level. In table X there are results about the modelling of lifetimes of Kevlar 49/Epoxy sphere boxes in %74 pressure level. p value of our distribution is 0.9047.

Table IX Data of lifetimes of Kevlar 49/Epoxy sphere boxes in %74 pressure level

225.2	503.6	1087.7	1134.3	1824.3	1920.1	2383.0	2442.5	3708.9
4908.9	5556.0	6271.1	7332.0	7918.7	7996.0	9240.3	9973.0	3708.9

Table X Results of lifetimes of Kevlar 49/Epoxy sphere boxes in %74 pressure level

Model	K-S	-2LL	AIC	AICc	A-D	C-W-M
BG-E	0.1258	337.49	341.4973	342.2973	0.596	0.0761

The last two data sets may show this new distributions capability of representing the system with the components which was introduced at the beginning of this study.

According to maximum likelihood estimation for %74 pressure level parameter estimation for p_0 is 1. Thus there are only 2 strength point for pressure. When the same examine done with similar components under %70 pressure parameter estimation for p_0 is 0.0961. Therefore, there may be more components than under %74 pressure in system. This means that there are some components which are capable in strength under %70 pressure on the other hand they are not capable under %74 pressure. This is very logical.

5.CONCLUSION

Related with our system, at first we gained lifetime function of system and with graph in Figure 1 we examined lifetime. Later using this lifetime function, probability density function was gained and some intelligence about distribution was told. Expected lifetime was gained via lifetime function either. While examining expected lifetime we screen that which parameter is efficient for the value of expected lifetime. Then hazard rate was gained and examined via graphs. Consequently for characteristics of distribution we looked for moments and variance.

After all these we try to reach a parameter estimation and a close form for LSE estimation was gained. Later we tried to decide for which data sets the distribution of our systematic structure was available.

By using data sets in literature we reach that our new distribution is efficiently available for strength data and influently modelling these type of data. The success of the new distribution which was studied in this paper, is more appropriate than other statistical distributions in modeling strength data sets of all disciplines in this study. Thus we reach a conclusion that this distribution suggests a capable model in this area.

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Computing Classification System:

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