

On the Efficiency of Multigrid Solver for Shifted Laplace Equation in a Heterogeneous Medium

W.A. Siyal^{1,2,3}, A.H. Sheikh^{3a}, K.B. Amur^{3b}, A.G. Shaikh^{3c} and R.A. Malookani^{3d}

¹Sukkur IBA University, Pakistan

²IBA Community College, Naushehro Feroze Pakistan;

Email : wajidsiyal.ccnf@iba-suk.edu.pk

^{3a,3b,3c,3d}Department of Mathematics & Statistics, Quaid-e-Awam University of Engineering, Science and Technology, Nawabshah Pakistan;

^{3a} Email : ah.sheikh@quest.edu.pk

^{3b} Email : kbamur@quest.edu.pk

^{3c} Email : agshaikh@quest.edu.pk

^{3d} Email : rajab_ali31@yahoo.com

ABSTRACT

In this paper, the computational efficiency of Multigrid solver is discussed with the comparison of basic solvers such as Gauss-Jacobi, Gauss-Seidel, and Generalized minimum residual (GMRES). The problem considered in work is elliptic partial differential equation, which immediately attracts Multigrid method. Multigrid is reluctant to perform well in heterogeneous geometries. Further shift in Helmholtz problem allows occurrence of negative eigenvalues, making problem indefinite. These two make Multigrid with standard components less favorable. In this work, smoothing parameter in Multigrid is tuned to get optimized results in heterogeneous domain. The results are obtained while taking three different relaxation parameters $\omega = 1$, $\omega = 2/3$ and $\omega = 1/2$ and different choices of real and imaginary shifts $a + ib$ are considered for e.g. (0,0), (0,1) and (1,1). Results showed that better choice of relaxation parameter in smoother is $\omega = 2/3$. Also Multigrid has better convergence with pure imaginary shift i.e. (0,1) as compared to rest of choices of shift, chosen in this work. The proposed technique helps to obtain comparably better results than existing solvers. This fact is affirmed by presented results, where problem with different grid size n and different shift " k " is experimented.

Keywords: Finite differences, Multigrid method, indefinite matrix, eigenvalues, Helmholtz

Mathematics Subject Classification: 65F05, 65F08, 65F10, 65N22, 65N06, 65N55,

1. INTRODCUTION

The real life models have changed the scenario of existing lifestyle like Drug Dosage, Time of death, Reactor Risk Assessment etc. the modern scientific community believes on the practical implementation and evidence of model problems which are to be solved by numerous numerical algorithms. Since the detection of radar to medical and seismic imaging analysis, the oscillative propagation is become a key role to transform the model into mathematical models either ordinary or partial differential equations furthermore, the models lead to system of linear equation so called discretization which is the main basic interest of research community. The methods have been developed targeting many applications in different areas of engineering and science involving wave propagation, especially in medical imaging, seismic imaging, radar imaging, global seismology or

many other engineering applications (Stenger 2007, Chaumont Frelet 2015, Sheikh, Lahaye et al. 2016, Shaikh, Sheikh et al. 2019). Multigrid is the numerical algorithm known as algorithms of algorithms, in a numerical analysis that uses its basic components to figure out the convergence of model problem (MP) in terms of less expensive CPU cost which is the first most attribute which is discussed in later section. Shifted Laplace equation (Helmholtz equation) one and two dimensional are read as

$$\begin{cases} -\nabla^2 u - (a - ib)k^2 u = g, & u \in \Omega = [0, 1] \\ u(0) = 3 \text{ and } u(1) = 5, & \text{On } \partial\Omega \end{cases} \quad (1)$$

where $g(x) = -48x^2$ and $u(x) = 4x^4 - 2x + 3$

and

$$\begin{cases} -\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) - (a - ib)k^2 u = g & , \quad u \in \Omega = [0, 1]^2 \\ u(0, y) = 0 = u(x, 0) \text{ and } u(1, y) = y^4, u(x, 1) = x^3, & \text{On } \partial\Omega \end{cases} \quad (2)$$

and $k^2 = -5$, the source function $g(x, y)$ and analytic solution $u(x, y)$ for Eq. (2) is chosen as

$$g(x, y) = -6xy^2(x^2 + 2y^2) + 5x^3y^4, \quad u(x, y) = x^3y^4$$

Eq. (1) and (2) are derived from the Newton's Law of motion and the Hook's Law, details can be found here (Erlangga 2005, Sheikh 2014, Sheikh 2014, Wang, Zou et al. 2014). The model problems are dealt with the different complex shift which is great interest of this paper. Computing the numerical solution to the Helmholtz equation for a heterogeneous medium is of prime importance in seismic imaging (Plessix 2007). In the low-frequency regime, MP can be solved efficiently using standard numerical methods for solving elliptic equations. In the high-frequency regime, efficient methods tend to rely on extra knowledge about the structure of the wave speed (Bayliss, Goldstein et al. 1983, Colton and Kress 2013, Sheikh, Lahaye et al. 2013).

The Multigrid methods are well known for the linear systems obtained from the discretization of elliptic partial differential equation (Trottenberg, Oosterlee et al. 2000, Sheikh, Vuik et al. 2011). However, the jumps in domain cause slow convergence for Multigrid methods (Erlangga and Nabben 2008). The geometry of model problem under discussion is carried out in one and two dimensional heterogeneous domain.

Heterogeneous domain for one –dimensional Helmholtz equation is given in Figure 1.

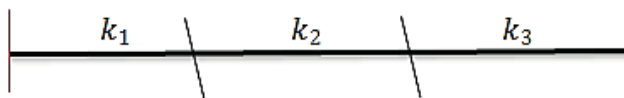


Figure 1: Heterogeneous domain for 1D problem

Heterogeneous domain for two –dimensional Helmholtz equation is given in Figure 2,

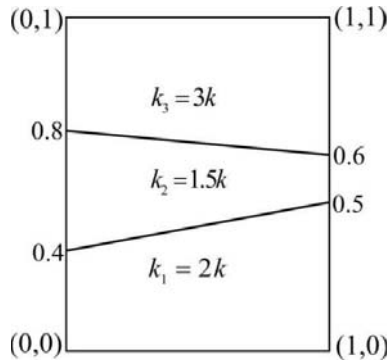


Figure 2: Heterogeneity in a square domain

Where $k_1 = 1.3k$, $k_2 = k$, $k_3 = 2k$

2. METHODOLOGY/MODEL PROBLEM/MATHEMATICAL MODEL

Due to lack of convergence of basic iterative methods (Jacobi and Gauss Seidel) the modern iterative methods are replaced one of them the robust and efficient method is Multigrid method which uses its basic components to enhance the capability of convergence of model problems which are under discussion it is evident in figure (a) and (b). Keeping in view literature review (Bayliss, Goldstein et al. 1983, Erlangga 2005, Erlangga and Turkel 2012) Multigrid method his adapted to maintain the convergence accuracy. The main component of Multigrid method is coarsening will be discussed separately that reduces the size of actual bandwidth of matrix A . Model problems (1) and (2) are discretized using finite difference method (FDM) which leads to models into system of linear equations of the form $Au = g$.

3. DISCRETIZATION OF 1D AND 2D MODEL PROBLEMS

First 1D problem is considered in unit interval $(0,1)$. Finite differences scheme is applied at interior points x_i , $i = 1, 2, 3, \dots, n-1$. The Eq (1) gives rise to simultaneous linear equations,

$$\frac{1}{h^2} \left[-u_n + (2 - k^2 h^2) u_{n-1} - u_{n-2} \right] = g_{n-1}$$

for values of $i = 1, 2$ and $n-1$. This can ultimately be written as the linear system $Au = g$.

Next, a 2D case is considered to explain discretization. The two-dimensional equation in spatial variables x and y is given

$$-u_{xx}(x, y) - u_{yy}(x, y) - k^2 u(x, y) = g(x, y)$$

where u_{xx} and u_{yy} represent second derivative with respect to x and y . Using finite difference approximations for differentiation, the equation gives rise to 5-point stencil, for interior (normal) nodes

$$\frac{1}{h^2} \begin{pmatrix} & -1 & \\ -1 & 4 - k^2 h^2 & -1 \\ & -1 & \end{pmatrix} u_{i,j} = g_{i,j},$$

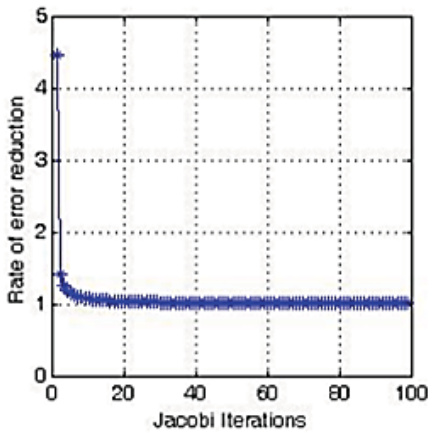
and 5-point stencil of equation on left boundary nodes

$$\frac{1}{h^2} \begin{pmatrix} & -1 \\ 0 & 4 - k^2 h^2 & -2 \\ & -1 \end{pmatrix} u_{i,j} = g_{i,j}$$

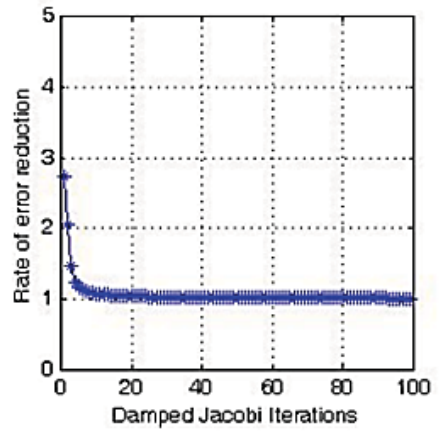
and 5-Point stencil of equation on left and bottom nodes

$$\frac{1}{h^2} \begin{pmatrix} & -2 \\ 0 & 4 - k^2 h^2 & -2 \\ & 0 \end{pmatrix} u_{i,j} = g_{i,j}$$

The above equations can be written into the matrix equation $Au = g$, conveniently, using lexicographical ordering, where co-efficient matrix is penta-diagonal sparse matrix and u is vectors of unknowns and g is right hand side vector.



(a)



(b)

Figure 3: History of iterative methods (a) Jacobi iterations (b) Damped Jacobi iterations

4. MULTIGRID METHOD

Multigrid method is a numerical method with ultra-robustness features of convergence properties to handle the sparse and large matrices. Many developments are carried out to accelerate the convergence of the system $Au = g$ arose from discretization of PDE's with different numerical approaches but are experienced as expensive in terms of bandwidth. Existing numerical solvers such as Jacobi (Jac), Gauss Seidel (GS), Conjugate gradient (CG), Generalized minimum residual (GMRES) are more expensive in terms of bandwidth of sparse and large matrices as well as number of operations to overcome this issue Multigrid method remains the superior of said methods that uses basic iterative methods as the smoothers along with the coarse grid correction properties. Jacobi method is used as the pre and post smoother as it removes the high frequency errors rapidly that helps the model problems to smooth out the curves using the basic components of Multigrid solver.

This fine-coarse-fine loop is a two-grid V-cycle. We call it a V-cycle (small v). Here are the steps (remember, the error solves $A_h(u - u_h) = g_h - A_h u_h = r_h$):

1. Iterate $A_h u = g_h$ to reach u_h (Say Jacobi or Gauss-Seidel steps).
2. Restrict the residual $r_h = g_h - A_h u_h$ to the coarse grid by $r_{2h} = R_h^{2h} r_h$
3. Solve $A_{2h} e_{2h} = r_{2h}$
4. Interpolate e_{2h} back to $e_h = I_{2h}^h e_{2h}$ add e_h to u_h .
5. Iterate 3 more times on $A_h u = g_h$ starting from the improved $u_h + e_h$.

Steps 2-4 are core theme of multigrid, which restricts errors on next coarse grid, solves there and carries back original grid. Recall the three matrices we are working with:

$$A = A_h = \text{original matrix}$$

$$A = R_h^{2h} = \text{restriction matrix}$$

$$A = I_{2h}^h = \text{interpolation matrix}$$

Step 3 involves another matrix A_{2h} . A_{2h} is analogy of original matrix A_h . It is smaller in dimension than the original A_h . It is called the coarse grid matrix and is often developed using Equation

$$A_{2h} = R_h^{2h} A_h I_{2h}^h. \quad (3)$$

Re-discretization of given equation on domain with discretization parameter $2h$ is another way to obtain coarse grid matrix, which is less preferred (Briggs and McCormick 2000, Trottenberg, Oosterlee et al. 2000).

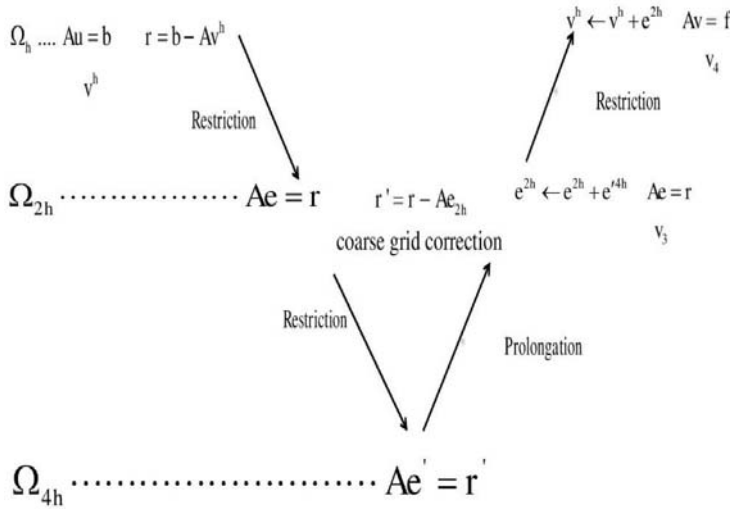


Figure 4: V-Cycle Algorithm

5. RESULTS

In this section, numerical results for all three problems introduced in Chapter 1 are presented. First of all, results concerning one-dimensional problem are presented. Tolerance of 10^{-4} has been used to stop iterations for all problems, unless mentioned exclusively. Results are divided into following section, considering problem nature.

 Table 5.1: Number of iteration for $k = 0$ and with different grid size

Parameters		Number of Iterations by various solvers						
k	N	Jacobi	Seidel	CG	PCG	GMRES	PGMRES	MG
0	8	104	50	7	7	7	5	5
	16	364	169	15	15	15	5	5
	24	750	342	23	23	23	6	6
	32	1243	561	31	31	31	6	6
	40	1834	821	39	39	39	6	-
	48	2513	1117	47	47	47	6	6
	56	3274	1445	55	55	55	6	-
	64	4110	1803	63	63	63	6	-
	72	5016	2188	71	71	71	6	-
	80	5987	2598	78	78	79	6	-

The first most result is about comparison of different solvers, given in Table 5. 1. While approximately solving one-dimensional (1D) Helmholtz equation, the comparison of different solvers with their number of iterations is presented in Table 5.1. CG is conjugate gradient method, whereas PCG is preconditioned (diagonally) conjugate gradient. Diagonal preconditioner seems ineffective at all, as the original problem is weakly diagonal dominant. Same is true for when grid size is halved, which is given in Table 5.2. In Table 5.2, when the mesh size is doubled, the number of iterations of Jacobi remains same while the iterations of CG and PCG are mutually equal. The missing number of

iterations in Multigrid (MG) solvers is because of non-availability of hierarchy of grids. It is important to have the original problem in size $2^n + 1$, unless there will not be a hierarchy of grids. Rapid increase in number of iterations of Jacobi and Seidel methods is because of same reason: weak diagonal dominance.

Table 5.2: Number of iteration for $k = 0$ and with different grid size n

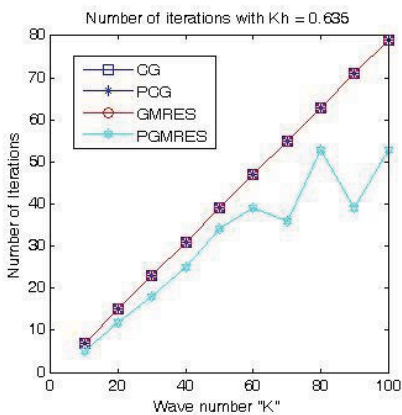
Parameters		Number of Iterations						
k	N	Jacobi	Seidel	CG	PCG	GMRES	PGMRES	MG
0	16	364	169	15	15	15	5	5
	32	1243	561	31	31	31	6	6
	48	2513	1117	47	47	47	6	6
	64	4110	1803	63	63	63	6	6
	80	5987	2598	78	78	79	6	-
	96	8110	3484	92	92	95	6	6
	112	10000+	4446	106	106	111	6	-
	128	10000+	5473	119	119	127	6	6
	144	10000+	6554	132	132	143	6	-
	160	10000+	7680	144	144	159	6	-

Figure 5 presented a comparison of solvers for varying wavenumber k . As wavenumber increases, CG and GMRES iterations increases. BIM simply diverges, hence no motivation for using multigrid. This is explained by the fact that when $k = 0$, the convergence indicator i.e. spectral radius

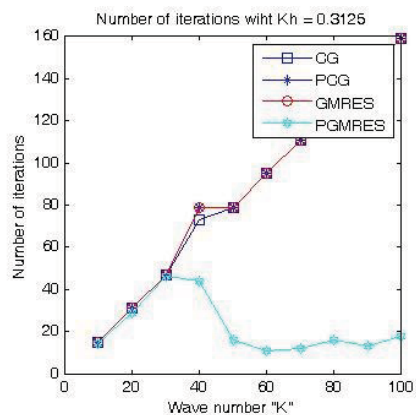
$$\rho(T) = \max \left\{ \left| \lambda \left(I - D^{-1}A \right) \right| \right\}$$

of iteration operator T is $\rho(T) < 1$. Whereas for $k \neq 0$, $\rho(T) > 1$, hence divergence. Same observations are concluded when mesh size is halved for certain wavenumber k , which is given in Figure

5



(a) Halved grid size



(b) Doubled grid size

Figure 5: The number of iterations by various solvers for a range of wavenumber "k" between 10 and 100 with different gridsizes.

In Table 5.1 – 5.2 and Figure 4, it is observed as number of grid points n increases number of iterations in GMRES. Reason is that GMRES (Generalized Minimal Residual) method for iteratively solving non-symmetric matrices, so the matrix used all above tables is symmetric matrix.

The heterogeneity is introduced in one dimensional problem, which, along with Dirichlet's boundary conditions, is given as

$$\begin{cases} -\nabla^2 u(x) - (a - ib)k^2 u(x) = g(x), & u \in \Omega = [0, 1] \\ u(0) = 3 \text{ and } u(1) = 5, & \text{On } \partial\Omega \end{cases} \quad (5.1)$$

A simple 1D numerical experiment is conducted to test the performance of Multigrid method. The heterogeneity is explained in Figure 1, where $k_1 = 1.3k$, $k_2 = k$, $k_3 = 2k$ the 1D problem with varying wavenumber k (Heterogeneity in medium) to measure the convergence behavior in a heterogeneous domain with various different values of k the wave frequency. The results are compared by adopting the Multigrid method (keeping $\omega = 2/3$) with other methods, such comparison is given in Table 5.3. It is notable that Jacobi iterations do not converge, which can be seen from number of iterations 10000 +.

Table 5.3: Multigrid results with fixed wave number $k=0$ or $a=b=0$ and $\omega = \frac{2}{3}$ with different grid size n .

Grid size (n)	Number of Iterations				
	Jacobi	G-Seidel	CG	GMRES	MG
16	364	169	15	15	5
32	1243	561	31	31	6
48	2513	1117	47	47	6
64	4110	1803	63	63	6
80	5987	2598	78	79	----
96	8110	3484	92	95	6
112	10000+	4446	106	111	----
128	10000+	5473	119	127	6
144	10000+	6554	132	143	----
160	10000+	7680	144	159	----

Table 5.4: Multigrid iterations for fixed value of wave number $k=0$, different parameter n and $a=1, b=0, \omega=1$

Parameter (n)	MG ($\omega=1$) Iterations	Spectral Radius
16	152	0.9808
32	152	0.9952
48	152	0.9979
64	152	0.9988

As the parameter i-e grid size is changed and the wave number $k=0$ is taken, Multigrid executes same number of iterations to reach accuracy already set in beginning of the chapter. That shows the

convergence of system at specific number of grids. Reason for consistent number of iterations is spectral radius is very close to one.

Table 5.5: Multigrid iteration for different values of wave number k , parameter n and $a = 1, b = 0, \omega = 1$

Parameter	wave number (k)	MG ($\omega=1$) Iterations	Spectral
16	10	10000+	1.2188
32	20	10000+	1.2367
48	30	10000+	1.2401
64	40	10000+	1.2412

The results, presented in Table 5.5, are taken by taking different values of wave number k , and $\omega = 1$. Since the spectral radius is more than one which does not help smoothers to smooth high frequency i.e causes smoothers diverge. In this case, even the adaptation in relaxation parameter ω will not help.

Next result is given in Figure 6. The Figure shows the Spectral radius analysis of Jacobi Method with variation of k different grid size and shifts (0,0),(0,1). It can be observed that Jacobi works better with purely imaginary shift is applied.

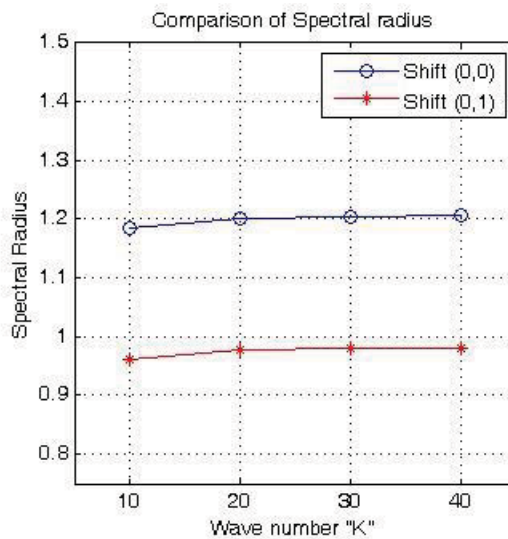


Figure 6. The Figure shows the Spectral radius analysis of Jacobi Method

Table 5.6: Comparison of different methods with various values of k and grid size n and $a = 0, b = 1$

Grid size(n)	Number of Iterations				
	Jacobi	G-Seidel	CG	GMRES	MG
16	186	78	16	16	5
32	256	130	28	32	6
48	260	124	28	35	6
64	253	129	32	37	6
80	251	128	33	39	----
96	252	128	34	38	6
112	252	127	12	21	----
128	252	127	12	20	6
144	254	127	12	20	----
160	254	127	12	20	----

Finally, multigrid iteration are compared with other solvers' iterations for different values of wavenumber, up to $k = 140$. Missing iterations of multigrid is due to unavailability of hierarchal grids. Multigrid seems leading while comparing with other solvers. .

Table 5.7: Comparison of different methods with various values of k and grid size n and $a = 1, b = 1$

Grid size	Number of Iterations				
	Jacobi	G-Seidel	CG	GMRES	MG
16	364	169	15	15	5
32	1243	561	31	31	6
48	2513	1117	47	47	6
64	4110	1803	63	63	6
80	5987	2598	78	79	----
96	8110	3484	92	95	6
112	10000+	4446	106	111	----
128	10000+	5473	119	127	6
144	10000+	6554	132	143	----
160	10000+	7680	144	159	----

Comparing Table 5.9 with Table 5.10, both tables make much difference the difference is due to the values pure imaginary shift and real and imaginary shift with different values of k and in this table real and imaginary part are defined as $a = 1, b = 1$

Impact of relaxation parameter with multigrid iterations is observed in Table 5.8. Relaxation parameter $\omega = \frac{2}{3}$ is best choice and $\omega = 1$ is worst choice. Further, a comparison of various shifts is given Table 5.9, while solving all with multigrid. Results are self-explanatory in said Table 5.9.

Table 5.8: Number of iterations by Multigrid with fixed wavelength $k = 0$ and different damping parameter in smoother Jacobi ω .

Gridsize (n)	MG $\omega=1/2$	MG $\omega=2/3$	MG $\omega=1$
16	7	5	152
32	7	6	152
48	7	6	152
64	7	6	152
80	---	---	152
96	7	6	152
112	---	---	152
128	7	6	152
144	---	---	152
160	---	---	152

Table 5.9: Multigrid iterations with different wavenumber k , and imaginary shift as (0, 0), (0, 1) and (1, 1)

wavenumber	Grid size	Number of Iterations		
(k)	(n)	MG(0,0)	MG(0,1)	MG(1,1)
10	16	5	5	10
20	32	6	5	14
30	48	6	5	14
40	64	6	5	14
60	96	6	5	14
80	128	6	5	14
120	192	6	5	14

Two dimensional problem in heterogeneous medium with Neumann Boundary condition is read as

$$\begin{cases} -\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) - (a - ib)k^2 u = g & , \quad u \in \Omega = [0, 1]^2 \\ u(0, y) = 0 = u(x, 0) \text{ and } u(1, y) = y^4, u(x, 1) = x^3, & \text{On } \partial\Omega \end{cases} \quad (5.2)$$

with different wave number, k , the source function $g(x, y)$ and analytic solution $u(x, y)$ for (3) are chosen as

$$g(x, y) = -6xy^2(x^2 + 2y^2) + 5x^3y^4, \quad u(x, y) = x^3y^4$$

Table 5.10: Multigrid iterations with different wavenumber k and (a,b) as $(0,0)$ and $(1,1)$

wavenumber	Grid size	Number of Iterations	
		MG(0,1)	MG(1,1)
10	18	8	14
20	34	7	11
40	66	7	10
120	130	7	16
160	258	7	10

6. DISCUSSION AND CONCLUSION

In light of the experimental results presented in Section 5, Jacobi is not recommended as solver, rather can be used as smoother in multigrid. Though, the optimization of relaxation parameter ω is required for better smoothing. In case of considered problems, relaxation parameter $\omega = 2/3$ is choice of preference. It is also concluded that multigrid thoroughly works better when pure imaginary shift is incorporated in problem.

The number of pre- and post-smoothing iterations does not influence convergence, however 2 pre- and two post-smoothing iterations are recommended in view of computation time and accuracy. Scalability is observed for problem without shift, however increasing shift bring small increase in number of iterations of multigrid. Scalability in terms of gridsize is highlighted result in this paper. The grid size independent results are confirming scalability, as no increase in number of iterations is seen even with increasing grid size.

7. FUTURE WORK

For future work, it is suggested that different shifts may be considered at lower level grids in Multigrid solver. Also a multilevel analysis of the method is possible to explore further in-depth convergence. This will certainly be interesting and challenging aspect of research and can be considered as future work.

8. REFERENCES

- Bayliss, A., C. I. Goldstein and E. Turkel (1983). "An iterative method for the Helmholtz equation." *Journal of Computational Physics* **49**(3): 443-457.
- Briggs, W. L. and S. F. McCormick (2000). *A multigrid tutorial*, Siam.
- Chaumont Frelet, T. (2015). *Finite element approximation of Helmholtz problems with application to seismic wave propagation*, INSA de Rouen.
- Colton, D. and R. Kress (2013). *Integral equation methods in scattering theory*, SIAM.
- Erlangga, Y. and E. Turkel (2012). "Iterative Schemes for High Order Compact Discretizations to the Exterior Helmholtz Equation." *Esaim-Mathematical Modelling and Numerical Analysis-Modelisation Mathematique Et Analyse Numerique* **46**(3): 647-660.

- Erlangga, Y. A. (2005). "A robust and efficient iterative method for the numerical solution of the Helmholtz equation." DIAM, TU Delft.
- Erlangga, Y. A. and R. Nabben (2008). "On a Multilevel Krylov Method for the Helmholtz Equation Preconditioned by Shifted Laplacian." *Electronic Transactions on Numerical Analysis* **31**: 403-424.
- Plessix, R.-E. (2007). "A Helmholtz iterative solver for 3D seismic-imaging problems." *Geophysics* **72**(5): SM185-SM194.
- Shaikh, A., G., A. H. Sheikh, A. Asif and S. Zeb (2019). "Critical Review of Preconditioners for Helmholtz Equation and their Spectral Analysis." *Indian Journal of Science and Technology* **12**(20).
- Sheikh, A. H. (2014). Development of Helmholtz Solver Based on Shifted Laplace Preconditioner and a Multigrid Deflation Technique. PhD, Delft University of Technology, .
- Sheikh, A. H. (2014). Development of Helmholtz Solver Based on Shifted Laplace Preconditioner and a Multigrid Deflation Technique. The Netherlands, Delft University of Technology.
- Sheikh, A. H., D. Lahaye, L. Garcia Ramos, R. Nabben and C. Vuik (2016). "Accelerating the Shifted Laplace Preconditioner for the Helmholtz Equation by Multilevel Deflation." *J. Comput. Phys.* **322**(C): 473–490.
- Sheikh, A. H., D. Lahaye and C. Vuik (2013). "On the convergence of shifted Laplace preconditioner combined with multilevel deflation." *Numerical Linear Algebra with Applications* **20**: 645–662.
- Sheikh, A. H., C. Vuik and D. Lahaye (2011). A scalable Helmholtz solver combining the shifted Laplace preconditioner with Multigrid deflation, DIAM, TU Delft Netherlands.
- Stenger, F. (2007). "Separation of variables solution of PDE via Sinc methods." *International Journal of Applied Mathematics and Statistics* **10**.
- Trottenberg, U., C. W. Oosterlee and A. Schüller (2000). *Multigrid*. London, Academic Press.
- Wang, X., J. Zou, F. Wang and R. Li (2014). "Mechanism and Analysis of Propagation of Ultrasonic Guided Wave in Rotating Shaft." *International Journal of Applied Mathematics & Statistics* **52**(5): 250-256.