

On generalized antieigenvalue pairs and antieigenvectors of linear two-parameter matrix eigenvalue problems

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ABSTRACT

This paper considers a general framework for generalized antieigenvalue pair and the corresponding antieigenvectors for linear two-parameter matrix eigenvalue problems. Structures of generalized antieigenvectors are discussed with the help of stationary vectors. Properties of generalized antieigenvalue pair with a special normal operator are also established.

Keywords: Linear two-parameter Matrix Eigenvalue Problems, generalized antieigenvalues, generalized antieigenvectors.

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1 Introduction

Let H be any Hilbert Space with corresponding inner product $\langle \cdot, \cdot \rangle$. Then, any operator T on the Hilbert Space H is called accretive if $\operatorname{Re} \langle Tx, x \rangle \geq 0$, $\forall x \neq 0$ and strictly accretive if $\operatorname{Re} \langle Tx, x \rangle > 0$, $\forall x \neq 0$ [Seddighin, et. al, 2005]. The concept of antieigenvalue was first notified in [Gustafson, 1968] for an accretive operator, while Gustafson studied some special problems of semi-group generators in perturbation theory. For a given accretive operator T on the Hilbert space H , Gustafson defined the first antieigenvalue of the operator T as follows:

$$\mu_1(T) = \min_{Tx \neq 0} \frac{\operatorname{Re} \langle Tx, x \rangle}{\|Tx\| \|x\|} \quad (1.1)$$

A vector x for which infimum of (1.1) can be obtained is called antieigenvector of T . Antieigenvalues are more difficult to calculate than eigenvalues due to their nonlinear Euler equations involved in the theory. The quantity $\mu_1(T)$ can also be denoted by $\cos(T)$ and is termed as cosine of the angle of T . It has its applications in diverse scientific domains, particularly in the area of numerical range of operator, numerical tools for optimization techniques as well as in statistics and econometrics [Gustafson, 2000; Gustafson, 2002 and Gustafson, et.al 1997]. Gustafson and Seddighin studied extensively the theories on antieigenvalues of normal operator in finite and infinite dimensional spaces [Gustafson, 1994, Seddighin, 2002; Seddighin, 2003 and Seddighin, 2005]. Researchers have been studied over the years to compute approximate antieigenvalue for operators on Hilbert spaces. In particular, computation aspects and

approximation of antieigenvalue for normal operators have been reported in [Mirman, 1983; Gustafson, et.al., 1997; Paul, et. al., 2015; Seddighin, et. al., 2005; Seddighin, 2002 and Seddighin, 2005].

The rest of the paper has been organized as follows. In Section 2 an abstract formulation Linear two-parameter matrix eigenvalue problem is presented. In Section 3 general framework of generalized antieigenvalue pair and the corresponding generalized antieigenvectors are discussed. Similarly, in Section 4 we present Structure of generalized antieigenvectors and in Section 5 some properties of generalized antieigenvalue pairs with (α, β) -normal operator are established. Finally, in Section 6 numerical example is given.

2 Linear two-parameter matrix eigenvalue problems (LTMEP)

We consider LTMEP of the form given below

$$\begin{aligned}\mathbb{W}_1(\lambda, \mu)x_1 &:= (T_1 - \lambda B_{11} - \mu B_{12})x_1 = 0 \\ \mathbb{W}_2(\lambda, \mu)x_2 &:= (T_2 - \lambda B_{21} - \mu B_{22})x_2 = 0\end{aligned}\quad (2.1)$$

where $\lambda, \mu \in \mathbb{C}$; $x_i \in \mathbb{C}^{n_i}$; and T_i, B_{ij} are $n_1 \times n_2$ over $\mathbb{C}$; $i, j = 1, 2$. We denote the problem (2.1) by $\mathbb{W}$. If for some λ, μ the problem $\mathbb{W}$ has a solution $x_i \neq 0$; $i := 1, 2$, then the pair (λ, μ) is called eigenvalue and the corresponding tensor product $x = x_1 \otimes x_2$ is called the eigenvector, where $\otimes$ stands for Kronecker product. The origin of the problem $\mathbb{W}$ can be traced back to mathematical physics [Volkmer, 1988 and Cottin, 2001]. The spectral theory and its related classical results can be found in the books [Sleeman, 1978 and Atkinson, 1972] and in the papers [Hochstenbach, et. al, 2003 and kosir, 1994]. The standard method to study the spectrum of $\mathbb{W}$ by transforming it into a commuting tuple of operators matrices by considering the following operator determinants

$$\Delta_0 := B_{11} \otimes B_{22} - B_{12} \otimes B_{21} \quad (2.2)$$

$$\Delta_1 := T_1 \otimes B_{22} - B_{12} \otimes T_2; \quad \Delta_2 := B_{11} \otimes T_2 - T_1 \otimes B_{21} \quad (2.3)$$

Generally, for spectral analysis the problem $\mathbb{W}$ is considered as nonsingular i.e when Δ_0 is nonsingular. A nonsingular system (2.1) can be transformed into a system of generalized eigenvalue problems (GEPs) [Atkinson, 1972] of the form

$$\begin{aligned}\Delta_1 x &= \lambda \Delta_0 x \\ \Delta_2 x &= \mu \Delta_0 x\end{aligned}\quad (2.4)$$

For nonsingular $\mathbb{W}$ the matrices $\Delta_0^{-1} \Delta_i$ commute for $i = 1, 2$, and all the eigenvalues of (2.1) are also the eigenvalues of (2.4).

3 Generalized antieigenvalue pairs and antieigenvectors

We consider that the problem $\mathbb{W}$ is nonsingular, i.e, the determinantal operator Δ_0 is nonsingular. In addition to this, we also consider that each operator Δ_i ; $i = 1, 2$ are nonsingular.

Extending the idea of [Paul, 2008], we define two parameters $\lambda_1(T)$ and $\mu_1(T)$ for the GEPs of the form (2.4) as follows:

$$\lambda_1(T) := \min \left\{ \frac{\operatorname{Re} \langle \Delta_1 x, \Delta_0 x \rangle}{\|\Delta_1 x\| \|\Delta_0 x\|} : x \in H, \Delta_1 x \neq 0, \Delta_0 x \neq 0 \right\} \quad (3.1)$$

$$\mu_1(T) := \min \left\{ \frac{\operatorname{Re} \langle \Delta_2 x, \Delta_0 x \rangle}{\|\Delta_2 x\| \|\Delta_0 x\|} : x \in H, \Delta_2 x \neq 0, \Delta_0 x \neq 0 \right\} \quad (3.2)$$

The pair $(\lambda_1(T), \mu_1(T))$ is called generalized antieigenvalue pair for the system (2.4), and the vector x for which $\min \left\{ \frac{\operatorname{Re} \langle \Delta_1 x, \Delta_0 x \rangle}{\|\Delta_1 x\| \|\Delta_0 x\|} \right\}$ and $\min \left\{ \frac{\operatorname{Re} \langle \Delta_2 x, \Delta_0 x \rangle}{\|\Delta_2 x\| \|\Delta_0 x\|} \right\}$ are attained is called generalized antieigenvectors corresponding to the pair $(\lambda_1(T), \mu_1(T))$. The inner products $\langle \Delta_i x, \Delta_0 x \rangle$ present in equations (3.1) and (3.2) can also be represented in following ways

$$\langle \Delta_i x, \Delta_0 x \rangle = \langle \Delta_i \Delta_0^{-1} y, y \rangle$$

where $\Delta_0 x = y$ and $i = 1, 2$. Thus, the expressions in equations (3.1) and (3.2) reduces to

$$\lambda_1(T) := \min \left\{ \frac{\operatorname{Re} \langle \Delta_1 \Delta_0^{-1} y, y \rangle}{\|\Delta_1 \Delta_0^{-1} y\| \|y\|} : y \in H, \Delta_1 \Delta_0^{-1} y \neq 0, y \neq 0 \right\} \quad (3.3)$$

$$\mu_1(T) := \min \left\{ \frac{\operatorname{Re} \langle \Delta_2 \Delta_0^{-1} y, y \rangle}{\|\Delta_2 \Delta_0^{-1} y\| \|y\|} : y \in H, \Delta_2 \Delta_0^{-1} y \neq 0, y \neq 0 \right\} \quad (3.4)$$

Since the matrices T_i and B_{ij} are of dimension $n_1 \times n_2$, and hence the dimension of Δ_i becomes $n_1^2 \times n_2^2$. This increase in dimension makes it difficult to calculate generalized antieigenvalue pairs $(\lambda_1(T), \mu_1(T))$ for the matrices of higher size. Again, if $A = [a_{ij}]$; $i, j = 1, \dots, n$ is any $n \times n$ matrix over $\mathbb{R}$ and $f = (x_1, x_2, \dots, x_n) \in \mathbb{R}^n$ be any vector, then the direct computation of the functional $\langle Af, f \rangle / \|Af\|$ takes the form [Seddighin, 2003] below:

$$J(x_1, x_2, \dots, x_n) = \frac{\sum_i^n (\sum_j^n a_{ij} x_j) x_i}{\sqrt{\sum_i^n (\sum_j^n a_{ij} x_j)^2}} \quad (3.5)$$

Thus in order to compute the pair $(\lambda_1(T), \mu_1(T))$, we need to minimize $J(x_1, x_2, \dots, x_n)$ for the matrix operators $\Delta_1 \Delta_0^{-1}$ and $\Delta_2 \Delta_0^{-1}$, respectively over the unit sphere $\sum_{i=1}^n x_i^2 = 1$. For that we have to optimize the problem. The usual approach to optimize $J(x_1, x_2, \dots, x_n)$ is by using Lagrange multiplier techniques. However, the computation becomes very difficult even for the matrices of small size due to the resulting nonlinear equations. For example, if $A = [a_{ij}]$ is any 4×4 matrix over $\mathbb{R}$, then substituting the entries of the matrix A in equation (3.5), we get

$$J(x_1, x_2, x_3, x_4) = \frac{p(x_1, x_2, x_3, x_4)}{q(x_1, x_2, x_3, x_4)} \quad (3.6)$$

where

$$p(x_1, x_2, x_3, x_4) = a_{11}x_1^2 + a_{22}x_2^2 + a_{33}x_3^2 + a_{44}x_4^2 + (a_{12} + a_{21})x_1x_2 + (a_{13} + a_{31})x_1x_3 + (a_{14} + a_{41})x_1x_4 + (a_{23} + a_{32})x_2x_3 + (a_{24} + a_{42})x_2x_4 + (a_{34} + a_{43})x_3x_4 \quad (3.7)$$

$$q(x_1, x_2, x_3, x_4) = \sqrt{[(a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + a_{14}x_4)^2 + (a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + a_{24}x_4)^2 + (a_{31}x_1 + a_{32}x_2 + a_{33}x_3 + a_{34}x_4)^2 + (a_{41}x_1 + a_{42}x_2 + a_{43}x_3 + a_{44}x_4)^2]} \quad (3.8)$$

A necessary condition for the vector $f = f(x_1, x_2, x_3, x_4)$ to be an optimum vector for $J(x_1, x_2, x_3, x_4)$ over $\sum_{i=1}^4 x_i^2 = 1$ is that the gradient of $J(x_1, x_2, x_3, x_4)$ and $\sum_{i=1}^4 x_i^2 = 1$ must be parallel. To optimize $J(x_1, x_2, x_3, x_4)$ over $\sum_{i=1}^4 x_i^2 = 1$, we need to solve the system

$$\frac{\partial J}{\partial x_1} = 2Kx_1, \quad \frac{\partial J}{\partial x_2} = 2Kx_2, \quad \frac{\partial J}{\partial x_3} = 2Kx_3, \quad \frac{\partial J}{\partial x_4} = 2Kx_4, \quad x_1^2 + x_2^2 + x_3^2 + x_4^2 = 1 \quad (3.9)$$

for some nonzero constant K. Elimination of K from (3.9), yields a nonlinear system. This system can be solved algebraically for some special matrices only, but numerically it can be solved for all cases [Gustafson, 1994]. Now, in expanded form system (3.9) reduces to

$$\frac{\partial J}{\partial x_1} = \frac{2a_{11}x_1 + (a_{12} + a_{21})x_2 + (a_{13} + a_{31})x_3 + (a_{14} + a_{41})x_4}{q(x_1, x_2, x_3, x_4)} - p(x_1, x_2, x_3, x_4) \times \frac{g_1(x_2, x_3, x_4)}{q(x_1, x_2, x_3, x_4)^3} = 2Kx_1 \quad (3.10)$$

$$\frac{\partial J}{\partial x_2} = \frac{2a_{22}x_2 + (a_{12} + a_{21})x_1 + (a_{23} + a_{32})x_3 + (a_{24} + a_{42})x_4}{q(x_1, x_2, x_3, x_4)} - p(x_1, x_2, x_3, x_4) \times \frac{g_2(x_1, x_3, x_4)}{q(x_1, x_2, x_3, x_4)^3} = 2Kx_2 \quad (3.11)$$

$$\frac{\partial J}{\partial x_3} = \frac{2a_{33}x_3 + (a_{13} + a_{31})x_1 + (a_{23} + a_{32})x_2 + (a_{43} + a_{34})x_4}{q(x_1, x_2, x_3, x_4)} - p(x_1, x_2, x_3, x_4) \times \frac{g_3(x_1, x_2, x_4)}{q(x_1, x_2, x_3, x_4)^3} = 2Kx_3 \quad (3.12)$$

$$\frac{\partial J}{\partial x_4} = \frac{2a_{44}x_4 + (a_{14} + a_{41})x_1 + (a_{24} + a_{42})x_2 + (a_{34} + a_{43})x_3}{q(x_1, x_2, x_3, x_4)} - p(x_1, x_2, x_3, x_4) \times \frac{g_4(x_1, x_2, x_3)}{q(x_1, x_2, x_3, x_4)^3} = 2Kx_4 \quad (3.13)$$

$$x_1^2 + x_2^2 + x_3^2 + x_4^2 = 1 \quad (3.14)$$

where

$$g_1(x_2, x_3, x_4) = (a_{11}^2 + a_{21}^2 + a_{31}^2 + a_{41}^2)x_1^2 + (a_{11}a_{12} + a_{21}a_{22} + a_{31}a_{32} + a_{41}a_{42})x_2 + (a_{11}a_{13} + a_{21}a_{23} + a_{31}a_{33} + a_{41}a_{43})x_3 + (a_{11}a_{14} + a_{21}a_{24} + a_{31}a_{34} + a_{41}a_{44})x_4$$

$$g_2(x_1, x_3, x_4) = (a_{12}^2 + a_{22}^2 + a_{32}^2 + a_{42}^2)x_2^2 + (a_{11}a_{12} + a_{21}a_{22} + a_{31}a_{32} + a_{41}a_{42})x_1 + (a_{13}a_{12} + a_{23}a_{22} + a_{33}a_{32} + a_{43}a_{42})x_3 + (a_{14}a_{12} + a_{24}a_{22} + a_{34}a_{32} + a_{44}a_{42})x_4$$

$$g_3(x_1, x_2, x_4) = (a_{13}^2 + a_{23}^2 + a_{33}^2 + a_{43}^2)x_3^2 + (a_{11}a_{13} + a_{21}a_{23} + a_{31}a_{33} + a_{41}a_{43})x_1 + (a_{12}a_{13} + a_{22}a_{23} + a_{32}a_{33} + a_{42}a_{43})x_2 + (a_{14}a_{13} + a_{24}a_{23} + a_{34}a_{33} + a_{44}a_{43})x_4$$

$$g_4(x_1, x_2, x_3) = (a_{14}^2 + a_{24}^2 + a_{34}^2 + a_{44}^2)x_4^2 + (a_{11}a_{14} + a_{21}a_{24} + a_{31}a_{34} + a_{41}a_{44})x_1 + (a_{12}a_{14} + a_{22}a_{24} + a_{32}a_{34} + a_{42}a_{44})x_2 + (a_{13}a_{14} + a_{23}a_{24} + a_{33}a_{34} + a_{43}a_{44})x_3$$

We can derive respective nonlinear equations of similar form as in equations (3.10)-(3.14) for the matrix operators $\Delta_1\Delta_0^{-1}$ and $\Delta_1\Delta_0^{-1}$, separately. By solving these, the solution sets (x_1, x_2, x_3, x_4) can be obtained. Then minimum values of $J(x_1, x_2, x_3, x_4)$ corresponding to $\Delta_1\Delta_0^{-1}$ and $\Delta_1\Delta_0^{-1}$ can be calculated directly and hence the pair $(\lambda_1(T), \mu_1(T))$. It is well known that, for a positive operator T, we have $\mu_1(T) = \frac{2\sqrt{mM}}{m+M}$, where m and M are respectively the smallest and largest eigenvalues of the operator T. This was first mentioned in [Gustafson, 1994]. Generalizing this results the following theorem can be proved.

Theorem 3.1. *Let the determinantal matrix operators $\Delta_i\Delta_0^{-1}$; $i = 1, 2$ are positive with eigenvalues $\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_k$ and $\mu_1 \geq \mu_2 \geq \dots \geq \mu_k$ respectively, then $(\lambda_1(T), \mu_1(T)) = (2\frac{\sqrt{\lambda_1\lambda_k}}{\lambda_1+\lambda_k}, 2\frac{\sqrt{\mu_1\mu_k}}{\mu_1+\mu_k})$.*

Using Theorem 3.1 generalized antieigenvectors for linear two-parameter matrix eigenvalue problems can be computed, with a restriction on the operators $\Delta_i\Delta_0^{-1}$; $i = 1, 2$ that they must be positive.

4 Structure of generalized antieigenvectors of $\mathbb{W}$

In this section, the structure of generalized antieigenvalue pairs and the corresponding generalized antieigenvectors of the problem $\mathbb{W}$ are studied using stationary vector. The concept stationary vector has been used extensively in [Das, Gupta and Paul, 1998] for strictly accretive operator and in [Paul, 2008] for generalized eigenvalue problems. Let us denote,

$$g(x) := \frac{Re \langle \Delta_1 x, \Delta_0 x \rangle}{\|\Delta_1 x\| \|\Delta_0 x\|}; x \in H, \Delta_1 x \neq 0, \Delta_0 x \neq 0 \quad (4.1)$$

Then, the functional $g(x)$ is said to have a stationary value at $0 \neq x \in H$, if the function $w_h(t)$ of real variable t , defined as

$$w_h(t) := g\left(\frac{x + th}{\|x + th\|}\right) \quad (4.2)$$

has a stationary value at $t = 0$ for any arbitrary but fixed vector $h \in H$. The vector x is called a stationary vector [Paul, et.al., 2015], (Definition 3). In other words, at a stationary vector we must have $w'_h(t) = 0$. A straightforward extension of the ideas presented in [Hossein, et. al., 2008] shows that $g(x)$ is stationary at x if and only if

$$\|\Delta_1 x\|^2 \|\Delta_0 x\|^2 (\Delta_1^* \Delta_0 + \Delta_0^* \Delta_1) x - \left\langle \frac{\Delta_1^* \Delta_0 + \Delta_0^* \Delta_1}{2} x, x \right\rangle \left\{ \|\Delta_1 x\|^2 \Delta_0^* \Delta_0 x + \|\Delta_0 x\|^2 \Delta_1^* \Delta_1 x \right\} = 0 \quad (4.3)$$

and

$$\|\Delta_2 x\|^2 \|\Delta_0 x\|^2 (\Delta_2^* \Delta_0 + \Delta_0^* \Delta_2) x - \left\langle \frac{\Delta_2^* \Delta_0 + \Delta_0^* \Delta_2}{2} x, x \right\rangle \left\{ \|\Delta_2 x\|^2 \Delta_0^* \Delta_0 x + \|\Delta_0 x\|^2 \Delta_2^* \Delta_2 x \right\} = 0 \quad (4.4)$$

$\forall x \in H$. Using this the following theorem can be proved .

Theorem 4.1. *Let $\Delta_i^* \Delta_0 = \Delta_0^* \Delta_i; i = 1, 2$ and x is the generalized antieigenvector of the system (2.4). Then, $\Delta_0 x$ can be expressed as a linear combination of either any two generalized eigenvectors of $\Delta_1^* x = \lambda \Delta_0^* x$ or $\Delta_2^* x = \mu \Delta_0^* x$.*

Proof. Since x is a generalized antieigenvector and hence x is also a stationary vector of $g(x)$. Hence x satisfies equations (4.3) and (4.4). As per our assumptions $\Delta_i^* \Delta_0 = \Delta_0^* \Delta_i$ and so equations (4.3) and (4.4) reduces to

$$\|\Delta_1 x\|^2 \|\Delta_0 x\|^2 2\Delta_1^* \Delta_0 x - \langle \Delta_1^* \Delta_0 x, x \rangle \left\{ \|\Delta_1 x\|^2 \Delta_0^* \Delta_0 x + \|\Delta_0 x\|^2 \Delta_1^* \Delta_1 x \right\} = 0 \quad (4.5)$$

$$\|\Delta_2 x\|^2 \|\Delta_0 x\|^2 2\Delta_2^* \Delta_0 x - \langle \Delta_2^* \Delta_0 x, x \rangle \left\{ \|\Delta_2 x\|^2 \Delta_0^* \Delta_0 x + \|\Delta_0 x\|^2 \Delta_2^* \Delta_2 x \right\} = 0 \quad (4.6)$$

On simplification, we get

$$\Delta_1^* \Delta_1 x - \frac{\|\Delta_1 x\|^2}{\langle \Delta_1^* \Delta_0 x, x \rangle} \Delta_1^* \Delta_0 x = \frac{\|\Delta_1 x\|^2}{\langle \Delta_1^* \Delta_0 x, x \rangle} \Delta_1^* \Delta_0 x - \frac{\|\Delta_1 x\|^2}{\|\Delta_0 x\|^2} \Delta_0^* \Delta_0 x \quad (4.7)$$

$$\Delta_2^* \Delta_2 x - \frac{\|\Delta_2 x\|^2}{\langle \Delta_2^* \Delta_0 x, x \rangle} \Delta_2^* \Delta_0 x = \frac{\|\Delta_2 x\|^2}{\langle \Delta_2^* \Delta_0 x, x \rangle} \Delta_2^* \Delta_0 x - \frac{\|\Delta_2 x\|^2}{\|\Delta_0 x\|^2} \Delta_0^* \Delta_0 x \quad (4.8)$$

Let us assume that $\Delta_i x = \lambda \Delta_0 x + h$ where $\langle \Delta_0 x, h \rangle = 0$, then

$$\|\Delta_i x\|^2 - \frac{|\langle \Delta_i x, \Delta_0 x \rangle|^2}{\langle \Delta_0 x, \Delta_0 x \rangle} = \|h\|^2$$

Using equation (4.7), we have

$$\begin{aligned} \Delta_1^* \Delta_1 x - \frac{\|\Delta_1 x\|^2}{\|\Delta_0 x\| g(x)} \Delta_1^* \Delta_0 x \pm \frac{\|h\|}{\|\Delta_0 x\| g(x)} \Delta_1^* \Delta_0 x \\ = \pm \frac{\|h\|}{\|\Delta_0 x\| g(x)} \Delta_1^* \Delta_0 x + \frac{\|\Delta_1 x\|}{\|\Delta_0 x\| g(x)} \Delta_1^* \Delta_0 x - \frac{\|\Delta_1 x\|^2}{\|\Delta_0 x\|^2} \Delta_0^* \Delta_0 x \end{aligned} \quad (4.9)$$

$$\begin{aligned} \Rightarrow \Delta_1^* \left[\Delta_1 x - \frac{\|\Delta_1 x\|}{\|\Delta_0 x\| g(x)} \Delta_0 x \pm \frac{\|h\|}{\|\Delta_0 x\| g(x)} \Delta_0 x \right] \\ = \frac{\|\Delta_1 x\| \pm \|h\|}{g(x) \|\Delta_0 x\|} \Delta_0^* \left[\Delta_1 x - \frac{\|\Delta_1 x\|}{\|\Delta_0 x\|^2 g(x)} \Delta_0^* x \pm \frac{\|h\|}{\|\Delta_0 x\|^2 g(x)} \Delta_0^* x \right] \end{aligned} \quad (4.10)$$

which implies

$$\Delta_1^* g_1 = \lambda_1 \Delta_0^* g_1, \quad \Delta_1^* g_2 = \lambda_2 \Delta_0^* g_2$$

where

$$\begin{aligned} z_1 = \Delta_0 x - \frac{\|\Delta_1 x\| - \|h\|}{\|\Delta_0 x\| g(x)} \Delta_0 x, \quad z_2 = \Delta_0 x - \frac{\|\Delta_1 x\| + \|h\|}{\|\Delta_0 x\| g(x)} \Delta_0 x \\ \lambda_1 = \frac{\|\Delta_1 x\| + \|h\|}{\|\Delta_0 x\|^2 g(x)} \Delta_0^* x, \quad \lambda_2 = \frac{\|\Delta_1 x\| - \|h\|}{\|\Delta_0 x\|^2 g(x)} \Delta_0^* x \end{aligned}$$

Thus z_1 and z_2 are two eigenvectors of $\Delta_1^* x = \lambda \Delta_0^* x$ with respective eigenvalues λ_1 and λ_2 . Hence

$$\Delta_0 x = \frac{\|\Delta_0 x\| g(x)}{2 \|h\|} (z_1 - z_2) \quad (4.11)$$

Similarly, using equation (4.8), we have

$$\Delta_0 x = \frac{\|\Delta_0 x\| g(x)}{2 \|h\|} (z_3 - z_4) \quad (4.12)$$

where z_3 and z_4 are two eigenvectors of $\Delta_2^* x = \lambda \Delta_0^* x$ defined by replacing Δ_1 by Δ_2 in z_1 and z_2 with respective eigenvalues μ_1 and μ_2 which has derived by replacing Δ_1 by Δ_2 in λ_1 and λ_2 , respectively. Equations (4.11) and (4.12) proves the theorem. $\square$

5 Properties with (α, β) -normal operator

In this section, some properties of generalized antieigenvector pair with (α, β) -normal operators are presented.

Definition 5.1 (Dragomir and Moslehian, 2008). An operator T acting on the Hilbert Space H is called (α, β) -normal for the real numbers α and β with $0 \leq \alpha \leq 1 \leq \beta$ if

$$\alpha^2 T^* T \leq T^* T \leq \beta^2 T^* T \quad (5.1)$$

From definition (5.1), it follows that

$$\alpha^2 \langle T^* T x, x \rangle \leq \langle T^* T x, x \rangle \leq \beta^2 \langle T^* T x, x \rangle \quad (5.2)$$

Theorem 5.1. Let the operators $\Delta_i \Delta_0^{-1}; i = 1, 2$ are (α, β) -normal on the Hilbert Space H such that $0 \leq \alpha \leq 1 \leq \beta$ and $z \in \mathbb{C}$. Then $z + \Delta_i \Delta_0^{-1}$ are (α, β) -normal if each component of the pair $(\lambda_1(\bar{z}T), \mu_1(\bar{z}T))$ is non negative.

Proof. First we claim that $|z|^2 + 2\operatorname{Re} \langle \Delta_i \Delta_0^{-1} y, y \rangle \geq 0$. Assume that each component of the pair $(\lambda_1(\bar{z}T), \mu_1(\bar{z}T))$ is non negative. Then,

$$\frac{\operatorname{Re} \langle \bar{z} \Delta_i \Delta_0^{-1} y, y \rangle}{\|\bar{z} \Delta_i \Delta_0^{-1} y\| \|y\|} \geq 0, \text{ for all } y \in H, \Delta_i \Delta_0^{-1} y \neq 0, y \neq 0$$

$$\frac{\operatorname{Re} \langle \bar{z} \Delta_i \Delta_0^{-1} y, y \rangle}{|z| \|\Delta_i \Delta_0^{-1} y\| \|y\|} \geq 0, \text{ for all } y \in H, \Delta_i \Delta_0^{-1} y \neq 0, y \neq 0$$

which implies $\operatorname{Re} \langle \Delta_i \Delta_0^{-1} y, y \rangle \geq 0$ and hence,

$$|z|^2 + 2\operatorname{Re} \langle \bar{z} \Delta_i \Delta_0^{-1} y, y \rangle \geq 0$$

Thus for any $y \in H$ with $y \neq 0$,

$$\begin{aligned} & \alpha^2 \langle (z + \Delta_i \Delta_0^{-1})^* (z + \Delta_i \Delta_0^{-1}) x, x \rangle \\ &= \alpha^2 \left[\langle |z|^2 x, x \rangle + \langle \bar{z} \Delta_i \Delta_0^{-1} x, x \rangle + \langle z \Delta_i \Delta_0^{-1} x, x \rangle \right] + \alpha^2 \langle (\Delta_i \Delta_0^{-1})^* \Delta_i \Delta_0^{-1} x, x \rangle \\ &\leq \alpha^2 \left[\langle |z|^2 x, x \rangle + \langle \bar{z} \Delta_i \Delta_0^{-1} x, x \rangle + \langle z \Delta_i \Delta_0^{-1} x, x \rangle \right] + \alpha^2 \langle (\Delta_i \Delta_0^{-1})^* \Delta_i \Delta_0^{-1} x, x \rangle \\ &= \langle (z + \Delta_i \Delta_0^{-1})^* (z + \Delta_i \Delta_0^{-1}) x, x \rangle \\ &\leq \beta^2 \left[\langle |z|^2 x, x \rangle + \langle \bar{z} \Delta_i \Delta_0^{-1} x, x \rangle + \langle z \Delta_i \Delta_0^{-1} x, x \rangle \right] + \beta^2 \langle (\Delta_i \Delta_0^{-1})^* \Delta_i \Delta_0^{-1} x, x \rangle \\ &= \beta^2 \langle (z + \Delta_i \Delta_0^{-1})^* (z + \Delta_i \Delta_0^{-1}) x, x \rangle \end{aligned}$$

Hence, $z + \Delta_i \Delta_0^{-1}$ are (α, β) -normal operators. □

6 Numerical Illustrations

Example 1: Consider the problem in equations (6.1) and (6.2) with real diagonal matrices:

$$\mathbb{W}_1(\lambda_1, \lambda_2)x_1 := \left[\begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix} - \lambda_1 \begin{pmatrix} 8 & 0 \\ 0 & 9 \end{pmatrix} - \lambda_2 \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix} \right] x_1 = 0 \quad (6.1)$$

$$\mathbb{W}_2(\lambda_1, \lambda_2)x_2 := \left[\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - \lambda_1 \begin{pmatrix} 3 & 0 \\ 0 & 1 \end{pmatrix} - \lambda_2 \begin{pmatrix} 6 & 0 \\ 0 & 7 \end{pmatrix} \right] x_2 = 0 \quad (6.2)$$

$$\text{Here } \Delta_0 = \begin{pmatrix} 45 & 0 & 0 & 0 \\ 0 & 55 & 0 & 0 \\ 0 & 0 & 48 & 0 \\ 0 & 0 & 0 & 61 \end{pmatrix}; \quad \Delta_1 \Delta_0^{-1} = \begin{pmatrix} 0.2444 & 0 & 0 & 0 \\ 0 & 0.2364 & 0 & 0 \\ 0 & 0 & 0.0833 & 0 \\ 0 & 0 & 0 & 0.0820 \end{pmatrix}$$

$$\Delta_2 \Delta_0^{-1} = \begin{pmatrix} 0.0444 & 0 & 0 & 0 \\ 0 & 0.1091 & 0 & 0 \\ 0 & 0 & 0.1250 & 0 \\ 0 & 0 & 0 & 0.1311 \end{pmatrix}$$

Clearly, Δ_0 is nonsingular and hence the problem considered is nonsingular. Moreover, the operators $\Delta_i \Delta_0^{-1}$ are positive. Computed generalized antieigenvector pair is shown in Table 1 using Theorem 3.1.

Eigenvalues of $\Delta_1\Delta_0^{-1}$	Eigenvalues of $\Delta_2\Delta_0^{-1}$	$(\lambda_1(T), \mu_1(T))$
+0.2444	+0.0444	(+0.0820, +0.0444)
+0.2364	+0.1091	
+0.0833	+0.1250	
+0.0820	+0.1311	

Table 1: Generalized antieigenvalue pair

7 Conclusion

In this paper a unified framework for generalized antieigenvalue pair and the corresponding antieigenvectors are presented for LTMEPs. Properties of generalized antieigenvalue pair with (α, β) -normal are established. Antieigenvalue pair is also evaluated for nonsingular LTMEP.

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