

Effect of Stage Structure in Predator-Prey Model with Crowley-Martin Functional Response

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ABSTRACT

In this paper, we study the effect of stage-structure on the dynamics of predator prey system with Crowley-Martin functional response. It is assumed that immature predators cannot catch the prey and their food is given to them by their parents (mature predators). We have obtained boundary and interior equilibrium points of the system. We have also obtained conditions ensuring the local stability of the equilibrium points, by analyzing eigenvalues of associated characteristic equation of linearized system. Permanence of the system has also been obtained. Numerical simulation are carried out based on four important control parameters: interference among predators, handling time, death rate of mature prey and capturing rate. We have proved that predators gradually decrease and die out as we increase b (interference among predators). Further, when stage structure is considered in both prey and predator population, predator die out faster in comparison with the case when there is no stage structure in predator. We have also proved that the predator population decreases with increase in handling rate and the decrease in predator population is more in system when stage structure in predator is considered. Based on control parameter, death rate of mature prey, we obtained the result that the prey and predator population will survive for much longer period when there is stage structuring in both prey and predator population, even if we increase the death rate to a large value. Finally, we proved that with the decrease in capturing rate, the predator population increases.

Keywords: Predator-Prey system, Crowley-Martin functional, Stage Structure, stability, Permanence.

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1 Introduction

Due to its universal existence, one of the dominant themes in both ecology and mathematical ecology is the study of dynamic relationship between predator and prey. In the depiction of dynamic associations, a significant component of all models is the classic definition of functional response. Many varieties of prey-predator models with Holling type, Leslie-Gower type etc. have been explored broadly by the scholars [(Crowley and Martin, 1989), (Upadhyay and Naji, 2009), (Jazar, 2013), (Dong, Ma and Sun, 2013), (Shi, Zhou and Song, 2011), (X.Y. Meng, 2011), (Hale and Waltman, 1989), (Thieme, 1993), (Aybar, 2018), (Indrajaya and Alghofari, 2016)]. C.S. Holling introduced three types of functional response: Type I assumes a linear increase in intake rate with food density. Type II is characterized by a decelerating intake rate. Type III functional response is similar to type II in that at high levels of prey density, saturation occurs. The Leslie Gower formulation is based on the assumption that reduction in a predator population has a reciprocal relationship with per capita availability of its preferred food and

stressed on the fact that there are upper limits to the confinement of increment of both prey and predator, which are not recognized in the Lotka Volterra model.

In this paper, we have studied the extension of the above models by incorporating both stages of prey and predator species, immature and mature of the Crowley and Martin functional response.

Two cases are possible which are as follows:

Case 1 : If $a > 0, b = 0$, then it reduces to a Michaelis-Menten functional response(or Holling type II functional response)(Freedman, 1980).

Case 2 : If $a = 0, b = 0$, then it reduces to a linear mass-action functional response(or Holling type I functional response)(Seifert, 1998).

The paper is organized as follows: In the second section we have described the model. In the third section, we have obtained boundary and interior the equilibrium points, the conditions under which unique interior equilibrium point exists also by analyzing the eigenvalues of associated characteristic equation of linearized system the conditions ensuring the local stability of the equilibrium points are obtained. In the fourth section, we have discussed permanence of the system. Later, in the fifth section we have performed numerical simulations. In the last section we have concluded the results.

2 Mathematical Model

In this section, we formulate the mathematical model describing the stage structured predator- prey system with stage structure for both prey and predator with Crowley-Martin functional response.

1. The immature prey population is growing logistically at the rate r with carrying capacity r/a_1 . Since the immature prey becomes mature at the rate d_1 , the equation becomes :

$$\frac{dx_1}{dt} = x_1(r - a_1x_1) - d_1x_1 \quad (2.1)$$

where $x_1(t)$ is the density of immature prey species at time t , r is the intrinsic growth rate of immature prey species, $\frac{r}{a_1}$ is the carrying capacity of the environment, d_1 is the transition rate for immature prey individuals to the mature prey.

2. The mature prey is growing at the transition rate d_1 . Also, since Crowley-Martin functional response is a predator-dependent functional response, which is a function of both prey and predator abundance because of predator interference, it is assumed that predator feeding rate decreases by higher predator density even when prey density is high and therefore the effects of predator interference on feeding rate is necessary whether an individual predator is handling or searching for a prey at a given instant of time. Thus, the per-capita feeding rate is given by:

$$\frac{\omega x_2}{1 + ax_2 + by_2 + abx_2y_2} \quad (2.2)$$

Here the effects of capture rate, handling time and the magnitude of interference among predators, on the feeding rate is explained by the positive parameters ω , a , b respectively.

Also, the mature prey dies at the rate d_2 . Thus, the equation formed is:

$$\frac{dx_2}{dt} = d_1x_1 - \frac{\omega x_2 y_2}{1 + ax_2 + by_2 + abx_2 y_2} - d_2x_2 \quad (2.3)$$

where, $x_2(t)$ is the density of mature prey species at time t , d_2 is the death rate of both mature prey and mature predator, ω , a and b are positive parameters that describe the effects of capture rate, handling time and the magnitude of interference among predators respectively, on the feeding rate.

3. The immature predator population dies at the rate d_3 while the transition rate of immature predator to mature predator is given by d_4 . Thus, the equation is as follows:

$$\frac{dy_1}{dt} = \frac{\omega x_2 y_2}{1 + ax_2 + by_2 + abx_2 y_2} - d_3y_1 - d_4y_1 \quad (2.4)$$

where, $y_1(t)$ is the density of immature predator species at time t , d_3 is the death rate of immature predator, d_4 is the transition rate for immature predator individual to the mature predator.

4. The immature predator is growing at the transition rate d_4 and is declining at the rate d_2 . Thus, the equation is given by:

$$\frac{dy_2}{dt} = d_4y_1 - d_2y_2 \quad (2.5)$$

where, $y_2(t)$ is the density of mature predator species at time t .

Thus, the mathematical model is:

$$\begin{aligned} \frac{dx_1}{dt} &= x_1(r - a_1x_1) - d_1x_1 \\ \frac{dx_2}{dt} &= d_1x_1 - \frac{\omega x_2 y_2}{1 + ax_2 + by_2 + abx_2 y_2} - d_2x_2 \\ \frac{dy_1}{dt} &= \frac{\omega x_2 y_2}{1 + ax_2 + by_2 + abx_2 y_2} - d_3y_1 - d_4y_1 \\ \frac{dy_2}{dt} &= d_4y_1 - d_2y_2 \end{aligned}$$

3 Dynamical Behaviour

In this section, we obtain boundary and interior equilibrium points of the system. Also, we analyze the stability of the system at the equilibrium points.

3.1 Existence of Boundary Equilibrium points

Equilibrium points of the system are obtained from the following set of equation [3.1-3.4].

$$x_1((r - a_1x_1) - d_1) = 0 \quad (3.1)$$

$$d_1x_1 - \frac{\omega x_2 y_2}{1 + ax_2 + by_2 + abx_2 y_2} - d_2x_2 = 0 \quad (3.2)$$

$$\frac{\omega x_2 y_2}{1 + ax_2 + by_2 + abx_2 y_2} - d_3y_1 - d_4y_1 = 0 \quad (3.3)$$

$$d_4y_1 - d_2y_2 = 0 \quad (3.4)$$

Thus, boundary equilibrium points are: $E_0(0, 0, 0, 0)$, $E_1(\frac{r-d_1}{a_1}, 0, 0, 0)$,

$E_2(\frac{r-d_1}{a_1}, \frac{d_1(r-d_1)}{a_1 d_2}, 0, 0)$. It is easy to see that E_1, E_2 exist when $r > d_1$, i.e., intrinsic growth rate is more than the transition rate of immature prey to mature prey.

3.2 Existence of Unique Interior Equilibrium Point

Since, $E_3(\tilde{x}_1, \tilde{x}_2, \tilde{y}_1, \tilde{y}_2)$ is an equilibrium point, it satisfies the equations [3.1-3.4]:

$$(r - a_1\tilde{x}_1) - d_1 = 0 \quad (3.5)$$

$$d_1\tilde{x}_1 - \frac{\omega\tilde{x}_2\tilde{y}_2}{1 + a\tilde{x}_2 + b\tilde{y}_2 + ab\tilde{x}_2\tilde{y}_2} - d_2\tilde{x}_2 = 0 \quad (3.6)$$

$$\frac{\omega\tilde{x}_2\tilde{y}_2}{1 + a\tilde{x}_2 + b\tilde{y}_2 + ab\tilde{x}_2\tilde{y}_2} - d_3\tilde{y}_1 - d_4\tilde{y}_1 = 0 \quad (3.7)$$

$$d_4\tilde{y}_1 - d_2\tilde{y}_2 = 0 \quad (3.8)$$

From (3.5) and (3.8), we get :

$$\tilde{x}_1 = \frac{r - d_1}{a_1},$$

$$\tilde{y}_2 = \frac{d_4\tilde{y}_1}{d_2}$$

Substituting the above values in (3.6), and simplifying the equation, we get:

$$\frac{d_1\tilde{x}_1 + ad_1\tilde{x}_1\tilde{x}_2 + bd_1\tilde{x}_1\tilde{y}_2 + abd_1\tilde{x}_1\tilde{x}_2\tilde{y}_2 - \omega\tilde{x}_2\tilde{y}_2 - d_2\tilde{x}_2 - ad_2\tilde{x}_2^2 - bd_2\tilde{x}_2\tilde{y}_2 - abd_2\tilde{x}_2^2\tilde{y}_2}{1 + a\tilde{x}_2 + b\tilde{y}_2 + ab\tilde{x}_2\tilde{y}_2} = 0$$

$$\tilde{x}_2^2(abd_2d_4) + \tilde{x}_2(bd_2d_4 + \omega d_4 - abd_1d_4\tilde{x}_1 - ad_2(d_3 + d_4)) - (d_2(d_3 + d_4) + bd_1d_4\tilde{x}_1) = 0 \quad (3.9)$$

Now, in the above equation, since the product of roots is < 0 , the system has a unique positive root.

Hence, the system has unique interior equilibrium point $E_3(\tilde{x}_1, \tilde{x}_2, \tilde{y}_1, \tilde{y}_2)$.

3.3 Local stability

In this section, we investigate the local stability of the equilibrium points. For sake of convenience, we give the following notation:

$$h = h(x_2, y_2) = \frac{x_2y_2}{1 + ax_2 + by_2 + abx_2y_2} \quad (3.10)$$

Let $\tilde{E} = (\tilde{x}_1, \tilde{x}_2, \tilde{y}_1, \tilde{y}_2)$ be any equilibrium of the system. By the translation: $X_1 = x_1 - \tilde{x}_1, X_2 = x_2 - \tilde{x}_2, Y_1 = y_1 - \tilde{y}_1, Y_2 = y_2 - \tilde{y}_2$ the system can be translated into a new system. Rewrite X_1, X_2, Y_1 and Y_2 as x_1, x_2, y_1 and y_2 , then the linearization of this system at the equilibrium point $\tilde{E}$ is :

$$\frac{dx_1}{dt} = rx_1 \left(1 - \frac{2a_1\tilde{x}_1}{r} \right) - d_1x_1 \quad (3.11)$$

$$\frac{dx_2}{dt} = d_1x_1 - (\omega h_{x_2} + d_2)x_2 - \omega h_{y_2}y_2 \quad (3.12)$$

$$\frac{dy_1}{dt} = -(d_3 + d_4)y_1 + \omega h_{x_2}x_2 + \omega h_{y_2}y_2 \quad (3.13)$$

$$\frac{dy_2}{dt} = d_4y_1 - d_2y_2 \quad (3.14)$$

where,

$$h_{x_2} = \frac{\partial h}{\partial x_2} = \frac{\tilde{y}_2(1 + b\tilde{y}_2)}{(1 + a\tilde{x}_2 + b\tilde{y}_2 + ab\tilde{x}_2\tilde{y}_2)^2} \quad (3.15)$$

$$h_{y_2} = \frac{\partial h}{\partial y_2} = \frac{\tilde{x}_2(1 + a\tilde{x}_2)}{(1 + a\tilde{x}_2 + b\tilde{y}_2 + ab\tilde{x}_2\tilde{y}_2)^2} \quad (3.16)$$

At any point, the jacobian matrix so formed is given by:

$$\begin{bmatrix} r \left(1 - \frac{2a_1\tilde{x}_1}{r} \right) - d_1 & 0 & 0 & 0 \\ d_1 & -(\omega h_{x_2} + d_2) & 0 & -\omega h_{y_2} \\ 0 & \omega h_{x_2} & -(d_3 + d_4) & \omega h_{y_2} \\ 0 & 0 & d_4 & -d_2 \end{bmatrix}$$

The jacobian corresponding to $E_0(0, 0, 0, 0)$ is

$$\begin{bmatrix} r - d_1 & 0 & 0 & 0 \\ d_1 & -d_2 & 0 & 0 \\ 0 & 0 & -(d_3 + d_4) & 0 \\ 0 & 0 & d_4 & -d_2 \end{bmatrix}$$

The corresponding characteristic equation with respect to jacobian is :

$$(r - d_1 - \lambda)(d_3 + d_4 + \lambda)(\lambda + d_2)^2 = 0.$$

The corresponding eigenvalues are:

$$\lambda = r - d_1, -(d_3 + d_4), -d_2, -d_2. \quad (3.17)$$

As $r - d_1 > 0$, there exists one positive eigenvalue. Thus, E_0 is unstable.

Now the jacobian corresponding to $E_1(\tilde{x}_1, 0, 0, 0)$ is

$$\begin{bmatrix} r \left(1 - \frac{2a_1\tilde{x}_1}{r} \right) - d_1 & 0 & 0 & 0 \\ d_1 & -d_2 & 0 & 0 \\ 0 & 0 & -(d_3 + d_4) & 0 \\ 0 & 0 & d_4 & -d_2 \end{bmatrix}$$

The corresponding characteristic equation with respect to jacobian is :

$$\left(r \left(1 - \frac{2a_1\tilde{x}_1}{r} \right) - d_1 - \lambda \right) (d_3 + d_4 + \lambda)(\lambda + d_2)^2 = 0$$

which reduces to,

$$(-(r - d_1) - \lambda)(d_3 + d_4 + \lambda)(\lambda + d_2)^2 = 0$$

The corresponding eigenvalues are:

$$\lambda = -(r - d_1), -(d_3 + d_4), -d_2, -d_2 \quad (3.18)$$

Since all four eigenvalues are negative, E_1 is stable.

Now the jacobian corresponding to $E_2(\tilde{x}_1, \tilde{x}_2, 0, 0)$ is

$$\begin{bmatrix} r \left(1 - \frac{2a_1\tilde{x}_1}{r} \right) - d_1 & 0 & 0 & 0 \\ d_1 & -d_2 & 0 & \frac{-\omega\tilde{x}_2}{1+a\tilde{x}_2} \\ 0 & 0 & -(d_3+d_4) & \frac{\omega\tilde{x}_2}{1+a\tilde{x}_2} \\ 0 & 0 & d_4 & -d_2 \end{bmatrix}$$

The corresponding characteristic equation with respect to the above jacobian is :

$$\left(r \left(1 - \frac{2a_1\tilde{x}_1}{r} \right) - d_1 - \lambda \right) (\lambda + d_2) [\lambda^2 + \lambda(d_2 + d_3 + d_4) + \left(d_2(d_3 + d_4) - \frac{\omega\tilde{x}_2 d_4}{1 + a\tilde{x}_2} \right)] = 0$$

which reduces to,

$$(-(r - d_1) - \lambda)(\lambda + d_2) \left[\lambda^2 + \lambda(d_2 + d_3 + d_4) + \left(d_2(d_3 + d_4) - \frac{\omega\tilde{x}_2 d_4}{1 + a\tilde{x}_2} \right) \right] = 0.$$

For the system to be local asymptotically stable, the constant term in the last bracket should be greater than zero i.e.

$$d_2(d_3 + d_4) - \frac{\omega\tilde{x}_2 d_4}{1 + a\tilde{x}_2} > 0 \quad (3.19)$$

resulting in,

$$(r - d_1)[\omega d_4 d_1 - a d_1 d_2 (d_3 + d_4)] < a_1 d_2^2 (d_3 + d_4). \quad (3.20)$$

Now, the jacobian corresponding to $E_3(\tilde{x}_1, \tilde{x}_2, \tilde{y}_1, \tilde{y}_2)$ is

$$\begin{bmatrix} r \left(1 - \frac{2a_1\tilde{x}_1}{r} \right) - d_1 & 0 & 0 & 0 \\ d_1 & -(\omega h_{x_2} + d_2) & 0 & -\omega h_{y_2} \\ 0 & \omega h_{x_2} & -(d_3 + d_4) & \omega h_{y_2} \\ 0 & 0 & d_4 & -d_2 \end{bmatrix}$$

The corresponding characteristic equation with respect to the above jacobian is :

$$\left[r \left(1 - \frac{2a_1\tilde{x}_1}{r} \right) - d_1 - \lambda \right] [\lambda^3 + A\lambda^2 + B\lambda + C] = 0;$$

which reduces to,

$$[-(r - d_1) - \lambda][\lambda^3 + A\lambda^2 + B\lambda + C] = 0; \quad (3.21)$$

where,

$$A = 2d_2 + d_3 + d_4 + \omega h_{x_2}. \quad (3.22)$$

$$B = d_2^2 + 2d_2 d_3 + 2d_2 d_4 + \omega((d_2 + d_3 + d_4)h_{x_2} - d_4 h_{y_2}). \quad (3.23)$$

$$C = d_2^2(d_3 + d_4) + \omega(d_2(d_3 + d_4)h_{x_2} - d_2 d_4 \omega \tilde{y}_2) \quad (3.24)$$

Note that equation (3.21), has one of the root which is $\lambda = -(r - d_1) < 0$. If $h_{x_2} > 0$, $h_{x_2}(d_2 + d_3 + d_4) > d_4 h_{y_2}$, $h_{x_2}(d_3 + d_4) > d_4 h_{y_2}$, then A, B, C are all greater than zero respectively. Thus, By Descartes' rule of sign change, equilibrium point is stable if $r > d_1, A, B, C > 0$

Thus, we have the following theorem:

Theorem 3.1. (i) E_0 is always unstable.

(ii) E_1 is always stable.

(iii) E_2 is local asymptotically stable if :

$$(r - d_1)[\omega d_4 d_1 - a d_1 d_2 (d_3 + d_4)] < a_1 d_2^2 (d_3 + d_4).$$

(iv) E_3 is stable if:

(a) $h_{x_2} > 0$.

(b) $h_{x_2}(d_2 + d_3 + d_4) > d_4 h_{y_2}$.

(c) $h_{x_2}(d_3 + d_4) > d_4 h_{y_2}$.

4 Permanence

Since all solutions of the system always exists and are positive under initial conditions, permanence of the system holds.

The system is said to be permanent if there are positive constants m and M such that:

$$m \leq \lim_{t \rightarrow +\infty} \inf x_1(t) \leq \lim_{t \rightarrow +\infty} \sup x_1(t) \leq M \quad (4.1)$$

$$m \leq \lim_{t \rightarrow +\infty} \inf x_2(t) \leq \lim_{t \rightarrow +\infty} \sup x_2(t) \leq M \quad (4.2)$$

$$m \leq \lim_{t \rightarrow +\infty} \inf y_1(t) \leq \lim_{t \rightarrow +\infty} \sup y_1(t) \leq M \quad (4.3)$$

$$m \leq \lim_{t \rightarrow +\infty} \inf y_2(t) \leq \lim_{t \rightarrow +\infty} \sup y_2(t) \leq M, \quad (4.4)$$

for all solutions $(x_1(t), x_2(t), y_1(t), y_2(t))$ of the system.

From eq.(1), we see that $\lim_{t \rightarrow +\infty} \sup x_1(t) \leq \frac{r-d_1}{a_1}$.

Then, $\exists T_1 > 0$ such that for $t > T_1, x_1(t) \leq \frac{r-d_1}{a_1} + \varepsilon$, for any sufficiently small $\varepsilon > 0$.

Theorem 4.1. If $d_2 < d_3$, then $\exists M > 0$ such that for any positive solution $(x_1(t), x_2(t), y_1(t), y_2(t))$ of the system, $x_2(t) \leq M$, $y_1(t) \leq M$ and $y_2(t) \leq M$ for all large t .

Proof. Let $V(t) = x_2(t) + y_1(t) + y_2(t)$. Then,

$$\begin{aligned} \frac{dV(t)}{dt} &= \frac{dx_2}{dt} + \frac{dy_1}{dt} + \frac{dy_2}{dt} \\ &= d_1 x_1 - d_2 x_2 - d_3 y_1 - d_4 y_1 + d_4 y_1 - d_2 y_2 \\ &= d_1 x_1 - d_2 x_2 - d_3 y_1 - d_2 y_2 + d_2 y_1 - d_2 y_1 \\ &= -d_2 V(t) + d_1 x_1 - d_3 y_1 + d_2 y_1 \\ &\leq -d_2 V(t) + M_1 \end{aligned}$$

where, $M_1 = d_1 \left(\frac{r-d_1}{a_1} \right)$, therefore, $\exists$ some positive $T(T > T_1)$, such that, $\frac{dV(t)}{dt} \leq M_1 - d_2 V(t)$, for all $t \geq T$. Then, we have $\lim_{t \rightarrow +\infty} \sup V(t) \leq \frac{M_1}{d_2}$. Thus, $x_2(t), y_1(t)$ and $y_2(t)$ are ultimately bounded. $\square$

Thus, we can define:

$$\Omega = \left\{ (x_1(t), x_2(t), y_1(t), y_2(t)) : 0 \leq x_1(t) \leq \frac{r-d_1}{a_1}, 0 \leq x_2(t), y_1(t), y_2(t) \leq \frac{M_1}{d_2} \right\}.$$

Theorem 4.2. Suppose that :

$$\mathbf{H1} : abd_1(r-d_1) > 2a_1\omega$$

$$\mathbf{H2} : [abd_1(r-d_1) - 2a_1\omega][\omega - a(d_3+d_4)] > 2a_1d_2[ab(d_3+d_4)]$$

$$\mathbf{H3} : d_4[abd_1(r-d_1) - 2a_1\omega][\omega - a(d_3+d_4)] > 2a_1d_2^2[ab(d_3+d_4)],$$

then the system is permanent.

Proof. Since,

$$\begin{aligned} \frac{dx_1}{dt} &= x_1(r - a_1x_1 - d_1) \\ \implies \lim_{t \rightarrow +\infty} \inf x_1(t) &\geq \frac{r-d_1}{a_1} \triangleq \bar{x}_1. \end{aligned}$$

Hence, there is a τ , such that :

$$x_1(t) \geq \frac{\bar{x}_1}{2} \text{ for } t > \tau.$$

Now, from (2.3), we have :

$$\begin{aligned} \frac{dx_2}{dt} &\geq \frac{d_1\bar{x}_1}{2} - \frac{\omega}{ab} - d_2x_2 \\ &= m_1 - d_2x_2; \end{aligned} \tag{4.5}$$

where

$$m_1 = \frac{d_1\bar{x}_1}{2} - \frac{\omega}{ab}.$$

Now, we can obtain that:

$$\lim_{t \rightarrow +\infty} \inf x_2(t) \geq \frac{m_1}{d_2} \geq \frac{1}{d_2} \left(\frac{d_1(r-d_1)}{2a_1} - \frac{\omega}{ab} \right) \triangleq \bar{x}_2 \text{ if } \mathbf{H1} \text{ holds.} \tag{4.6}$$

Hence, we have :

$$\lim_{t \rightarrow +\infty} \inf x_2(t) \geq \bar{x}_2 \tag{4.7}$$

Similarly, from (2.4), we have:

$$\frac{dy_1}{dt} \geq \frac{\omega\bar{x}_2y_2}{1+a\bar{x}_2+ab\bar{x}_2y_2} - d_3y_1 - d_4y_1. \tag{4.8}$$

Now, define the equation as :

$$\frac{dp}{dt} = \frac{\omega\bar{x}_2p}{1+a\bar{x}_2+ab\bar{x}_2p} - d_3p - d_4p. \tag{4.9}$$

If **H2** holds, then the non-negative equilibrium $p = \frac{\omega}{ab(d_3+d_4)} - \frac{1}{ab\bar{x}_2} - \frac{1}{b}$ of (4.9) exists and is locally asymptotically stable. Hence, all the solutions of the system (4.9) are bounded.

It follows that:

$$\lim_{t \rightarrow +\infty} \inf y_1(t) \geq \frac{\omega}{ab(d_3 + d_4)} - \frac{1}{ab\tilde{x}_2} - \frac{1}{b} \triangleq \tilde{y}_1. \quad (4.10)$$

Finally, from (2.5), we have:

$$\frac{dy_2}{dt} \geq d_4\tilde{y}_1 - d_2y_2. \quad (4.11)$$

Now, from above equation, we see that,

$$y_2 \geq \frac{d_4}{d_2} \left(\frac{\omega}{ab(d_3 + d_4)} - \frac{1}{ab\tilde{x}_2} - \frac{1}{b} \right) \triangleq \tilde{y}_1 \quad (4.12)$$

Now, if H3 holds, then,

$$\lim_{t \rightarrow +\infty} \inf y_2(t) \geq \frac{d_4\tilde{y}_1}{d_2} \triangleq \tilde{y}_2. \quad (4.13)$$

From Theorem 4.1 and the above arguments, we have proved that the system is permanent. $\square$

5 Numerical Example

In this section, we have shown the rich dynamics of stage structure predator-prey system with Crowley-Martin functional response. Numerical simulations show the following results for the following set of parameters:

$$r = 3.5, a_1 = 0.29, d_1 = 0.5, w = 10, a = 0.2, b = 12, d_3 = 0.5, d_4 = 0.5$$

1. Keeping the above parameters fixed, we study the behaviour of the system based on control parameter d_2 . For $d_2 = 2, 3, 5$, the equilibrium point for system with stage structure in prey and predator both are $E^*(10.2860, 2.0008, 1.1409, 1.9015)$, $E^*(10.2857, 1.4235, 0.8732, 1.4554)$, $E^*(10.2857, 0.9103, 0.5917, 0.9862)$ and for system with stage structure in prey only are $E^*(10.2858, 1.5781, 1.9159)$, $E^*(10.2857, 1.1295, 1.4522)$, $E^*(10.2857, 0.7256, 0.9727)$ respectively. From these equilibrium points we conclude that the number of prey as well as predator population has increased when stage structuring is considered in both prey and predator population in comparison with the system when there is no stage structuring in predator population. The result is because of the reason that lesser number of mature predators are available for preying prey population. The population will survive for much longer period when there is stage structuring in both prey and predator population, even if we increase the death rate to a large value [Figure 7, 8, 9].
2. Again, keeping all the parameters fixed except the control parameter b (interference among predators), we see that predators gradually decrease and die out as we increase b [Figure 1, 2]. Further, when stage structure is considered in both prey and predator population, predator die out faster in comparison with the case when there is no stage structure in predator. Biologically, if the interference among mature predators is more, then predators will die out.
3. We also analyse the change in behaviour of the system with change in control parameter a (handling time). The predator population decreases with increase in handling rate. Also, we note that the decrease in predator population is more in system with stage structure in predator in comparison with system without stage structure in predator [Figure 5, 6].

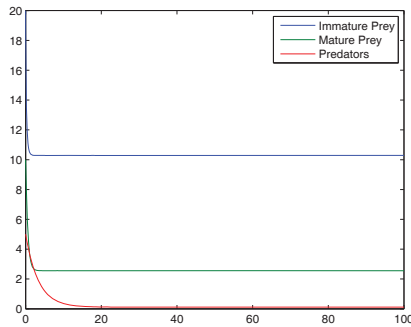


Figure 1: Graph of the system for $b=500$ without stage structure in predator

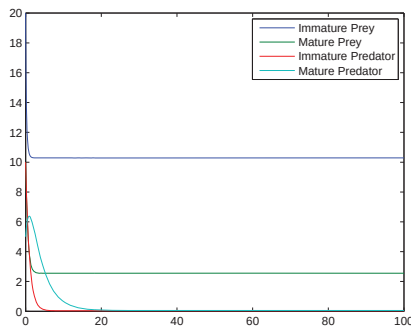


Figure 2: Graph of the system for $b=500$ with stage structure in predator

4. Lastly, numerical simulation has also been carried out for the capturing rate w and we notice that with the decrease in capturing rate, the predator population increases [Figure 3, 4].

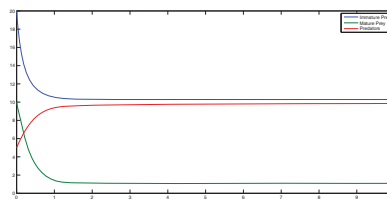


Figure 3: Graph of the system for $w=40$ without stage structure in predator

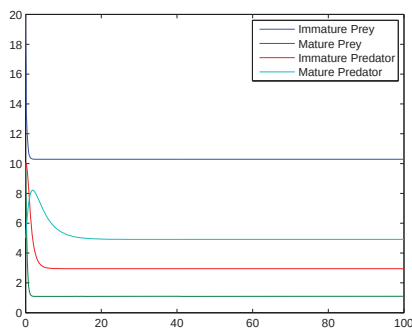


Figure 4: Graph of the system for $w=40$ with stage structure in predator

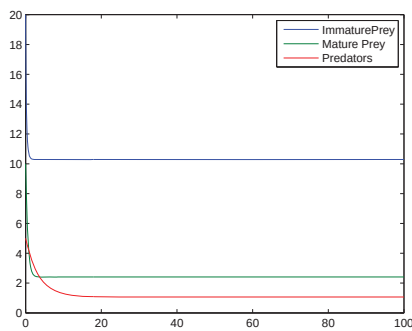


Figure 5: Graph of the system for $a=2$ without stage structure in predator

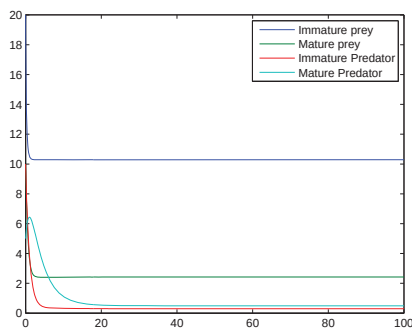


Figure 6: Graph of the system for $a=2$ with stage structure in predator

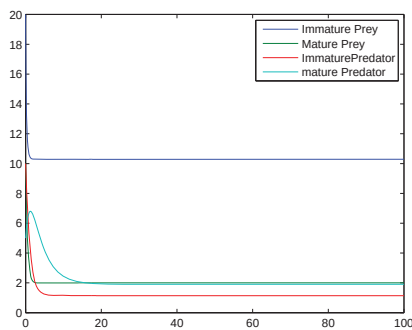


Figure 7: Graph of the system for $d=2$ with stage structure in predator

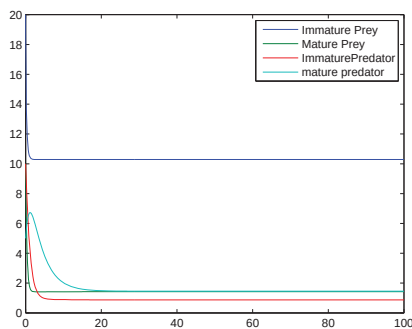


Figure 8: Graph of the system for $d=3$ with stage structure in predator

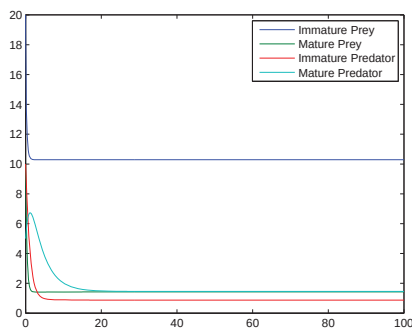


Figure 9: Graph of the system for $d=5$ with stage structure in predator

6 Conclusion

In this paper, we have studied the stage-structured prey-predator system with Crowley-Martin functional response. We have considered stage structuring in both prey and predator population. We obtained boundary and interior equilibrium points and studied their local stability by linearizing the model. The permanence of the system is established under certain conditions. It is shown that the model exhibits rich dynamics based on four important control parameters: interference among predators, handling time, death rate of mature prey and capturing rate. We noticed that when stage structure is considered in both prey and predator population, predator die out faster in comparison with the case when there is no stage structure in predator. We have also proved that the predator population decreases with increase in handling rate and the decrease in predator population is more in the system when stage structure in predator is considered. We have proved that although predator dies out faster if stage structure is considered but if the death rate of mature prey is controlled then not only prey but also predator population will survive for much longer period when stage structuring is considered in both prey and predator population. Thus, stage structure plays an important role in shaping the dynamics of the model.

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