

Adjoint Poisson-Charlier and Related polynomials

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ABSTRACT

In recent papers, new sets of Sheffer and Brenke polynomials based on higher order Bell numbers have been studied, and several integer sequences related to them have been introduced. In this article other types of Sheffer polynomials are considered, by introducing the adjoint Poisson-Charlier and the adjoint Related polynomials.

Keywords: generating functions, Sheffer polynomials, combinatorial analysis, Poisson-Charlier and Related polynomials.

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1 Introduction

In recent articles (Bretti, Natalini and Ricci, in press), (Ricci, Natalini and Bretti, 2017), we have studied new sets of Sheffer (Roman, 1939) and Brenke (Brenke, 1945) polynomials related to higher order Bell numbers (Natalini and Ricci, 2016).

Furthermore, several integer sequences (Sloane, 2016) associated with the considered polynomials sets both of exponential (Bell, 1934), (Bell, 1938) and logarithmic type have been introduced (Bretti et al., in press).

It is worth to note that exponential and logarithmic polynomials have been recently studied in the multidimensional case (Qi, Niu, Lim and Guo, 2018), (Qi, 2018a) (Qi, 2018b).

Sheffer sets (Roman, 1939) are a wide class of polynomials generalizing the Appell polynomial sequences (Appell, 1880). A list of known Sheffer polynomials can be found in (Boas and Buck, 1958). In preceding articles we have extended the Sheffer sets by introducing the notion

of adjointness.

Adjointness is a general method of associating a family of Sheffer polynomials with another family of Sheffer's (Ricci, 2018a). This definition is based on an equivalent characterization of the Sheffer sets introduced in (Roman, 1984, p. 18). It follows by simply exchanging two basic elements of the generating function.

Using the general and efficient method already applied in previous articles (Ricci, 2018b) - (Ricci, 2019), (Natalini and Ricci, 2019b)-(Natalini and Ricci, 2019a), it is possible to build the fundamental properties of polynomials adjoint to the classical ones, such as the recurrence relation, the shift operators and the differential equation.

We have already applied this technique in several cases (Bretti, Natalini and Ricci, 2018), (Natalini, Bretti and Ricci, 2018), (Ricci, Bretti and Natalini, 2019).

In this article we apply the same technique to other types of Sheffer polynomials, by introducing the adjoint Poisson-Charlier and the adjoint Related polynomials.

2 Sheffer polynomials

The Sheffer polynomials $\{s_n(x)\}$ are introduced (Roman, 1939) by means of the exponential generating function (Srivastava and Manocha, 1984) of the type:

$$A(t) \exp(xH(t)) = \sum_{n=0}^{\infty} s_n(x) \frac{t^n}{n!}, \quad (2.1)$$

where

$$A(t) = \sum_{n=0}^{\infty} a_n \frac{t^n}{n!}, \quad (a_0 \neq 0), \quad H(t) = \sum_{n=0}^{\infty} h_n \frac{t^n}{n!}, \quad (h_0 = 0). \quad (2.2)$$

According to a different characterization (see (Roman, 1984, p. 18)), the same polynomial sequence can be defined by means of the pair $(g(t), f(t))$, where $g(t)$ is an invertible series and $f(t)$ is a delta series:

$$g(t) = \sum_{n=0}^{\infty} g_n \frac{t^n}{n!}, \quad (g_0 \neq 0), \quad f(t) = \sum_{n=0}^{\infty} f_n \frac{t^n}{n!}, \quad (f_0 = 0, f_1 \neq 0). \quad (2.3)$$

Denoting by $f^{-1}(t)$ the compositional inverse of $f(t)$ (i.e. such that $f(f^{-1}(t)) = f^{-1}(f(t)) = t$), the exponential generating function of the sequence $\{s_n(x)\}$ is given by

$$\frac{1}{g[f^{-1}(t)]} \exp(xf^{-1}(t)) = \sum_{n=0}^{\infty} s_n(x) \frac{t^n}{n!}, \quad (2.4)$$

so that

$$A(t) = \frac{1}{g[f^{-1}(t)]}, \quad H(t) = f^{-1}(t). \quad (2.5)$$

When $g(t) \equiv 1$, the Sheffer sequence corresponding to the pair $(1, f(t))$ is called the associated Sheffer sequence $\{\sigma_n(x)\}$ for $f(t)$, and its exponential generating function is given by

$$\exp(xf^{-1}(t)) = \sum_{n=0}^{\infty} \sigma_n(x) \frac{t^n}{n!}. \quad (2.6)$$

A list of known Sheffer polynomial sequences and their associated ones can be found in (Boas and Buck, 1958).

3 Adjointness for Poisson-Charlier polynomials

According to the above considerations, Sheffer polynomials are characterized both by the ordered couples $(A(t), H(t))$, or by $(g(t), f(t))$.

Definition 3.1. - Adjoint Sheffer polynomials are defined by interchanging the ordered couple $(A(t), H(t))$ with $(g(t), f(t))$, when writing the generating function.

Here and in the following the *tilde* “ $\sim$ ”, above the symbol of a polynomial set stands for the adjective “adjoint”.

Here we consider the adjoint Poisson-Charlier polynomials, defined through their generating function, i.e. by putting

$$\begin{aligned} A(t) &= \exp(a(e^t - 1)), & H(t) &= a(e^t - 1), \\ G(t, x) &= \exp(a(1+x)(e^t - 1)) = \sum_{n=0}^{\infty} \tilde{c}_n(x; a) \frac{t^n}{n!}, \end{aligned} \quad (3.1)$$

1. - A differential identity

Theorem 3.1. For any $k \geq 0$, the polynomials $\tilde{c}_k(x, a)$ satisfy the differential identity:

$$\tilde{c}'_k(x, a) = a \sum_{h=0}^{k-1} \binom{k}{h} \tilde{c}_h(x, a). \quad (3.2)$$

Proof. - By differentiating both sides of equation (3.1)₂ with respect to x , and putting $\tilde{c}'_0(x, a) = 0$, we find

$$\begin{aligned} \frac{\partial G}{\partial x} &= ae^t G(t, x) - aG(t, x) = \\ &= a \sum_{k=0}^{\infty} \sum_{h=0}^k \binom{k}{h} \tilde{c}_h(x, a) \frac{t^k}{k!} - a \sum_{k=0}^{\infty} \tilde{c}_k(x, a) \frac{t^k}{k!} = \sum_{k=0}^{\infty} \tilde{c}'_k(x, a) \frac{t^k}{k!}, \end{aligned} \quad (3.3)$$

and therefore

$$a \sum_{h=0}^k \binom{k}{h} \tilde{c}_h(x, a) - a \tilde{c}_k(x, a) = \tilde{c}'_k(x, a),$$

so that the differential identity (3.2) follows. $\square$

2. - Recurrence relation

Theorem 3.2. For any $k \geq 0$, the polynomials $\tilde{c}_k(x, a)$ satisfy the following recurrence relation:

$$\tilde{c}_{k+1}(x, a) = a(1+x) \sum_{h=0}^k \binom{k}{h} \tilde{c}_h(x, a). \quad (3.4)$$

Proof. Differentiating $G(t, x)$ with respect to t , we have

$$\frac{\partial G(t, x)}{\partial t} = a(1+x)e^t G(t, x) = \sum_{k=1}^{\infty} \tilde{c}_k(x, a) \frac{t^{k-1}}{(k-1)!}. \quad (3.5)$$

i.e.

$$\begin{aligned} \sum_{k=0}^{\infty} \tilde{c}_{k+1}(x, a) \frac{t^k}{k!} &= a(1+x) \sum_{k=0}^{\infty} \frac{t^k}{k!} \sum_{k=0}^{\infty} \tilde{c}_k(x, a) \frac{t^k}{k!} = \\ &= a(1+x) \sum_{k=0}^{\infty} \sum_{h=0}^k \binom{k}{h} \tilde{c}_h(x, a) \frac{t^k}{k!}, \end{aligned}$$

so that the recurrence relation (3.4) follows. □

Generating function's PDE

Theorem 3.3. *The generating function (3.1₂) satisfies the homogeneous linear PDE*

$$(1+x) \frac{\partial G}{\partial x} + (e^{-t} - 1) \frac{\partial G}{\partial t} = 0. \quad (3.6)$$

Proof. It is sufficient to eliminate $a G(t, x)$ between the partial derivatives of $G(t, x)$:

$$\frac{\partial G(t, x)}{\partial x} = a G(t, x) (e^t - 1), \quad \frac{\partial G(t, x)}{\partial t} = (1+x) e^t a G(t, x).$$

□

3.1 Shift operators

We recall that a polynomial set $\{p_n(x)\}$ is called quasi-monomial if and only if there exist two operators $\hat{P}$ and $\hat{M}$ such that

$$\hat{P}(p_n(x)) = n p_{n-1}(x), \quad \hat{M}(p_n(x)) = p_{n+1}(x), \quad (n = 1, 2, \dots). \quad (3.7)$$

$\hat{P}$ is called the *derivative* operator and $\hat{M}$ the *multiplication* operator, as they act in the same way of classical operators on monomials.

This definition traces back to a paper by J.F. Steffensen (Steffensen, 1941), recently improved by G. Dattoli ("Dattoli, 2000") and widely used in several applications (see e.g. (Dattoli, Ricci and Srivastava, 2000), (Dattoli, Germano, B. and Ricci, 2009)).

Y. Ben Cheikh (Ben Cheikh, 2003) proved that every polynomial set is quasi-monomial under the action of suitable derivative and multiplication operators. In particular, in the same article (Corollary 3.2), the following result is proved

Theorem 3.4. *Let $(p_n(x))$ denote a Boas-Buck polynomial set, i.e. a set defined by the generating function*

$$A(t) \psi(xH(t)) = \sum_{n=0}^{\infty} p_n(x) \frac{t^n}{n!}, \quad (3.8)$$

where

$$A(t) = \sum_{n=0}^{\infty} \tilde{a}_n t^n, \quad (\tilde{a}_0 \neq 0), \quad \psi(t) = \sum_{n=0}^{\infty} \tilde{\gamma}_n t^n, \quad (\tilde{\gamma}_n \neq 0 \quad \forall n), \quad (3.9)$$

with $\psi(t)$ not a polynomial, and lastly

$$H(t) = \sum_{n=0}^{\infty} \tilde{h}_n t^{n+1}, \quad (\tilde{h}_0 \neq 0). \quad (3.10)$$

Let $\sigma \in \Lambda^{(-)}$ the lowering operator defined by

$$\sigma(1) = 0, \quad \sigma(x^n) = \frac{\tilde{\gamma}_{n-1}}{\tilde{\gamma}_n} x^{n-1}, \quad (n = 1, 2, \dots). \quad (3.11)$$

Put

$$\sigma^{-1}(x^n) = \frac{\tilde{\gamma}_{n+1}}{\tilde{\gamma}_n} x^{n+1} \quad (n = 0, 1, 2, \dots). \quad (3.12)$$

Denoting, as before, by $f(t)$ the compositional inverse of $H(t)$, the Boas-Buck polynomial set $\{p_n(x)\}$ is quasi-monomial under the action of the operators

$$\hat{P} = f(\sigma), \quad \hat{M} = \frac{A'[f(\sigma)]}{A[f(\sigma)]} + x D_x H'[f(\sigma)] \sigma^{-1}, \quad (3.13)$$

where prime denotes the ordinary derivatives with respect to t .

Note that in the present case we are dealing with a Sheffer polynomial set, so that since we have $\psi(t) = e^t$, the operator σ defined by equation (3.11) simply reduces to the derivative operator D_x . Furthermore, we have:

$$A(t) = \exp(a(e^t - 1)), \quad H(t) = a(e^t - 1),$$

$$f(t) = H^{-1}(t) = \log\left(\frac{t}{a} + 1\right) = \sum_{k=1}^{\infty} (-1)^{k+1} \frac{t^k}{k a^k},$$

$$H'(t) = a e^t, \quad \frac{A'(t)}{A(t)} = a e^t,$$

so that we have the theorem

Theorem 3.5. *The adjoint Poisson-Charlier polynomial set $\{\tilde{c}_n(x, a)\}$ is quasi-monomial under the action of the operators*

$$\hat{P} = \log\left(\frac{D_x}{a} + 1\right) = \sum_{k=1}^{\infty} (-1)^{k+1} \frac{D_x^k}{k a^k}, \quad (3.14)$$

$$\hat{M} = a e^{\log(\frac{D_x}{a} + 1)} + x a \left(\frac{D_x}{a} + 1\right) = (1 + x)(D_x + a),$$

3.2 Differential equation

According to the results of monomiality principle ("Dattoli, "2000"), the quasi-monomial polynomials $\{p_n(x)\}$ satisfy the differential equation

$$\hat{M} \hat{P} p_n(x) = n p_n(x). \quad (3.15)$$

In the present case, we have

Theorem 3.6. *The Sheffer polynomials $\{\tilde{c}_n(x, a)\}$ satisfy the differential equation*

$$(1+x)(D_x + a) \sum_{k=1}^{\infty} (-1)^{k+1} \frac{D_x^k}{k a^k} \tilde{c}_n(x, a) = n \tilde{c}_n(x, a), \quad (3.16)$$

i.e.

$$(1+x)(D_x + a) \sum_{k=1}^n (-1)^{k+1} \frac{D_x^k}{k a^k} \tilde{c}_n(x, a) = n \tilde{c}_n(x, a), \quad (3.17)$$

because, for any fixed n , the last series expansion reduces to a finite sum, with upper limit n , when it is applied to a polynomial of degree n

3.3 First few values

$$\begin{aligned} \tilde{c}_0(x, a) &= 1 \\ \tilde{c}_1(x, a) &= a(x+1) \\ \tilde{c}_2(x, a) &= a(x+1)[a(x+1)+1] \\ \tilde{c}_3(x, a) &= a(x+1)[a^2(x+1)^2+3a(x+1)+1] \\ \tilde{c}_4(x, a) &= a(x+1)[a^3(x+1)^3+6a^2(x+1)^2+7a(x+1)+1] \\ \tilde{c}_5(x, a) &= a(x+1)[a^4(x+1)^4+10a^3(x+1)^3+25a^2(x+1)^2+15a(x+1)+1] \\ \tilde{c}_6(x, a) &= a(x+1)[a^5(x+1)^5+15a^4(x+1)^4+65a^3(x+1)^3+90a^2(x+1)^2 \\ &\quad +31a(x+1)+1] \\ \tilde{c}_7(x, a) &= a(x+1)[a^6(x+1)^6+21a^5(x+1)^5+140a^4(x+1)^4+350a^3(x+1)^3 \\ &\quad +301a^2(x+1)^2+63a(x+1)+1] \\ \tilde{c}_8(x, a) &= a(x+1)[a^7(x+1)^7+28a^6(x+1)^6+266a^5(x+1)^5+1050a^4(x+1)^4 \\ &\quad +1701a^3(x+1)^3+966a^2(x+1)^2+127a(x+1)+1] \end{aligned}$$

Further values can be easily achieved by using Wolfram Alpha[®].

4 Adjointness for Related polynomials

Here we consider the adjoint Related polynomials, defined through their generating function, i.e. by putting

$$\begin{aligned} A(t) &= \frac{e^t + 1}{2}, & H(t) &= e^t - 1, \\ G(t, x) &= \frac{e^t + 1}{2} \exp[x(e^t - 1)] = \sum_{k=0}^{\infty} \tilde{r}_k(x) \frac{t^k}{k!}, \end{aligned} \quad (4.1)$$

1. - A differential identity

Theorem 4.1. - *For any $k \geq 0$, the polynomials $\tilde{r}_k(x)$ satisfy the differential identity:*

$$\tilde{r}'_k(x) = \sum_{h=0}^{k-1} \binom{k}{h} \tilde{r}_h(x). \quad (4.2)$$

Proof. By differentiating both sides of equation (3.1)₂ with respect to x , and putting $\tilde{r}'_0(x) = 0$, we find

$$\frac{\partial G}{\partial x} = (e^t - 1) G(t, x) = \sum_{k=0}^{\infty} \tilde{r}'_k(x) \frac{t^k}{k!}, \quad (4.3)$$

and therefore

$$\sum_{k=0}^{\infty} \tilde{r}'_k(x) \frac{t^k}{k!} = e^t G(t, x) - G(t, x) = \sum_{k=0}^{\infty} \sum_{h=0}^k \binom{k}{h} \tilde{r}_h(x) \frac{t^k}{k!} - \sum_{k=0}^{\infty} \tilde{r}_k(x) \frac{t^k}{k!},$$

$$\tilde{r}'_k(x) = \sum_{h=0}^k \binom{k}{h} \tilde{r}_h(x) - \tilde{r}_k(x),$$

so that the differential identity (3.2) follows. $\square$

2. - Recurrence relation

Theorem 4.2. - For any $k \geq 0$, the polynomials $\tilde{r}_k(x)$ satisfy the following recurrence relation:

$$\tilde{r}_{k+1}(x) = \sum_{h=0}^k \binom{k}{h} [x + a_{k-h}] \tilde{r}_h(x), \quad (4.4)$$

where the coefficients a_k are the solution of the triangular system

$$\begin{cases} a_0 = 1/2, \\ \sum_{h=0}^{k-1} \binom{k}{h} a_h + 2a_k = 1, & \forall k \geq 1. \end{cases} \quad (4.5)$$

Proof. Differentiating $G(t, x)$ with respect to t , we have

$$\begin{aligned} \frac{\partial G(t, x)}{\partial t} &= x G(t, x) e^t + \frac{e^t}{2} \exp [x (e^t - 1)] = \\ &= x G(t, x) e^t + \frac{e^t}{e^t + 1} G(t, x) = \sum_{k=1}^{\infty} \tilde{r}_k(x) \frac{t^{k-1}}{(k-1)!}, \end{aligned} \quad (4.6)$$

putting

$$\frac{e^t}{e^t + 1} = \sum_{k=0}^{\infty} a_k \frac{t^k}{k!},$$

it is easily seen that the coefficients a_k are the solution of the system (3.5), so that

$$\sum_{k=0}^{\infty} \tilde{r}_{k+1}(x) \frac{t^k}{k!} = x \sum_{k=0}^{\infty} \frac{t^k}{k!} \sum_{k=0}^{\infty} \tilde{r}_k(x) \frac{t^k}{k!} + \sum_{k=0}^{\infty} a_k \frac{t^k}{k!} \sum_{k=0}^{\infty} \tilde{r}_k(x) \frac{t^k}{k!},$$

and therefore the recurrence relation (3.4) follows. $\square$

Note that even in the present case we are dealing with a Sheffer polynomial set, so that since we have $\psi(t) = e^t$, the operator σ defined by equation (3.11) simply reduces to the derivative operator D_x . Furthermore, we have:

$$\begin{aligned} A(t) &= \frac{e^t + 1}{2}, & H(t) &= e^t - 1, \\ f(t) &= H^{-1}(t) = \log(t + 1) = \sum_{k=1}^{\infty} (-1)^{k+1} \frac{t^k}{k}, \\ H'(t) &= e^t, & \frac{A'(t)}{A(t)} &= \frac{e^t}{e^t + 1}. \end{aligned}$$

Therefore, we have the theorem

Theorem 4.3. *The adjoint Related polynomial set $\{\tilde{r}_n(x)\}$ is quasi-monomial under the action of the operators*

$$\begin{aligned} \hat{P} = \log(D_x + 1) &= \sum_{k=1}^{\infty} (-1)^{k+1} \frac{D_x^k}{k}, \\ \hat{M} &= \frac{D_x + 1}{D_x + 2} + x(D_x + 1). \end{aligned} \quad (4.7)$$

4.1 Differential equation

First, note the following Mc-Laurin expansion:

$$\frac{t+1}{t+2} = \frac{1}{2} + \sum_{k=1}^{\infty} \frac{(-1)^{k+1} t^k}{2^{k+1}}. \quad (4.8)$$

Recalling equation (3.14), in the present case, we have

Theorem 4.4. *The Sheffer polynomials $\{\tilde{r}_n(x)\}$ satisfy the differential equation*

$$\left[\frac{D_x + 1}{D_x + 2} + x(D_x + 1) \right] \sum_{k=1}^{\infty} (-1)^{k+1} \frac{D_x^k}{k} \tilde{r}_n(x) = n \tilde{r}_n(x), \quad (4.9)$$

or, in equivalent form, recalling (4.8)

$$\left[\frac{1}{2} + \sum_{k=1}^{\infty} \frac{(-1)^{k+1} D_x^k}{2^{k+1}} + x(D_x + 1) \right] \left[\sum_{k=1}^{\infty} (-1)^{k+1} \frac{D_x^k}{k} \tilde{r}_n(x) \right] = n \tilde{r}_n(x), \quad (4.10)$$

that is

$$\left[\frac{1}{2} + \sum_{k=1}^n \frac{(-1)^{k+1} D_x^k}{2^{k+1}} + x(D_x + 1) \right] \left[\sum_{k=1}^n (-1)^{k+1} \frac{D_x^k}{k} \tilde{r}_n(x) \right] = n \tilde{r}_n(x), \quad (4.11)$$

because, for any fixed n , the series expansions in equation (4.10) reduce to finite sums, with upper limit n , when are applied to a polynomial of degree n .

4.2 First few values

$$\tilde{r}_0(x) = 1$$

$$\tilde{r}_1(x) = x + \frac{1}{2}$$

$$\tilde{r}_2(x) = x^2 + 2x + \frac{1}{2}$$

$$\tilde{r}_3(x) = x^3 + \frac{9}{2}x^2 + 4x + \frac{1}{2}$$

$$\tilde{r}_4(x) = x^4 + 8x^3 + 16x^2 + 8x + \frac{1}{2}$$

$$\tilde{r}_5(x) = x^5 + \frac{25}{2}x^4 + 45x^3 + \frac{105}{2}x^2 + 16x + \frac{1}{2}$$

$$\tilde{r}_6(x) = x^6 + 18x^5 + \frac{205}{2}x^4 + 220x^3 + 166x^2 + 32x + \frac{1}{2}$$

$$\tilde{r}_7(x) = x^7 + \frac{49}{2}x^6 + 203x^5 + 700x^4 + 1001x^3 + \frac{1029}{2}x^2 + 64x + \frac{1}{2}$$

$$\tilde{r}_8(x) = x^8 + 32x^7 + 364x^6 + 1848x^5 + 4326x^4 + 4368x^3 + 1576x^2 + 128x + \frac{1}{2}$$

Further values can be easily achieved by using Wolfram Alpha[©].

5 Conclusion

We have introduced two set of adjoint Sheffer polynomials, called adjoint Charkier and adjoint Related polynomials, in the framework of a general technique associating a new Sheffer polynomial set starting from a given one. We have derived their main characteristics by using the monomiality property and the possibility to construct the shift operators by means of a preceding result by Y. Ben Cheikh.

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