

A New Procedure for Detecting the Population Having More New Better than Used Property Based on U-statistic

¹Kala Nayak, ²Parameshwar V. Pandit and ²Shubhashree Joshi

¹Don Bosco College of Engineering, Margao, Goa-403602, India
¹kalanayak@gmail.com

²Department of Statistics, Bangalore University, Bengaluru-560056, India
² panditpv12@gmail.com; ²shubhashreejoshi13@gmail.com
Corresponding Author: Parameshwar V. Pandit

ABSTRACT

In this paper, a new test procedure based on U-statistic for detecting a distribution possessing 'more new better than used (NBU)-ness' property of life distributions is proposed to compare two life distributions possessing NBU-ness property. The distributional properties of the proposed U-statistic are studied. The asymptotic relative efficiencies of the proposed procedure is evaluated with respect to the tests available in the literature for this problem. It is observed that the members of proposed class perform well as compared to the test due to Hollander, Park and Proschan (1986).

Keywords: NBU, Two-sample test, Pitman ARE, U-statistic.

Mathematics Subject Classification: 62N05, 62G08

1. Introduction

A problem of selecting a population which possesses more positive ageing property among two populations possessing positive ageing property is an area of interest in survival analysis. The problem of testing exponentiality against the alternative of NBU is studied by many researchers. This problem has been studied by Hollander and Proschan (1972), Koul (1977), Kumazawa (1983), Ahmed (1994, 2004) among others. However, the problem of testing whether one distribution possess more 'positive ageing' property than the other distribution has received less attention. Hollander, Park and Proschan(1986) is the first procedure developed specifically for detecting NBU-ness property. Pandit and Gudaganavar (2007) proposed tests for detecting positive ageing (IFR and NBU ness) property of distributions. Pandit and Gudaganavar (2009) proposed a test procedure for selecting a distribution possessing more new better than used at specified age t_0 (NBU- t_0) property between two NBU- t_0 populations Pandit and Gudaganavar (2010) studied the problem of detecting more NBU-ness property of life distributions. In this paper, we consider a test procedure to test the hypothesis that the two life distributions are identical against the alternative that one possesses more 'new better than used' property than the other.

The following situation illustrates how a test for the hypothesis that ‘the two life distributions are identical against the alternative that one possesses more ‘new better than used’ property than the other’ might prove useful.

A police department is considering buying a fleet of used cars for its detectives. The department is trying to choose between a fleet of Camaros and a fleet of Firebirds of comparable ages which are being offered for sale by a rental Cam company. It is known that the mean life times of the two types of cars are equal. In this case the department would prefer the Firebird fleet if it were less NBU than the Camaro fleet. The primary information that the means are equal may be known a prior, or could be checked by application of a standard two-sample test for location differences.

2. NEW TEST PROCEDURE

Let $X_1, X_2, \dots, X_k$ and $Y_1, Y_2, \dots, Y_n$ denote two random samples from continuous life distributions F and G , respectively. Here, a class of tests are developed for testing

$H_0: F = G$ (the common distribution is not specified) against $H_1: F$ is ‘more NBU’ than G .

Consider the parameter, for an integer $m > 1$

$$\gamma(F, G) = \gamma(G) - \gamma(F),$$

$$\text{Where } \gamma(F) = \int_0^\infty \bar{F}(mx) dF(x) \quad \text{and} \quad \gamma(G) = \int_0^\infty \bar{G}(my) dG(y).$$

Here, $\gamma(F)$ and $\gamma(G)$ can be considered as the measure of degree of the NBU-ness. Ahmed (2004) test used this measure as basis for their test statistic. If $F(G)$ belongs to NBU, then $\gamma(F) < 0$ ($\gamma(G) < 0$) and $\gamma(F, G)$ can be taken as a measure by which F is ‘more NBU’ than G . Under H_0 , $\gamma(F, G) = 0$ and it is strictly greater than zero under H_1 .

An unbiased estimator for $\gamma(F, G)$, which is defined as

$$T_{k,n} = T_n - T_k$$

where T_k and T_n are U -statistics with kernels of degree $(m+1)$ which are defined as

$$h_1(x_1, x_2) = \frac{1}{2} [I(x_1 > mx_2) + I(x_2 > mx_1)]$$

$$\text{and } h_2(y_1, y_2) = \frac{1}{2} [I(y_1 > my_2) + I(y_2 > my_1)] \text{ respectively.}$$

Here $I(\cdot)$ is an indicator function.

2.1. Asymptotic normality of the test $T_{k,n}$

In this subsection, we study the asymptotic distribution of $T_{k,n}$. For that define

$$\xi_1(F) = E[\psi_1(X_1)]^2 - [\gamma(F)]^2,$$

$$\text{where } \psi_1(X_1) = E[h_1(x, X_2)] = \frac{1}{2} \left[F\left(\frac{x}{m}\right) + \bar{F}(mx) \right]$$

Next, $\xi_1(G)$ is defined as

$$\xi_1(G) = E[\psi_1^*(Y_1)]^2 - [\gamma(G)]^2,$$

where $\psi_1^*(Y_1) = E[h_2(y, Y_2)] = \frac{1}{2} \left[G\left(\frac{y}{m}\right) + \bar{G}(my) \right]$

The asymptotic normality of the test $T_{k,n}$ is presented in the following theorem.

Theorem 2.1: The asymptotic distribution of $\sqrt{N}[T_{k,n} - \gamma(F, G)]$ is normal with mean zero and

variance given by $\sigma^2(T_{k,n}) = \sigma_1^2 + \sigma_2^2$,

where $\sigma_1^2 = \frac{4\xi_1(F)}{\lambda}$ and $\sigma_2^2 = \frac{4\xi_1(G)}{1-\lambda}$,

where $\xi_1(F)$ and $\xi_1(G)$ are as defined above.

Under $H_0: F = G = F_0$, then $\xi_1(F) = \xi_1(G) = \xi_1(F_0)$ and if F_0 is exponential distribution function then,

$$4\xi_1(F_0) = \frac{1}{2m+1} - \frac{2(m-1)}{m+1} - \frac{2m}{m(m+1)+1} - \frac{4}{(m+1)^2} + \frac{2(m+1)}{m+2}$$

Proof: Proof follows from Hoeffding(1948).

The approximate α -level test rejects H_0 in favour of H_1 , if $\frac{\sqrt{N}T_{k,n}}{\sigma^2(T_{k,n})} > Z_\alpha$, where Z_α is the upper α -percentile point of standard normal distribution. Since, $\gamma(F, G) > 0$ under H_1 and from the asymptotic normality of $T_{k,n}$, the test based on $T_{k,n}$ is consistent against the alternative F is 'more NBU than' G .

3. CONSISTENT ESTIMATOR OF $\sigma^2(T_{k,n})$:

Here, we find a consistent estimator for $\sigma^2(T_{k,n})$, which is necessary to implement the test procedure developed here. For that let,

$$\hat{\xi}_1(F) = \frac{1}{k-1} \sum_{i=1}^k [\psi_1(X_i) - T_k]^2,$$

where $\psi_1(X_i) = \frac{1}{k-1} \sum_i h_1(X_i, X_\alpha)$

and the summation $\sum_i$ extends over all possible $i \leq \alpha \leq k$ with $\alpha \neq i$.

We define $\hat{\xi}_1(G) = \frac{1}{n-1} \sum_{j=1}^n [\psi_1^*(Y_j) - T_n]^2$, where $\psi_1^*(Y_j) = \frac{1}{k-1} \sum_j h_1(Y_j, Y_\beta)$ and the summation $\sum_j$ is over all possible $j \leq \beta \leq n$ with $\beta \neq j$.

It can easily be shown that

$$T_k = \frac{1}{k} \sum_{i=1}^k \psi_1(X_i) \text{ and } T_n = \frac{1}{n} \sum_{j=1}^n \psi_1^*(Y_j).$$

Then, from Puri and Sen (1971) $\hat{\xi}_1(F)$ and $\hat{\xi}_1(G)$ are consistent estimators of $\xi_1(F)$ and $\xi_1(G)$ respectively and hence the consistent estimator $\hat{\sigma}^2(T_{k,n})$ of $\sigma^2(T_{k,n})$ is obtained by replacing $\xi_1(F)$ and $\xi_1(G)$ by $\hat{\xi}_1(F)$ and $\hat{\xi}_1(G)$ respectively in the expression $\sigma^2(T_{k,n})$. That is,

$$\hat{\sigma}^2(T_{k,n}) = \frac{4N\hat{\xi}_1(F)}{\lambda} + \frac{4N\hat{\xi}_1(G)}{1-\lambda}, \quad N = k + n.$$

By Slutsky's theorem, we have $\frac{\sqrt{N}T_{k,n}}{\hat{\sigma}^2(T_{k,n})}$ is asymptotically $N(0,1)$ under H_0 .

4. ASYMPTOTIC RELATIVE EFFICIENCY

We study the asymptotic relative efficiency of $T_{k,n}$, relative to the $V_{k,n}$ test statistic given by Hollander, Park and Proschan(1986) for the two pairs of distributions $(F_{i,\theta}, G)$. Here, we assume that G is an exponential distribution with mean one. We consider $F_{1,\theta}$ as Weibull distribution and $F_{2,\theta}$ as Linear failure rate distribution as defined below:

1. Weibull Distribution:

$$\bar{F}_{1,\theta} = \exp(-x^\theta), \quad x, \theta > 0.$$

2. Linear Failure Rate Distribution:

$$\bar{F}_{2,\theta} = \exp\left(-\left(x + \theta \frac{x^2}{2}\right)\right), \quad x > 0, \theta \geq 0.$$

The Pitman asymptotic efficacy is

$$eff(T_{k,n}, F_{i,\theta}, G) = \frac{1}{\sigma_0^2(T_{k,n})} \left\{ \frac{d}{d\theta} \gamma(F_{i,\theta}, G) \right\}_{\theta \rightarrow \theta_0}$$

The ARE's of the proposed tests with respect to the $V_{k,n}$ test of Hollander, Park and Proschan(1986) for Weibull distribution with parameter θ , and Linear failure rate distribution are respectively 1.8432 and 2.6923.

Next, we compute the efficiency of the two sample test based on $T_{k,n}$ proposed in section 2, by specifying the common null distribution in the null hypothesis as F_θ with $\theta \geq 1$ and considering sequence of alternatives $(F_{\theta\phi_N}, F_\theta)$, where $\phi = 1 + \frac{\alpha}{\sqrt{N}}$, α being arbitrary positive constant. Note that as $N \rightarrow \infty$, the sequence of alternatives converges to the null hypothesis. The efficacy of the $T_{k,n}$ test is given by

$$eff(T_{k,n}) = \frac{[\gamma'(F_{\theta\phi_N}, F_\theta)]^2}{\sigma_0^2(T_{k,n})}$$

where $\sigma_0^2(T_{k,n})$ is null asymptotic variance of $\sqrt{N} T_{k,n}$ and $\gamma'(F, G) = \left[\frac{d\gamma(F_{\theta\phi_N}, F_\theta)}{d\phi} \right]_{\phi=1}$

The sequence of alternatives considered here are $(F_{1,\theta\phi_N}, F_{1,\theta})$, $(F_{2,\theta\phi_N}, F_{2,\theta})$ and $(F_{3,\theta\phi_N}, F_{3,\theta})$

The Asymptotic relative efficiencies of the proposed test $T_{k,n}$ relative to the test due to Hollander, Park and Proschan (1986) $V_{k,n}$ for the various alternatives are presented in table 1, table 2 and table 3 respectively.

Table 1

ARE of $T_{k,n}$ w.r.t. $V_{k,n}$ for $(F_{1,\theta\phi_N}, F_{1,\theta})$

$m \rightarrow$ $\theta \downarrow$	2	3	4	5
2	1.3522	1.2725	1.1414	1.0246
3	4.7063	3.6523	2.8459	2.2945
4	1.432	0.9118	0.6191	0.4482
5	2.3486	1.2304	0.7358	0.4699
6	1.2243	0.5354	0.2804	0.1574
7	1.3871	0.5089	0.2463	0.1144
8	0.9226	0.2865	0.1247	0.0485
9	0.6982	0.2443	0.0912	0.0294
10	0.6258	0.1573	0.0519	0.0135

Table 2

ARE of $T_{k,n}$ w.r.t. $V_{k,n}$ for $(F_{2,\theta\phi_N}, F_{2,\theta})$

$m \rightarrow$ $\theta \downarrow$	2	3	4	5
2	1.6799	1.6586	1.6383	1.6212
3	1.6641	1.6371	1.6146	1.5958
4	1.6587	1.6279	1.6034	0.7128
5	1.6582	1.6243	1.5983	1.5776
6	1.6601	1.6237	1.5964	1.5749
7	1.6632	1.6247	1.5963	1.5741
8	1.6671	1.6268	1.5974	1.5746
9	1.6715	1.6295	1.5992	1.5759
10	1.6762	1.6327	1.6016	1.5778

Table 3

ARE of $T_{k,n}$ w.r.t. $V_{k,n}$ for $(F_{3,\theta\phi_N}, F_{3,\theta})$

$m \rightarrow$ $\theta \downarrow$	2	3	4	5
2	2.5776	2.8234	2.8750	2.8412
3	1.0864	0.8754	0.6946	0.5582
4	1.4023	0.9043	0.5902	0.4003
5	1.8609	1.0739	0.5855	0.3397
6	3.3181	1.4374	0.6602	0.3295
7	2.4501	2.1549	0.8368	0.3608
8	3.4573	3.5692	1.1751	0.4379
9	5.1083	5.4312	1.2905	0.3606
10	2.5374	1.2801	0.2912	2.4733

5. SOME REMARKS

1. The Asymptotic relative efficiencies of the proposed test with respect to the test due to Hollander, Park and Proschan (1986) is computed for two pairs of distributions (F_θ, G) with G is exponential with mean one and F_θ as Weibull and Linear failure rate distributions.
2. It is observed that the proposed test performs better for the alternatives considered F_0 is either Weibull or Linear failure rate distributions when G is exponential.
3. The asymptotic relative efficiencies of the test proposed with respect to Hollander, Park and Proschan (1986) is computed for three pairs of distributions $(F_{1,\theta\phi_N}, F_{1,\theta})$, $(F_{2,\theta\phi_N}, F_{2,\theta})$ and $(F_{3,\theta\phi_N}, F_{3,\theta})$ with F_1, F_2, F_3 as Weibull, Makeham and Gamma distributions respectively.
4. It is observed that the proposed test performs better than the test due to Hollander, Park and Proschan (1986) when the underlying distribution is Makeham or Gamma.
5. The test due to Hollander, Park and Proschan (1986) performs better for Weibull and Linear failure rate distributions.
6. Hence, if the data under consideration is exactly NBU, the new test proposed would be a better choice.

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