

Multitravelling Wave Solutions for some Nonlinear Fractional Equations of Mathematical Physics

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ABSTRACT

In this paper, the nonlinear space-time conformable fractional partial differential equations are transformed into Schrödinger equations based on the solitary wave variable and Malfliet method. This method is used to construct several classes of the multisoliton wave, multi triangular type, and multi Jacobi elliptic doubly periodic type solutions of the space-time conformable fractional Korteweg–de Vries, regularized long wave equation, and Boussinesq equations.

Keywords: conformable fractional differential equations; Malfliet method; solitary wave, Jacobi elliptic doubly periodic.

Mathematics Subject Classification: 26A33, 34A08, 35R11

1. INTRODUCTION

The fractional calculus became an important field in applied sciences in the late decades. The applications of fractional calculus including a wide and various sciences as applied mathematics, physics, biology, and engineering, Hilfer (2000), Kilbas et al.(2006), Sun et al. (2018), Ghosh et al. (2016). The fractional derivatives have different definitions and types. For example, Riemann–Liouville, Caputo, Riesz, Kolwankar–Gangal, Podlubny (1999). In the last few years, Khalil et al. (2014) proposed a new fractional derivative that is called conformable fractional derivative. The conformable fractional derivatives have the advantages of satisfying the properties of the classical integer derivatives, Khalil (2014). The definition of the conformable fractional derivative of a function f of order α is given in terms of limit as

$$D^\alpha f(t) = \lim_{\varepsilon \rightarrow 0} \frac{f(t+\varepsilon t^{1-\alpha}) - f(t)}{\varepsilon}, t > 0, \alpha \in (0,1], \quad (1)$$

The function f is said to be α -differentiable.

$$f^\alpha(0) = \lim_{t \rightarrow 0^+} f^\alpha(t). \quad (2)$$

Assume that f and g are α -differentiable functions and c is an arbitrary constant, then the properties of the conformable fractional derivative are given as (Khalil (2014)

$$\text{i. } D^\alpha t^p = p t^{p-\alpha}, p \in \mathbb{R}.$$

$$\text{ii. } D^\alpha c = 0, \text{ where } c \text{ is a constant.}$$

$$\text{iii. } D^\alpha f(t) = t^{1-\alpha} \frac{df}{dt}.$$

$$\text{iv. } D^\alpha (af + bg) = a D^\alpha f + b D^\alpha g, \quad a, b \in \mathbb{R}.$$

$$\text{v. } D^\alpha (fg) = f D^\alpha g + g D^\alpha f.$$

$$\text{vi. } D^\alpha \left(\frac{f}{g} \right) = \frac{g D^\alpha f - f D^\alpha g}{g^2}.$$

$$\text{vii. } D^\alpha (f(g(t))) = \frac{df}{dg} D^\alpha g(t) = t^{1-\alpha} \frac{df}{dg} \frac{dg}{dt}.$$

Following are the conformable fractional derivatives of some elementary functions

$$\begin{cases} D^\alpha e^{ct} = ct^{1-\alpha} e^{ct}, D^\alpha \sin(ct) = ct^{1-\alpha} \cos(ct), D^\alpha \cos(ct) = -ct^{1-\alpha} \sin(ct) \\ D^\alpha e^{ct^\alpha} = c^\alpha e^{ct^\alpha}, D^\alpha \sin(ct^\alpha) = c^\alpha \cos(ct^\alpha), D^\alpha \cos(ct^\alpha) = -c^\alpha \sin(ct^\alpha) \end{cases}, \quad (3)$$

The rest of the paper is outline as follows: In Sec. 2, we describe the steps of constructing the solution method. In Sec. 3 the solutions of the space-time conformable fractional Korteweg–de Vries are derived. In Sec. 4, the solutions of the space-time conformable fractional regularized long wave equation are obtained. Further, the solutions of the space-time conformable fractional Boussinesq equation are given in Sec. 5. We conclude the paper in Sec. 6.

2. DESCRIPTION OF THE METHOD

Consider the conformable space-time fractional partial differential equation in two independent variables x and t as

$$G(u, D_t^\alpha u, D_x^\beta u, D_t^{\alpha\alpha} u, D_x^{\beta\beta} u, D_x^{\beta\beta\beta} u, \dots) = 0. \quad (4)$$

By introducing the solitary wave variable

$$u(x, t) = u(\xi), \quad \xi = k \left(\frac{x^\beta}{\beta} - \omega \frac{t^\alpha}{\alpha} \right), \quad (5)$$

where k, ω are arbitrary constants, and α, β are the orders of the conformable derivatives where $0 < \alpha, \beta \leq 1$. By applying the chain rule on Eq. 5) we have

$$\left. \begin{aligned} D_t^\alpha u &= -\omega k u', D_x^\beta u = k u', D_t^{\alpha\alpha} u = k^2 \omega^2 u'', D_x^{\beta\beta} u = k^2 u'' \\ D_x^{\beta\beta\beta} u &= k^3 u''', D_t^\alpha D_x^{\beta\beta} u = -\omega k^3 u''' \end{aligned} \right\}, \quad ' \equiv \frac{d}{d\xi} \quad (6)$$

Based on the equations (5) and (6) the space-time conformable fractional partial differential equation Eq. (4) becomes an ordinary differential equation given by

$$H(u, u', u'', u''', \dots) = 0. \quad (7)$$

The solution of Eq. (7) is assumed as a product of two functions (Malfliet, 1981) that expressed as

$$u(\xi) = p(\xi)q(\xi), \quad (8)$$

where p and q are arbitrary functions in ξ . By substituting Eq. (8) into Eq. (7) where assuming the existence of the derivatives of p and q with respect to ξ . By applying the techniques of refs (Hassanien et al. (2003), Abdel-Rahman (1983), Zhang et al. (2000)) then Eq. (7) converts to Schrödinger equation satisfied by the functions p and q given as.

$$\frac{d^2\psi}{d\xi^2} + (f(u) + \lambda)\psi = 0, \quad (9)$$

where ψ stands for both p and q , λ is the eigenvalue, $f(u)$ is the scattering potential. Let us assume that there are N distinct eigenvalues λ_n , $n = 1, 2, \dots, N$ in the Schrödinger Eq. (9) with corresponding wave functions ψ_n . At this stage we use the wave functions ψ_n to construct the solution of the nonlinear Eq. (7) as

$$u(\xi) = \sum_{n=1}^N \psi_n^2(\xi_n), \quad \xi_n = k_n \left(\frac{x^\beta}{\beta} - \omega_n \frac{t^\alpha}{\alpha} \right), \quad (10)$$

If the wave functions ψ_n do not overlap with each other, then Eq. (9) can be written as

$$\frac{d^2\psi_n}{d\xi^2} + (f(\psi_n^2) + \lambda_n)\psi_n = 0. \quad (11)$$

Our main goal here is to find the wave functions ψ_n that satisfy Eq. (11) to get the multisoliton solution of the considered problem. In order to find the wave functions ψ_n as solution of Eq. (11), we assume that ψ_n are expressed in the form

$$\psi_n = \sum_{j=0}^S A_j F^j(\xi_n), \quad F' = \sqrt{c_0 + c_1 F + c_2 F^2 + c_3 F^3 + c_4 F^4}, \quad (12)$$

where S is determined by balancing the highest power of the linear term with the highest power of the nonlinear term in Eq. (11), A_j are parameters to be determined. Several kinds of solutions can be generated from Eq. (12) by freely choose of possible different values of the constants c_0 , c_1 , c_2 , c_3 , and c_4 .

3. THE SPACE-TIME CONFORMABLE FRACTIONAL KORTEWEG-DE VRIES

In this section we discuss the solution of the space-time fractional conformal Korteweg-de Vries (KdV) equation given as

$$D_t^\alpha u - 6uD_x^\beta u + D_x^{\beta\beta\beta} u = 0. \quad (13)$$

Substitution of Eq. (5) and formula (6) into Eq. (13) transforms the space-time fractional KdV into the following ordinary differential equation

$$-\omega u' - 6uu' + k^2 u''' = 0, \quad ' \equiv \frac{d}{d\xi}. \quad (14)$$

Based on the formula (8) we get

$$u' = pq' + (p \leftrightarrow q), \quad u''' = (pq''' + 3p''q') + (p \leftrightarrow q). \quad (15)$$

We express the nonlinear term $6uu'$ as

$$6uu' = auu' + buu' = aupq' + \frac{1}{2}bpqu' + (p \leftrightarrow q), \quad (16)$$

where $a + b = 6$. Now by substituting Eq. (8), Eq. (15), and Eq. (16) into Eq. (14) we obtain

$$p \left[k^2 q''' + 3k^2 \frac{p''}{p} q' - a \left(u + \frac{\omega}{a} \right) q' - \frac{1}{2} bqu' \right] + (p \leftrightarrow q) = 0, \quad (17)$$

motivated by the occurrence of q' . Now set

$$k^2 \frac{p''}{p} = u + \frac{\omega}{a}, \quad (18)$$

with a similar equation for q .

For $\psi = p, q$ we obtain from Eq. (18) the Schrödinger equation

$$k^2 \psi'' - \left(u + \frac{\omega}{a} \right) \psi = 0. \quad (19)$$

Eq. (19) has scattering potential $\left(\frac{u}{k^2} \right)$ and the eigenvalue $-\left(\frac{\omega}{ak^2} \right)$. Insertion of Eq. (18) into Eq. (17) yields

$$p \left[k^2 q''' + (3 - a) \left(u + \frac{\omega}{a} \right) q' - \frac{1}{2} bqu' \right] + (p \leftrightarrow q) = 0. \quad (20)$$

Differentiating Eq. (19) w.r.t. ξ we get

$$k^2 \psi''' - \left(u + \frac{\omega}{a} \right) \psi' - \psi u' = 0. \quad (21)$$

Eq. (20) coincides with Eq. (21) in case $a = 4$ and $b = 2$, then Eq. (19) can be written as

$$k^2 \psi'' - \left(u + \frac{\omega}{4} \right) \psi = 0. \quad (22)$$

Assuming that the potential $\left(\frac{u}{k^2} \right)$ is attractive, and there are N distinct discrete eigenvalues $\left(\frac{\omega_n}{4k^2} \right)$, $n = 1, 2, \dots, N$ associated with it. Thus we rewrite Eq. (22) as

$$k^2 \psi_n'' - \left(u + \frac{\omega_n}{4} \right) \psi_n = 0, \quad \xi_n = k_n \left(\frac{x^\beta}{\beta} - \frac{\omega_n t^\alpha}{\alpha} \right), \quad (23)$$

Assuming that the wave functions do not overlap with each other, then Eq. (23) can be expressed as

$$k_n^2 \psi_n'' - \left(\psi_n^2 + \frac{\omega_n}{4} \right) \psi_n = 0, \quad (24)$$

By balancing ψ_n'' and ψ_n^3 then from Eq. (12) we get $s = 1$, therefore the series solution is given from Eq. (12) as

$$\psi_n = A_0 + A_1 F(\xi_n), \quad F' = \sqrt{c_0 + c_1 F + c_2 F^2 + c_3 F^3 + c_4 F^4}. \quad (25)$$

Substituting (25) into (24) and setting the coefficients of powers of F to zero, we get an algebraic system with solution

$$A_0 = 0, \quad c_1 = 0, \quad \omega_n = 4c_2 k_n^2, \quad c_3 = 0, \quad A_1 = \pm \sqrt{2c_4} k_n. \quad (26)$$

If $c_0 = 0$, we obtain the bell-shaped wave solution

$$\psi_{n1} = \pm \sqrt{-2c_2} k_n \operatorname{sech} \sqrt{c_2} \xi_n \quad c_2 > 0, c_4 < 0, \quad (27)$$

$$\psi_{n2} = \pm \sqrt{-2c_2} k_n \operatorname{sec} \sqrt{-c_2} \xi_n \quad c_2 < 0, c_4 > 0. \quad (28)$$

Therefore, the corresponding solutions for the space time conformal fractional KdV equation are given as multisoliton wave and multi triangular type as

$$u_1 = -2c_2 \sum_{n=1}^N k_n^2 \operatorname{sech}^2(\sqrt{c_2} \xi_n), \quad \xi_n = k_n \left(\frac{x^\beta}{\beta} - \frac{4c_2 k_n^2 t^\alpha}{\alpha} \right), \quad c_2 > 0, \quad (29)$$

$$u_2 = -2c_2 \sum_{n=1}^N k_n^2 \sec^2(\sqrt{-c_2} \xi_n), \quad \xi_n = k_n \left(\frac{x^\beta}{\beta} - \frac{4c_2 k_n^2 t^\alpha}{\alpha} \right), \quad c_2 > 0. \quad (30)$$

If $c_1 \neq 0$ we obtain kink-shaped solitary wave solution, a triangular type solution and Jacobi elliptic doubly periodic type solutions, Lawden (1989), Fan and Hon (2002), Yan (2003)

$$\psi_{n3} = \pm \sqrt{-2c_2} k_n \tanh \sqrt{-\frac{c_2}{2}} \xi_n \quad c_2 < 0, \quad c_4 > 0, \quad c_0 = \frac{c_2^2}{4c_4}, \quad (31)$$

$$\psi_{n4} = \pm \sqrt{2c_2} k_n \tan \sqrt{\frac{c_2}{2}} \xi_n \quad c_2 > 0, \quad c_4 > 0, \quad c_0 = \frac{c_2^2}{4c_4}, \quad (32)$$

$$\psi_{n5} = \pm \sqrt{-\frac{2c_2 m^2}{2m^2-1}} k_n \operatorname{cn} \sqrt{\frac{c_2}{2m^2-1}} \xi_n, \quad c_2 > 0, \quad c_4 < 0, \quad c_0 = \frac{c_2^2 m^2 (1-m^2)}{4c_4 (2m^2-1)^2}, \quad (33)$$

$$\psi_{n6} = \pm \sqrt{-\frac{2m^2}{2-m^2}} k_n \operatorname{dn} \sqrt{\frac{c_2}{2-m^2}} \xi_n, \quad c_2 > 0, \quad c_4 < 0, \quad c_0 = \frac{c_2^2 (1-m^2)}{4c_4 (2-m^2)^2}, \quad (34)$$

$$\psi_{n7} = \pm \sqrt{-\frac{2c_2 m^2}{1+m^2}} k_n \operatorname{sn} \sqrt{-\frac{c_2}{m^2+1}} \xi_n, \quad c_2 < 0, \quad c_4 > 0, \quad c_0 = \frac{c_2^2 m^2}{c_4 (m^2+1)^2}, \quad (35)$$

where m is a modulus. Eqs. (31-35) lead to the following multisoliton wave, multi triangular and multi Jacobi elliptic doubly periodic type solutions of Eq. (13) respectively given as

$$u_3 = 2|c_2| \sum_{n=1}^N k_n^2 \tanh^2 \left(\sqrt{-\frac{c_2}{2}} \xi_n \right), \quad \xi_n = k_n \left(\frac{x^\beta}{\beta} - \frac{4c_2 k_n^2 t^\alpha}{\alpha} \right), \quad c_2 < 0, \quad c_0 = \frac{c_2^2}{4c_4}, \quad (36)$$

$$u_4 = 2c_2 \sum_{n=1}^N k_n^2 \tan^2 \left(\sqrt{\frac{c_2}{2}} \xi_n \right), \quad \xi_n = k_n \left(\frac{x^\beta}{\beta} - \frac{4c_2 k_n^2 t^\alpha}{\alpha} \right), \quad c_2 > 0, \quad c_0 = \frac{c_2^2}{4c_4}, \quad (37)$$

$$u_5 = \frac{2c_2 m^2}{1-2m^2} \sum_{n=1}^N k_n^2 \operatorname{cn}^2 \left(\sqrt{\frac{c_2}{2m^2-1}} \xi_n \right), \quad \xi_n = k_n \left(\frac{x^\beta}{\beta} - \frac{4c_2 k_n^2 t^\alpha}{\alpha} \right), \quad c_2 > 0, \quad c_0 = \frac{c_2^2 m^2 (1-m^2)}{4c_4 (2m^2-1)^2}, \quad (38)$$

$$u_6 = \frac{2m^2}{(m^2-2)} \sum_{n=1}^N k_n^2 \operatorname{dn}^2 \left(\sqrt{\frac{c_2}{2-m^2}} \xi_n \right), \quad \xi_n = k_n \left(\frac{x^\beta}{\beta} - \frac{4c_2 k_n^2 t^\alpha}{\alpha} \right), \quad c_2 > 0, \quad c_0 = \frac{c_2^2 (1-m^2)}{4c_4 (2-m^2)^2}, \quad (39)$$

$$u_7 = \frac{2|c_2| m^2}{m^2+1} \sum_{n=1}^N k_n^2 \operatorname{sn}^2 \left(\sqrt{\frac{-c_2}{2-m^2}} \xi_n \right), \quad \xi_n = k_n \left(\frac{x^\beta}{\beta} - \frac{4c_2 k_n^2 t^\alpha}{\alpha} \right), \quad c_2 < 0, \quad c_0 = \frac{c_2^2 m^2}{c_4 (1+m^2)^2}, \quad (40)$$

4. THE SPACE-TIME CONFORMABLE FRACTIONAL REGULARIZED LONG WAVE EQUATION

Consider the space-time conformable fractional regularized long wave (RLW) equation

$$D_t^\alpha u + D_x^\beta u + 12uD_x^\beta u - D_t^\alpha D_x^{\beta\beta} u = 0. \quad (41)$$

Consider the transformations $u \rightarrow \frac{u}{\lambda}$, then Eq. (41) becomes

$$D_t^\alpha u + D_x^\beta u + \frac{12}{\lambda} u D_x^\beta u - D_t^\alpha D_x^{\beta\beta} u = 0. \quad (42)$$

Substituting Eq. (5) and Eq. (6) into (42) will obtain the ordinary equation

$$(1 - \omega)u' + \frac{12}{\lambda}uu' + \omega ku''' = 0, \quad ' \equiv \frac{d}{d\xi}. \quad (43)$$

By rewriting the nonlinear $\frac{1}{12}uu'$ as

$$\frac{12}{\lambda}uu' = auu' + buu' = aupq' + \frac{1}{2}bpqu' + (p \leftrightarrow q), \quad (44)$$

where $a + b = \frac{12}{\lambda}$. Substitution of the Eqs. (8), (15) and (44) into (43) yield

$$p \left[k\omega q''' + 3\omega k \frac{p''}{p} q' - a \left(\frac{\omega}{a} - \frac{1}{a} - u \right) q' + \frac{1}{2}bpqu' \right] + (p \leftrightarrow q) = 0. \quad (45)$$

Following the same steps as in section (3), we set

$$\omega k \frac{p''}{p} = \frac{\omega}{a} - \frac{1}{a} - u. \quad (46)$$

For $\psi = p, q$ we have from Eq. (46) the Schrödinger equation

$$\omega k \psi'' + \left(u + \frac{1-\omega}{a} \right) \psi = 0, \quad (47)$$

with the scattering potential $\left(-\frac{u}{k\omega} \right)$ and the eigenvalue $\left(\frac{1-\omega}{k\omega a} \right)$. Using Eq. (46) to write equation Eq. (45) as

$$p \left[k\omega q''' + (3 - a) \left(\frac{\omega}{a} - \frac{1}{a} - u \right) q' + \frac{1}{2}bpqu' \right] + (p \leftrightarrow q) = 0. \quad (48)$$

After differentiating Eq. (47) w.r.t. ξ and comparing with Eq. (48), we find that they are coincide when $a = 4$ and $b = 2$, thus $\lambda = 2$. Thus Eq. (47) becomes

$$k\omega \psi'' + \left(u + \frac{1-\omega}{4} \right) \psi = 0. \quad (49)$$

As before assuming that the potential $\left(-\frac{u}{k\omega} \right)$ is attractive, and there are N distinct discrete eigenvalues $\left(\frac{\omega_n}{4k^2} \right), n = 1, 2, \dots, N$ associated with it. Thus we rewrite Eq. (49) as

$$\omega_n k_n \psi_n'' + \left(u + \frac{1-\omega_n}{4} \right) \psi_n = 0, \quad \xi_n = k_n \left(\frac{x^\beta}{\beta} - \frac{\omega_n t^\alpha}{\alpha} \right), \quad (50)$$

In case the wave functions do not overlap with each other, then Eq. (50) can expressed as

$$\omega_n k_n \psi_n'' + \left(\psi_n^2 + \frac{1-\omega_n}{4} \right) \psi_n = 0. \quad (51)$$

To obtain the wave functions ψ_n satisfied by Eq. (51), following the same procedure as in section (3) above we have

$$\psi_n = A_0 + A_1 F(\xi_n), \quad F' = \sqrt{c_0 + c_1 F + c_2 F^2 + c_3 F^3 + c_4 F^4}. \quad (52)$$

Substituting Eq. (52) into Eq. (51) and setting the coefficients of powers of F to zero, we obtain algebraic system with the solution

$$A_0 = 0, c_1 = 0, \omega_n = \frac{1}{1-4c_2k_n}, c_3 = 0, A_1 = \pm \sqrt{\frac{2c_4k_n}{4c_2k_n-1}}. \quad (53)$$

In case $c_0 = 0$, we have the bell-shaped wave solution, a triangular type solution

$$\psi_{n1} = \pm \sqrt{\frac{2c_2k_n}{1-4c_2k_n}} \operatorname{sech} \sqrt{c_2} \xi_n \quad c_2 > 0, c_4 < 0, \quad (54)$$

$$\psi_{n2} = \pm \sqrt{\frac{2c_2k_n}{1-4c_2k_n}} \operatorname{sec} \sqrt{-c_2} \xi_n \quad c_2 < 0, c_4 > 0. \quad (55)$$

Therefore, the solutions of the space-time conformal fractional RLW equation are given in terms of multisoliton wave and multi triangular type as

$$u_1 = \sum_{n=1}^N \frac{2c_2 k_n}{1-4c_2 k_n} \operatorname{sech}^2(\sqrt{c_2} \xi_n), \quad \xi_n = k_n \left(\frac{x^\beta}{\beta} - \frac{t^\alpha}{\alpha(1-4c_2 k_n)} \right), \quad c_2 > 0, \quad (56)$$

$$u_2 = \sum_{n=1}^N \frac{2c_2 k_n}{1-4c_2 k_n} \operatorname{sec}^2(\sqrt{-c_2} \xi_n), \quad \xi_n = k_n \left(\frac{x^\beta}{\beta} - \frac{t^\alpha}{\alpha(1-4c_2 k_n)} \right), \quad c_2 < 0, \quad (57)$$

If $c_0 \neq 0$ we obtain kink-shaped solitary wave solution, a triangular type solution, and Jacobi elliptic doubly periodic type solutions

$$\psi_{n3} = \pm \sqrt{\frac{2c_2 k_n}{1-4c_2 k_n}} \tanh \sqrt{-\frac{c_2}{2}} \xi_n \quad c_2 < 0, \quad c_4 > 0, \quad c_0 = \frac{c_2^2}{4c_4}, \quad (58)$$

$$\psi_{n4} = \pm \sqrt{\frac{2c_2 k_n}{4c_2 k_n - 1}} \tan \sqrt{\frac{c_2}{2}} \xi_n \quad c_2 > 0, \quad c_4 > 0, \quad c_0 = \frac{c_2^2}{4c_4}, \quad (59)$$

$$\psi_{n5} = \pm \sqrt{\frac{2c_2 m^2 k_n}{(2m^2 - 1)(1 - 4c_2 k_n)}} cn \sqrt{\frac{c_2}{2m^2 - 1}} \xi_n, \quad c_2 > 0, \quad c_4 < 0, \quad c_0 = \frac{c_2^2 m^2 (1 - m^2)}{4c_4 (2m^2 - 1)^2}, \quad (60)$$

$$\psi_{n6} = \pm \sqrt{\frac{2m^2 k_n}{(2 - m^2)(1 - 4c_2 k_n)}} dn \sqrt{\frac{c_2}{2 - m^2}} \xi_n, \quad c_2 > 0, \quad c_4 < 0, \quad c_0 = \frac{c_2^2 (1 - m^2)}{4c_4 (2 - m^2)^2}, \quad (61)$$

$$\psi_{n7} = \pm \sqrt{\frac{2c_2 m^2 k_n}{(1 + m^2)(1 - 4c_2 k_n)}} sn \sqrt{-\frac{c_2}{m^2 + 1}} \xi_n, \quad c_2 < 0, \quad c_4 > 0, \quad c_0 = \frac{c_2^2 m^2}{c_4 (m^2 + 1)^2}, \quad (62)$$

Therefore, the solutions of the space-time conformable fractional RLW due to the Eqs. (58-62) are the multisoliton wave, multi triangular, and multi Jacobi elliptic doubly periodic type given as

$$u_3 = \sum_{n=1}^N \frac{2c_2 k_n}{1-4c_2 k_n} \tanh^2 \left(\sqrt{-\frac{c_2}{2}} \xi_n \right), \quad \xi_n = k_n \left(\frac{x^\beta}{\beta} - \frac{t^\alpha}{\alpha(1-4c_2 k_n)} \right), \quad c_2 < 0, \quad c_0 = \frac{c_2^2}{4c_4}, \quad (63)$$

$$u_4 = \sum_{n=1}^N \frac{2c_2 k_n}{4c_2 k_n - 1} \tan^2 \left(\sqrt{\frac{c_2}{2}} \xi_n \right), \quad \xi_n = k_n \left(\frac{x^\beta}{\beta} - \frac{t^\alpha}{\alpha(1-4c_2 k_n)} \right), \quad c_2 > 0, \quad c_0 = \frac{c_2^2}{4c_4}, \quad (64)$$

$$u_5 = \sum_{n=1}^N \frac{2c_2 m^2 k_n}{(1 - 2m^2)(1 - 4c_2 k_n)} cn^2 \left(\sqrt{\frac{c_2}{2m^2 - 1}} \xi_n \right), \quad (65)$$

$$\xi_n = k_n \left(\frac{x^\beta}{\beta} - \frac{t^\alpha}{\alpha(1-4c_2 k_n)} \right), \quad c_2 > 0, \quad c_0 = \frac{c_2^2 m^2 (1 - m^2)}{4c_4 (2m^2 - 1)^2}.$$

$$u_6 = \sum_{n=1}^N \frac{2c_2 m^2 k_n}{(m^2 - 2)(1 - 4c_2 k_n)} dn^2 \left(\sqrt{\frac{c_2}{2 - m^2}} \xi_n \right), \quad (66)$$

$$\xi_n = k_n \left(\frac{x^\beta}{\beta} - \frac{t^\alpha}{\alpha(1-4c_2 k_n)} \right), \quad c_2 > 0, \quad c_0 = \frac{c_2^2 (1 - m^2)}{4c_4 (2 - m^2)^2}.$$

$$u_7 = \sum_{n=1}^N \frac{2c_2 m^2 k_n}{(m^2 + 1)(1 - 4c_2 k_n)} sn^2 \left(\sqrt{\frac{-c_2}{2 - m^2}} \xi_n \right), \quad (67)$$

$$\xi_n = k_n \left(\frac{x^\beta}{\beta} - \frac{t^\alpha}{\alpha(1-4c_2 k_n)} \right), \quad c_2 < 0, \quad c_0 = \frac{c_2^2 m^2}{c_4 (1 + m^2)^2}.$$

5. THE SPACE-TIME CONFORMABLE FRACTIONAL BOUSSINESQ EQUATION

In this section we introduce the space-time conformable fractional Boussinesq equation that is given in the form

$$D_t^{\alpha\alpha}u - D_x^{\beta\beta}u - 6D_x^{\beta\beta}(u^2) - D_x^{\beta\beta\beta\beta}u = 0. \quad (68)$$

Using Eq. (5) and Eq. (6) in Eq. (68) we have

$$(\omega^2 - 1)u'' - 6(2uu')' - k^2u^{(4)} = 0, \quad ' \equiv \frac{d}{d\xi}. \quad (69)$$

Consider the transformation $u' \rightarrow \frac{1}{2}W$, thus Eq. (69) becomes

$$(\omega^2 - 1)W' - 6WW' - k^2W''' = 0, \quad ' \equiv \frac{d}{d\xi}. \quad (70)$$

In a similar manner as in sections (3) and (4) we obtain the Schrödinger equation

$$k_n^2\psi_n'' + \left(\psi_n^2 + \frac{1-\omega_n}{4}\right)\psi_n = 0. \quad (71)$$

By following the same procedure as in the above sections we obtain

$$\psi_n = A_0 + A_1F(\xi_n), \quad F' = \sqrt{c_0 + c_1F + c_2F^2 + c_3F^3 + c_4F^4}. \quad (72)$$

Inserting Eq. (72) into Eq. (71) and equating the coefficient of powers of F to zero, we end with algebraic system with the following solution

$$A_0 = 0, c_1 = 0, \omega_n = 4c_2k_n^2 + 1, c_3 = 0, A_1 = \pm\sqrt{-2c_4}k_n. \quad (73)$$

If $c_0 = 0$, we get the solution as a bell-shaped wave and a triangular type given by

$$\psi_{n1} = \pm\sqrt{2c_2}k_n \operatorname{sech}\sqrt{c_2}\xi_n \quad c_2 > 0, c_4 < 0, \quad (74)$$

$$\psi_{n2} = \pm\sqrt{2c_2}k_n \operatorname{sec}\sqrt{-c_2}\xi_n \quad c_2 < 0, c_4 > 0, \quad (75)$$

From Eq. (74) and Eq. (75) we have respectively the solutions

$$W_1 = \sum_{n=1}^N 2c_2k_n^2 \operatorname{sech}^2(\sqrt{c_2}\xi_n), \xi_n = k_n\left(\frac{x^\beta}{\beta} - (1 + 4c_2k_n^2)\frac{t^\alpha}{\alpha}\right), \quad c_2 > 0, \quad (76)$$

$$W_2 = \sum_{n=1}^N 2c_2k_n^2 \operatorname{sec}^2(\sqrt{-c_2}\xi_n), \xi_n = k_n\left(\frac{x^\beta}{\beta} - (1 + 4c_2k_n^2)\frac{t^\alpha}{\alpha}\right), \quad c_2 < 0, \quad (77)$$

If $c_0 \neq 0$ we have kink-shaped solitary wave solution, a triangular type solution and three Jacobi elliptic doubly periodic type solutions

$$\psi_{n3} = \pm\sqrt{2c_2}k_n \tanh\sqrt{-\frac{c_2}{2}}\xi_n \quad c_2 < 0, c_4 > 0, \quad c_0 = \frac{c_2^2}{4c_4}, \quad (78)$$

$$\psi_{n4} = \pm\sqrt{-2c_2}k_n \tan\sqrt{\frac{c_2}{2}}\xi_n \quad c_2 > 0, c_4 > 0, \quad c_0 = \frac{c_2^2}{4c_4}, \quad (79)$$

$$\psi_{n5} = \pm\sqrt{\frac{2c_2m^2}{2m^2-1}}k_n \operatorname{cn}\sqrt{\frac{c_2}{2m^2-1}}\xi_n, c_2 > 0, c_4 < 0, \quad c_0 = \frac{c_2^2m^2(1-m^2)}{4c_4(2m^2-1)^2}, \quad (80)$$

$$\psi_{n6} = \pm\sqrt{\frac{2m^2}{2-m^2}}k_n \operatorname{dn}\sqrt{\frac{c_2}{2-m^2}}\xi_n, c_2 > 0, c_4 < 0, \quad c_0 = \frac{c_2^2(1-m^2)}{4c_4(2-m^2)^2}, \quad (81)$$

$$\psi_{n7} = \pm\sqrt{\frac{2c_2m^2}{1+m^2}}k_n \operatorname{sn}\sqrt{-\frac{c_2}{m^2+1}}\xi_n, c_2 < 0, c_4 > 0, \quad c_0 = \frac{c_2^2m^2}{c_4(m^2+1)^2}, \quad (82)$$

Thus, we obtain the following multisoliton wave, multi triangular and multi Jacobi elliptic doubly periodic type solutions of the problem ads

$$u_3 = \sum_{n=1}^N 2ck_n^2 \tanh^2 \left(\sqrt{-\frac{c_2}{2}} \xi_n \right), \xi_n = k_n \left(\frac{x^\beta}{\beta} - (1 + 4c_2 k_n^2) \frac{t^\alpha}{\alpha} \right), c_2 < 0, c_0 = \frac{c_2^2}{4c_4}, \quad (83)$$

$$u_4 = \sum_{n=1}^N -2ck_n^2 \tan^2 \left(\sqrt{\frac{c_2}{2}} \xi_n \right), \xi_n = k_n \left(\frac{x^\beta}{\beta} - (1 + 4c_2 k_n^2) \frac{t^\alpha}{\alpha} \right), c_2 > 0, c_0 = \frac{c_2^2}{4c_4}, \quad (84)$$

$$u_5 = \sum_{n=1}^N \frac{2c_2 m^2 k_n^2}{(2m^2 - 1)} cn^2 \left(\sqrt{\frac{c_2}{2m^2 - 1}} \xi_n \right),$$

$$\xi_n = k_n \left(\frac{x^\beta}{\beta} - (1 + 4c_2 k_n^2) \frac{t^\alpha}{\alpha} \right), \quad (85)$$

$$c_2 > 0, \quad c_0 = \frac{c_2^2 m^2 (1 - m^2)}{4c_4 (2m^2 - 1)^2}.$$

$$u_6 = \sum_{n=1}^N \frac{2c_2 m^2 k_n^2}{(2 - m^2)} dn^2 \left(\sqrt{\frac{c_2}{2 - m^2}} \xi_n \right),$$

$$\xi_n = k_n \left(\frac{x^\beta}{\beta} - (1 + 4c_2 k_n^2) \frac{t^\alpha}{\alpha} \right), \quad (86)$$

$$c_2 > 0, c_0 = \frac{c_2^2 (1 - m^2)}{4c_4 (2 - m^2)^2}.$$

$$u_7 = \sum_{n=1}^N \frac{2|c_2| m^2 k_n^2}{(m^2 + 1)} sn^2 \left(\sqrt{\frac{-c_2}{2 - m^2}} \xi_n \right),$$

$$\xi_n = k_n \left(\frac{x^\beta}{\beta} - (1 + 4c_2 k_n^2) \frac{t^\alpha}{\alpha} \right), \quad (87)$$

$$c_2 < 0, \quad c_0 = \frac{c_2^2 m^2}{c_4 (1 + m^2)^2}.$$

6. CONCLUSION

In this work, we successfully employed the Malfliet method with solitary wave solution to covert a nonlinear space-time conformable fractional partial differential equation into Schrödinger type equation. We derive various classes of the multisoliton wave, multi triangular type, and multi Jacobi elliptic doubly periodic type solutions of the space-time conformable fractional Korteweg–de Vries, regularized long wave equation, and Boussinesq equations. Further applications of this technique can be used to obtain solutions to other nonlinear fractional partial differential equations that arising in mathematical physics and other branches of science.

ACKNOWLEDGMENTS

The authors gratefully acknowledge the approval and the support of this research study by the grant no. SCI-2018-3-9-F-7644 from the Deanship of Scientific Research at Northern Border University, Arar, Saudi Arabia.

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