

Estimation of Density and Reliability Function of Inverse Lomax Distribution under Type-II Censoring and the Sampling Scheme of Bartholomew

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ABSTRACT

In this paper, the probability density and reliability functions of the Inverse Lomax probability distribution (ILD) under Type-II censoring scheme has been estimated. We have measures of reliability, $R(t) = P(X > t)$ and $P = P_r(X > Y)$ based on Type-II censoring and the sampling scheme of Bartholomew of ILD. Further two type of estimators viz., Uniformly Minimum Variance Unbiased Estimator (UMVUE) and Maximum Likelihood Estimators (MLE) have been derived for parameters of ILD. The constructed the confidence intervals for the parameter α , $R(t)$ and P are also constructed based on Type-II censoring and scheme of Bartholomew has been constructed. The performance of the obtained estimators has been compared through simulation study.

Keywords: Inverse Lomax Distribution, Point estimation, Confidence Interval, Type-II censoring, Sampling Scheme of Bartholomew, Markov Chain Monte Carlo.

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1 Introduction

Inverse Lomax distribution has found applications in various field of studies like stochastic modeling, economics, actuarial sciences and reliability analysis (Kleiber and Kotz (2003)). It is a special case of generalized beta distribution of second kind and is considered good alternative to frequently used families of distribution such as Weibull, Gamma, Generalized Exponential, Inverse Weibull, Generalized Inverted Exponential, etc. Since the Inverse Lomax distribution belongs to an inverted family of distributions so it is very flexible use it in the situation where the non-monotonicity of the failure rate has been realized, see(Singh *et. al* (2012)). Singh(Singh *et. al* studied survival estimators of inverse lomax under Type-II censoring scheme. If a random variable Y has Lomax distribution, then $X = \frac{1}{Y}$ follows the Inverse Lomax distribution (ILD) with cumulative distribution function and probability density function defined as

$$F(x) = \left(1 + \frac{\beta}{x}\right)^{-\alpha}; x \geq 0, \alpha, \beta \geq 0 \quad (1.1)$$

where α is the shape parameter and β is the scale parameter of the distribution,

$$f(x) = \frac{\alpha\beta}{x^2} \left(1 + \frac{\beta}{x}\right)^{-(1+\alpha)}; x \geq 0, \alpha, \beta \geq 0 \quad (1.2)$$

and

$$R(t) = 1 - \left(1 + \frac{\beta}{t}\right)^{-\alpha}; \alpha, \beta \geq 0 \quad (1.3)$$

From equation (1.2) and (1.3), the hazard rate of the distribution is given by

$$H(t) = \frac{\alpha\beta \left(1 + \frac{\beta}{x}\right)^{-(1+\alpha)}}{x^2 \left(1 - \left(1 + \frac{\beta}{t}\right)^{-\alpha}\right)} \quad (1.4)$$

$R(t)$ is defined as the probability of failure-free operation of an item until time t . Bartholomew (1957, 1963), Tong (1974), Johnson (1975), Kelley *et.al* (1976), Sathe and Shah (1981), Chao (1982), Awad and Gharraf (1986) have extensively studied the reliability of different life time distributions. The measure $P = P_r(X > Y)$, represents the reliability of an item or system of random strength X subject to random stress Y . It is widely used in the field of mechanical engineering and manufacturing sectors. For the recent literature one may see, Classical and Bayes estimation of stress-strength reliability (SSR) parameter for inverse Weibull distribution under progressive Type-II censored studied samples by Yadav *et.al* (2018). Reliability Functions of Nakagami Distribution under stress-strength reliability (SSR) model have been studied by Chaturvedi *et. al* (2019).

As censoring is one of the most common situation in actuarial sciences medical science and toxicological experiments, where we need to consider only those values which are above / below the cutoff point. In this paper, we have explored the stress-strength reliability measure of Inverse Lomax Distribution. In sections 2 and 3, an inferential procedure for the parameter as well as reliability measures of Inverse Lomax distribution under Type-II censoring scheme has been derived. In section 4, we have provided Asymptotic confidence intervals are for different parameters. In sections 5, we have provided point estimator of reliability function and parameter under Bartholonew scheme. In sections 6, some Asymptotic confidence intervals for parameters have been provided. In Section 7, sample has been generated using mcmc simulation techniques to compare the performance of obtained estimators.

2 Estimation under Type-II Censoring Scheme

In life testing experiment, the failure times of all units in the test are not always observed by the experimenter. Samples that are produced from such cases are called censored samples. The most common censoring schemes are: Type-I censoring scheme we put n items on experiment and the experiment is terminated at a prefix time t and the number of failure r becomes a random and Type-II censoring scheme we put n items experiment will be terminated experiment

r^{th} item failure.

Let n items are put on life testing and the failures times of first r ($\leq n$) units are observed. Let $X_{(1)} \leq X_{(2)} \leq X_{(3)} \leq \dots \leq X_{(r)}$, ($0 < r \leq n$) are the observed lifetimes of first r units. Obviously, remaining $(n - r)$ items survived until $X_{(r)}$. The joint pdf n order statistics $X_{(1)} \leq X_{(2)} \leq X_{(3)} \leq \dots \leq X_{(n)}$ is

$$f(x_{(1)}, x_{(2)}, \dots, x_{(n)}; \alpha, \beta) = n! \prod_{i=1}^n f(x_{(i)}, \alpha, \beta) \quad (2.1)$$

If we rewrite the pdf as follows

$$f(x; \alpha, \beta) = \frac{\alpha\beta}{x^2 \left(1 + \frac{\beta}{x}\right)} \exp\left(-\alpha \log\left(1 + \frac{\beta}{x}\right)\right) \quad ; x \geq 0, \alpha, \beta \geq 0 \quad (2.2)$$

Using (2.2), the joint pdf of n order statistics will be

$$f(x_{(1)}, x_{(2)}, \dots, x_{(n)}; \alpha, \beta) = n! (\alpha\beta)^n \prod_{i=1}^n \frac{1}{x_{(i)}^2 \left(1 + \frac{\beta}{x_{(i)}}\right)} \exp\left(-\alpha \log\left(1 + \frac{\beta}{x_{(i)}}\right)\right) \quad ; x \geq 0, \alpha, \beta \geq 0 \quad (2.3)$$

The joint pdf of $X_{(1)} \leq X_{(2)} \leq X_{(3)} \leq \dots \leq X_{(r)}$, is obtained by integrating out $X_{(r+1)} \leq X_{(r+2)} \leq X_{(r+3)} \leq \dots \leq X_{(n)}$, from (2.3) which lead to

$$f(x_{(1)}, x_{(2)}, \dots, x_{(r)}; \alpha, \beta) = \frac{n!}{(n-r)!} (\alpha\beta)^r \prod_{i=1}^r \frac{1}{x_{(i)}^2 \left(1 + \frac{\beta}{x_{(i)}}\right)} \exp\left(-\alpha S_r\right) \quad (2.4)$$

we use S_r for simplicity, where

$$S_r = \sum_{i=1}^r \left[\log\left(1 + \frac{\beta}{x_{(i)}}\right) \right] + (n-r) \log\left(1 + \frac{\beta}{x_{(r)}}\right)$$

Next, we prove completeness and sufficiency properties of S_r as Lemma

Lemma S_r is the complete and sufficient estimator of α for the distribution represented by (1.2). The pdf of S_r is given by

$$f_1(S_r; \alpha) = \frac{\alpha^r}{\Gamma_r} S_r^{r-1} \exp(-\alpha S_r) \quad (2.5)$$

Proof. It follows from (2.4) and factorization theorem, that S_r is sufficient for α . We take transformation $Y_{(i)} = -\log\left(1 + \frac{\beta}{x_{(i)}}\right)$ follows exponential distribution with mean life $\frac{1}{\alpha}$. Let us consider the transformation $U_i = (n-i+1)\{Y_{(i)} - Y_{(i-1)}\}$, $i = 1, 2, \dots, r$. Obviously, $\sum_{i=1}^r U_i = S_r$. Since U_i 's are exponential random variable with mean life $(\frac{1}{\alpha})$, using the additive property of gamma variate's, result follows (2.5) (see Johnson and Kotz 1970). $\square$

The following theorem provides the UMVUE of power of α .

Theorem 2.1. For $p \in (-\infty, \infty)$, ($p \neq 0$) the UMVUE of α^p is given by

$$\tilde{\alpha}_{II}^p = \begin{cases} \left\{ \frac{\Gamma(r)}{\Gamma(r-p)} \right\} S_r^{-p} & ; (p < r) \\ 0 & ; \text{Otherwise} \end{cases} \quad (2.6)$$

where $\tilde{\alpha}_{II}$ is UMVUE of α under Type-II censoring.

Proof. Since S_r is sufficient for α Lemma (1). As the distribution of S_r belongs to exponential family, so, it is also complete (Rohtagi (1976)). The proof now follows from the fact that

$$E(S_r^{-p}) = \frac{\Gamma(r-p)}{\Gamma(p)} \alpha^p \quad \square \quad (2.7)$$

Theorem 2.2. The UMVUE of $\tilde{R}_{II}(t)$ is given by

$$\tilde{R}_{II}(t) = \begin{cases} \left\{ 1 - \left(1 + \frac{\log(1+\frac{\beta}{t})}{S_r} \right)^{(r-1)} \right\} & ; if -\log(1+\frac{\beta}{t}) < S_r \\ 0 & ; \text{ Otherwise} \end{cases} \quad (2.8)$$

where, $\tilde{R}_{II}(t)$ is reliability of ILD under Type-II censoring scheme.

Proof. Reliability is the probability that a device will perform its intended function during a specified period of time such as $R(t) = P(T > t) = \int_t^\infty \tilde{f}(x)dx$

We know, $R(t) = 1 - F(t)$

$$\begin{aligned} \tilde{R}_{II}(t) &= 1 - \exp\left(-\alpha \log\left(1 + \frac{\beta}{t}\right)\right) \\ &= 1 - \sum_{i=0}^{\infty} \frac{(-1)^i}{i!} \left(\log\left(1 + \frac{\beta}{t}\right)\right)^i \hat{\alpha}_{II}^i \end{aligned} \quad (2.9)$$

Now from Lemma 1 of Chaturvedi and Tomer (2002) and Theorem(2.1), we have,

$$\tilde{R}_{II}(t) = 1 - \sum_{i=0}^{r-1} (-1)^i \binom{r-1}{i} \left(\frac{\log(1+\frac{\beta}{t})}{S_r}\right)^i$$

using binomial series $(1+z)^k = \prod_{n=0}^{\infty} \binom{k}{n} z^n$, n is integer and $|z| < 1$
hence proof follows. $\square$

Corollary 2.1 The UMVUE of sampled pdf $f(x; \alpha, \beta)$ at a specified point x is given by

$$\tilde{f}_{II}(x; \alpha, \beta) = \begin{cases} \frac{\beta(r-1)}{x^2(1+\frac{\beta}{x})S_r} \left(1 + \frac{\log(1+\frac{\beta}{x})}{S_r}\right)^{r-2} & ; if -\log(1+\frac{\beta}{x}) < S_r \\ 0 & ; \text{ Otherwise} \end{cases} \quad (2.10)$$

Proof. As we know that the expectation of $\int_t^\infty \tilde{f}_{II}(x; \alpha, \beta)dx$ with respect to S_r is $R(t)$. Thus

$$\tilde{R}_{II}(t) = \int_t^\infty \tilde{f}_{II}(x; \alpha, \beta)dx$$

or

$$\tilde{f}_{II}(x; \alpha, \beta) = -\frac{d\tilde{R}_{II}}{dx} \quad (2.11)$$

The result follows from Theorem (2.2) $\square$

Let X and Y be two independent random variables form the ILD. Suppose n items on X and m items on Y are put on a life test and failure times of first r_1 and r_2 units are observed on X and

Y respectively then from Corollary (2.1), it follows that UMVUEs of $f_1(x; \alpha, \beta)$ and $f_2(y; \alpha, \beta)$ at specified point x and y are given by

$$\tilde{f}_{1II}(x_1; \alpha_1, \beta_1) = \begin{cases} \frac{\beta_1(r_1-1)}{x_1^2(1+\frac{\beta_1}{x_1})S_{r_1}} \left(1 + \frac{\log(1+\frac{\beta_1}{x_1})}{S_{r_1}}\right)^{r_1-2} & ; if -\log(1 + \frac{\beta_1}{x_1}) < S_{r_1} \\ 0 & ; otherwise \end{cases} \quad (2.12)$$

and

$$\tilde{f}_{2II}(y_2; \alpha_2, \beta_2) = \begin{cases} \frac{\beta_2(r_2-1)}{y_2^2(1+\frac{\beta_2}{y_2})T_{r_2}} \left(1 + \frac{\log(1+\frac{\beta_2}{y_2})}{T_{r_2}}\right)^{r_2-2} & ; if -\log(1 + \frac{\beta_2}{y_2}) < T_{r_2} \\ 0 & ; otherwise \end{cases} \quad (2.13)$$

SSR model is used to measure performance of the system when strength X and stress Y is applied and the system will stop working when the applied stress is greater than its strength. So, we too have considered the stress-strength reliability SSR model under censoring scheme considering the Inverse Lomax Distribution.

Theorem 2.3. The UMVUE of Stress -Strength reliability (SSR) P is given by

$$\tilde{P}_{II} = \begin{cases} (r_2 - 1) \int_0^U (1 - Z)^{(r_2-2)} \left\{ 1 - \left(1 + S_{r_1}^{-1} \left(\log(1 + \beta_1(\beta_2^{-1}(\exp(-ZT_{r_2}) + 1))) \right) \right) \right\}^{(r_1-1)} dZ \\ \quad if \left(\beta_1(\exp(-S_{r_1}) - 1)^{-1} \right) < \left(\beta_2(\exp(-T_{r_2}) - 1)^{-1} \right) \\ (r_2 - 1) \int_0^1 \left\{ (1 - Z)^{(r_2-2)} \left\{ 1 - \left(1 + S_{r_1}^{-1} \left(\log(1 + \beta_1(\beta_2^{-1}(\exp(-ZT_{r_2}) + 1))) \right) \right) \right\} \right\}^{(r_1-1)} dZ \\ \quad if \left(\beta_1(\exp(-S_{r_1}) - 1)^{-1} \right) > \left(\beta_2(\exp(-T_{r_2}) - 1)^{-1} \right) \end{cases}$$

where $U = T_{r_2}^{-1} \log \left(1 + \beta_2 \left(\beta_1^{-1}(\exp(-S_{r_1}) - 1) \right) \right)$

Proof. From Corollary (2.1), we have

$$\begin{aligned} \tilde{P}_{II} &= \int_{y=0}^{\infty} \int_{x=y}^{\infty} \tilde{f}_{1II}(x_1; \alpha_1, \beta_1) \tilde{f}_{2II}(y_2; \alpha_2, \beta_2) dy dx \\ &= \int_{y=0}^{\infty} \tilde{R}_{1II}(y) \left\{ \frac{-d}{dy} \tilde{R}_{2II}(y) \right\} dy \end{aligned}$$

Now, using Theorem (2.2) and (2.10), we have

$$= \int_{y=0}^M 1 - \left(1 + \frac{\log(1 + \frac{\beta_1}{y})}{S_{r_1}} \right)^{r_1-1} \frac{(r-1)\beta}{y_2^2(1+\frac{\beta}{y})T_{r_2}} \left(1 + \frac{\log(1 + \frac{\beta_2}{y})}{T_{r_2}} \right)^{r_2-2} dy \quad (2.14)$$

where $M = \min \left\{ \left(\beta_1(\exp(-S_{r_1}) - 1)^{-1} \right), \left(\beta_2(\exp(-T_{r_2}) - 1)^{-1} \right) \right\}$

When $M = \left(\beta_1(\exp(-S_{r_1}) - 1)^{-1} \right)$, putting $\frac{-\log(1+\frac{\beta_2}{y})}{T_{r_2}} = Z$ in (2.14), we get

$$\tilde{P}_{II} = (r_2 - 1) \int_0^U (1 - Z)^{(r_2-2)} \left\{ 1 - \left(1 + S_{r_1}^{-1} \left(\log(1 + \beta_1(\beta_2^{-1}(\exp(-ZT_{r_2}) - 1))) \right) \right) \right\}^{(r_1-1)} dZ \quad (2.15)$$

When $M = \left(\beta_2 (\exp(T_{r_2}) - 1)^{-1} \right)$ for the same transformation of the variables, we get

$$\tilde{P}_{II} = (r_2 - 1) \int_0^1 (1 - Z)^{(r_2-2)} \left\{ 1 - \left(1 + S_{r_1}^{-1} \left(\log(1 + \beta_1 (\beta_2^{-1} (\exp(-ZT_{r_2}) - 1))) \right) \right)^{(r_1-1)} \right\} dZ \quad (2.16)$$

The theorem follows from (2.15) and (2.16). $\square$

Corollary 2.2 When $\beta_1 = \beta_2 = 1$, UMVUE of P is given by

$$\tilde{P} = \begin{cases} (r_2 - 1) \int_0^{\frac{S_{r_1}}{T_{r_2}}} (1 - Z)^{(r_2-2)} \left\{ 1 - \left(1 - \frac{T_{r_2}}{S_{r_1}} Z \right)^{(r_1-1)} \right\} dZ & \text{if } S_{r_1} < T_{r_2} \\ (r_2 - 1) \int_0^1 (1 - Z)^{(r_2-2)} \left\{ 1 - \left(1 - \frac{T_{r_2}}{S_{r_1}} Z \right)^{(r_1-1)} \right\} dZ & \text{if } S_{r_1} > T_{r_2} \end{cases}$$

3 Maximum Likelihood Estimation under Type-II Censoring Scheme

Let $X_1, X_2, \dots, X_r$ be a random sample of size r from a population with p.d.f. $f(x_i|\theta)$ then the likelihood function is given by

$$L(\theta|x_i) = \prod_{i=1}^r f(x_i|\theta_i) \quad (3.1)$$

Maximum likelihood estimation (MLE) of the unknown parameters of a distribution is obtained by its maximizing a likelihood function *i.e.*

$$\frac{\partial L(x_i|\alpha, \beta)}{\partial \theta_i} = 0 \quad (3.2)$$

provided that

$$\frac{\partial^2 L(x_i|\alpha, \beta)}{\partial \theta_i^2} < 0, \quad i = 1, 2, \dots, r \quad (3.3)$$

In this section, we have derived the expression for MLE of power α . We have considered two cases.

Case1: When β is known and α is unknown

Theorem 3.1. For $p \in (-\infty, \infty)$, ($p \neq 0$) the MLE of α^p is given by

$$\hat{\alpha}_{II}^p = \left(\frac{r}{S_r} \right)^p$$

Proof. Taking logarithm on both sides of the equation (2.4), we get

$$\log f(x_{(i)}; \alpha, \beta) = \log \left(\frac{n!}{(n-r)!} \right) + r \log(\alpha\beta) + \sum_{i=1}^r \log \left(\frac{x_{(i)}^{-2}}{\left(1 + \frac{\beta}{x_{(i)}} \right)} \right) - \alpha S_r \quad (3.4)$$

Maximizing the log-likelihood function, differentiating it with respect to α and equating the differential equation to zero, we get

$$\hat{\alpha}_{II} = \frac{r}{S_r} \quad (3.5)$$

The result follows from the invariance properties of MLE's see(Rao 1973) $\square$

Theorem 3.2. The MLE of reliability function $R(t)$ is given by

$$\hat{R}_{II}(t) = 1 - \exp\left(-\frac{r}{S_r} \log\left(1 + \frac{\beta}{t}\right)\right) \quad (3.6)$$

Proof The result follows from the expression of $R(t)$, Theorem (3.1) and the invariance property of the MLE. $\square$

Corollary 3.1 The MLE of the p.d.f. $f(x; \alpha, \beta)$ at specified point x is given by

$$\hat{f}_{II}(x; \alpha, \beta) = \frac{r\beta}{x^2\left(1 + \frac{\beta}{x}\right)S_r} \exp\left(-\frac{r}{S_r} \log\left(1 + \frac{\beta}{x}\right)\right) \quad (3.7)$$

Proof The Proof is similar to that of Corollary (2.1). $\square$

Theorem 3.3. The MLE of P is given by

$$\hat{P}_{II} = \int_0^\infty 1 - \exp\left\{-\frac{r_1}{S_{r_1}} \log\left(1 + \beta_1\left(\beta_2^{-1}\left(\exp\left(\frac{zT_{r_2}}{r_2}\right) - 1\right)\right)\right)\right\} e^{-z} dz \quad (3.8)$$

Proof.

$$\begin{aligned} \hat{P}_{II} &= \int_{y=0}^\infty \int_{x=y}^\infty \hat{f}_{II}(x_1; \alpha_1, \beta_1) \hat{f}_{II}(y_2; \alpha_2, \beta_2) dy dx \\ &= \int_{y=0}^\infty \hat{R}_{II}(y) \left\{ \frac{-d}{dy} \hat{R}_{II}(y) \right\} dy \end{aligned}$$

Now, using (3.6) and (3.7), we have

$$= \int_{y=0}^\infty 1 - \exp\left(-\frac{r_1}{S_{r_1}} \log\left(1 + \frac{\beta_1}{y_1}\right)\right) \frac{r_2\beta_2}{y^2\left(1 + \frac{\beta_2}{x_2}\right)T_{r_2}} \exp\left(-\frac{r_2}{T_{r_2}} \log\left(1 + \frac{\beta_2}{y_2}\right)\right) dy \quad (3.9)$$

proof follows on substituting $\frac{r_2}{T_{r_2}} \log\left(1 + \frac{\beta_2}{y_2}\right) = z$

$$\hat{P}_{II} = \int_0^\infty 1 - \exp\left\{-\frac{r_1}{S_{r_1}} \log\left(1 + \beta_1\left(\beta_2^{-1}\left(\exp\left(\frac{zT_{r_2}}{r_2}\right) - 1\right)\right)\right)\right\} e^{-z} dz \quad \square \quad (3.10)$$

Corollary 3.2 Let X and Y are two independent variables from the same family of distribution and $\beta_1 = \beta_2$, then MLE of P is given as

$$\hat{P}_I = \frac{r_2 S_{r_1}}{r_2 S_{r_1} + r_1 T_{r_2}} \quad (3.11)$$

Case 2: When both β and α are unknown

We provide the MLE of α^p , $R_{II}(t)$ and P_{II} in the following theorems.

Differentiating (3.4) w.r.t α and β and equating to 0, we get

$$\frac{\partial L}{\partial \alpha} = \frac{r}{\alpha} - S_r = 0 \quad (3.12)$$

$$\frac{\partial L}{\partial \beta} = \frac{r}{\beta} - \sum_{i=1}^n \frac{1}{x_{(i)}\left(1 + \frac{\beta}{x_{(i)}}\right)} = 0 \quad (3.13)$$

As closed form solution of above does not exist, therefore any iterative procedure can be used to compute the MLE.

Theorem 3.4. The MLE of $R(t)$ is given by

$$\hat{R}_{II}(t) = 1 - \exp\left(-\frac{r}{s_r} \log\left(1 + \frac{\hat{\beta}}{t}\right)\right) \quad (3.14)$$

Proof. The proof follows from the expression of $R(t)$ and using MLE of α and β and the invariance property of MLE. $\square$

Corollary 3.3 The MLE of the pdf $f(x; \alpha, \beta)$ at specified point x is

$$\hat{f}_{II}(x; \hat{\alpha}, \hat{\beta}) = \frac{r\hat{\beta}}{x^2\left(1 + \frac{\hat{\beta}}{x}\right)S_r} \exp\left(-\frac{r}{S_r} \log\left(1 + \frac{\hat{\beta}}{x}\right)\right)$$

Theorem 3.5. The MLE of P is given by

$$\hat{P}_{II} = \int_0^\infty 1 - \exp\left\{-\frac{r_1}{S_{r_1}} \log\left(1 + \hat{\beta}_1\left(\hat{\beta}_2^{-1}\left(\exp\left(\frac{zT_{r_2}}{r_2}\right) - 1\right)\right)\right)\right\} e^{-z} dz \quad \square \quad (3.15)$$

Proof. Proof follows from Theorem (3.3) and invariance property of MLE.

4 Interval Estimation of the parameters under Type-II Censoring Scheme

In this section, we have derived asymptotic confidence intervals of the parameters based on expected Fisher information matrix. The Fisher information matrix (α, β) is given by

$$I(\hat{\alpha}, \hat{\beta}) = E \begin{vmatrix} -\frac{\partial^2 L}{\partial \alpha^2} & -\frac{\partial^2 L}{\partial \alpha \partial \beta} \\ -\frac{\partial^2 L}{\partial \beta \partial \alpha} & -\frac{\partial^2 L}{\partial \beta^2} \end{vmatrix}_{(\alpha, \beta) = (\hat{\alpha}, \hat{\beta})}$$

We use the observed information matrix obtained. The inverse of $I(\hat{\alpha}, \hat{\beta})$ is the variance covariance matrix. After obtaining the inverse matrix, we get variance of the parameter α and β . These values of variance are used to construct respective confidence intervals.

Taking second derivative of (3.12) and (3.13) with respect to α and β respectively. We have

$$\frac{\partial^2 L}{\partial \alpha \partial \beta} = -\frac{r}{\alpha^2} \quad (4.1)$$

$$\frac{\partial^2 L}{\partial \beta \partial \alpha} = -\frac{r}{\beta^2} + \sum_{i=1}^r \frac{1}{x_{(i)}^2 \left(1 + \frac{\beta}{x_{(i)}}\right)^2} \quad (4.2)$$

Assuming the asymptotic normality of MLE's, confidence interval for α and β are constructed. Let $\sigma^2(\hat{\alpha})$ and $\sigma^2(\hat{\beta})$ be the estimated variance of the parameters α and β then asymptotic $100(1 - \theta)\%$ confidence interval for the parameter α and β are given by $(\alpha \pm Z_{\epsilon/2} \hat{\sigma}(\hat{\alpha}))$ and $(\beta \pm Z_{\epsilon/2} \hat{\sigma}(\hat{\beta}))$ respectively, where $Z_{\epsilon/2}$ is the upper $100(1 - \epsilon)$ percentile point of standard normal distribution

5 Point Estimation under Censoring Scheme Using Bartholomew Scheme

Let n items are on life testing and if we terminate experiment at a pre-fixed time t_0 then it is called Type-I censoring. Let $X_{(1)} \leq X_{(2)} \leq X_{(3)} \leq \dots \leq X_{(r)}$ be the life times of first r units as defined in (1.2). the test start, at time $X_{(0)}$ and operates till $X_{(1)} = x_{(1)}$, the first failure time. The failed item is replaced by the new one and system operates till the second failure occurs at time $X_{(2)} = x_{(2)}$ and so on. Bartholomew (1963) considered only the number of failures occurred during $[0, t_0]$ and ignored the times when failures occurred and showed that under such a plan, the number of failures $N(t_0)$ during the interval $[0, t_0]$ follows Poisson distribution.

Theorem 5.1. Let $Y_{(1)} \leq Y_{(2)} \leq Y_{(3)} \leq \dots \leq Y_{(r)}$, $(0 < r \leq n)$ are the observed lifetimes of first t_0 time. If $N(t_0)$ be the number of items that failed before times t_0 then $N(t_0)$ follows the Poisson distribution.

Proof. For $Y'_i, i = 1, 2, \dots, n$ defined in section 2, let us use the transformation $K_1 = Y_{(1)}, K_2 = Y_{(2)} - Y_{(1)}, \dots, K_n = Y_{(n)} - Y_{(n-1)}$.

In section 2, we have shown that K'_i 's are i.i.d r.v.'s having exponential distribution with mean $(\frac{1}{\alpha})$. By using the definition of $N(t_0)$,

$$\begin{aligned} P[N(t_0)] &= P[Y_{(r)} \leq t_0] - P[Y_{(r+1)} \leq t_0] \\ &= P\left[Y_{(r)} \leq -\log\left(1 + \frac{\beta}{t_0}\right)\right] - P\left[Y_{(r+1)} \leq -\log\left(1 + \frac{\beta}{t_0}\right)\right] \\ &= P\left[n\alpha \sum_{i=1}^{r+1} K_i \geq -n\alpha \log\left(1 + \frac{\beta}{t_0}\right)\right] - P\left[n\alpha \sum_{i=1}^r K_i \geq -n\alpha \log\left(1 + \frac{\beta}{t_0}\right)\right] \quad (5.1) \end{aligned}$$

Using a result of Patel *et. al*(1976, pp. 244), we have

$$\begin{aligned} P\left[N(t_0) = \frac{r}{t_0}\right] &= \frac{1}{\Gamma(r+1)} \int_{-n\alpha \log(1+\frac{\beta}{t_0})}^{\infty} u^r \exp^{-u} du - \frac{1}{\Gamma(r)} \int_{-n\alpha \log(1+\frac{\beta}{t_0})}^{\infty} u^{r-1} \exp^{-u} du \\ &= \exp\left\{n\alpha \log\left(1 + \frac{\beta}{t_0}\right)\right\} \left[\sum_{j=0}^r \frac{\{-n\alpha \log(1 + \frac{\beta}{t_0})\}^j}{j!} - \sum_{j=0}^{r-1} \frac{\{-n\alpha \log(1 + \frac{\beta}{t_0})\}^j}{j!} \right] \quad (5.2) \end{aligned}$$

On simplification,

$$P\left[N(t_0) = \frac{r}{t_0}\right] = \frac{\exp\left\{n\alpha \log\left(1 + \frac{\beta}{t_0}\right)\right\} \left\{-n\alpha \log\left(1 + \frac{\beta}{t_0}\right)\right\}^r}{r!} \quad (5.3)$$

the proof follows. $\square$

Next, we provide the UMVU and MLE of $\alpha^p, R_I(t)$ and P_I under the sampling scheme of Bartholomew.

Theorem 5.2. For $p \in (0, \infty)$ the UMVUE of α^p is given by

$$\tilde{\alpha}_I^p = \begin{cases} \frac{r!}{(r-p)!} \left\{ -n \log\left(1 + \frac{\beta}{t_0}\right) \right\}^{-p} & ; \text{if } p \leq r \\ 0 & ; \text{Otherwise} \end{cases} \quad (5.4)$$

Proof. From the theorem (5.1) and factorization theorem it can be deduced that r is complete and sufficient for α (see Rohatgi 1976, p.341) and r belongs to exponential family, it is also complete (see Rohatgi 1976, p.347). The theorem now follows from the result that the p^{th} factorial moment of distribution of r is given by

$$E \left[r(r-1) \dots (r-p+1) \right] = \left\{ -n \alpha \log\left(1 + \frac{\beta}{t_0}\right) \right\}^p$$

So,

$$E \left[\frac{r!}{(r-p)!} \right] = \left(-n \alpha \log\left(1 + \frac{\beta}{t_0}\right) \right)^p \text{ if } p \leq r \quad (5.5)$$

and the proof follows.

Theorem 5.3. The UMVUE of $R_I(t)$ is given by

$$\tilde{R}_I(t) = \begin{cases} 1 - \left[1 - \frac{\log(1 + \frac{\beta}{t})}{n \log(1 + \frac{\beta}{t_0})} \right]^r & ; \text{if } \log(1 + \frac{\beta}{t}) > n \log(1 + \frac{\beta}{t_0}) \\ 0 & ; \text{Otherwise} \end{cases} \quad (5.6)$$

Proof. Using Theorem (2.1)

$$\tilde{R}_I(t) = 1 - \exp \left[-\alpha \log\left(1 + \frac{\beta}{t}\right) \right]$$

where $\tilde{R}_I(t)$ is the Reliability under Bartholomew scheme.

Using Lemma 1 of Chaturvedi and Tomer (2002), we have

$$\begin{aligned} \tilde{R}_I(t) &= 1 - \sum_{i=0}^{\infty} \frac{(-1)^i}{i!} \left[\log\left(1 + \frac{\beta}{t}\right) \right]^i \tilde{\alpha}_i \\ &= 1 - \sum_{i=0}^r (-1)^i \binom{r}{i} \left(\frac{\log(1 + \frac{\beta}{t})}{n \log(1 + \frac{\beta}{t_0})} \right)^i \end{aligned} \quad (5.7)$$

using $(1+z)^k = \prod_{n=0}^{\infty} \binom{k}{n} z^n$, n is integer and $|z| < 1$

hence proof follows. □

Corollary 5.1 The UMVUE of the sampled pdf at a specified point x is

$$\tilde{f}_I(x; \alpha, \beta) = \begin{cases} \frac{-r \beta \left[1 - \frac{\log(1 + \frac{\beta}{x})}{n \log(1 + \frac{\beta}{t_0})} \right]^{r-1}}{n x^2 \log(1 + \frac{\beta}{t_0}) (1 + \frac{\beta}{x})} & ; \text{if } \log(1 + \frac{\beta}{x}) > n \log(1 + \frac{\beta}{t_0}) \\ 0 & ; \text{otherwise} \end{cases} \quad (5.8)$$

Proof. The proof follows from the theorem (5.3) on using the techniques as was adopted in the proof the Corollary (2.1) $\square$

Let X and Y be the two independent random variables with pdf $f(x; \alpha_1, \beta_1)$ and $f(y; \alpha_2, \beta_2)$ respectively. Let n items on X and m items on Y are put on life test and let r_1 and r_2 be the number of failures before time t_0 and t_{00} respectively. It follows from Corollary (5.1) that the UMVUE of $f_1(x; \alpha_1, \beta_1)$ and $f_2(y; \alpha_2, \beta_2)$ at a specified point x and y respectively based on Bartholomew sampling scheme, are given by

$$\tilde{f}_{1I}(x; \alpha_1, \beta_1) = \frac{-r_1 \beta_1 \left[1 - \frac{\log\left(1 + \frac{\beta_1}{x}\right)}{n \log\left(1 + \frac{\beta_1}{t_0}\right)} \right]^{r_1-1}}{n x^2 \log\left(1 + \frac{\beta_1}{t_0}\right) \left(1 + \frac{\beta_1}{x}\right)} ; \text{ if } \log\left(1 + \frac{\beta_1}{x}\right) > n \log\left(1 + \frac{\beta_1}{t_0}\right) \quad (5.9)$$

and

$$\tilde{f}_{2I}(y; \alpha_2, \beta_2) = \frac{-r_2 \beta_2 \left[1 - \frac{\log\left(1 + \frac{\beta_2}{y}\right)}{m \log\left(1 + \frac{\beta_2}{t_{00}}\right)} \right]^{r_2-1}}{m y^2 \log\left(1 + \frac{\beta_2}{t_{00}}\right) \left(1 + \frac{\beta_2}{y}\right)} ; \text{ if } \log\left(1 + \frac{\beta_2}{y}\right) > m \log\left(1 + \frac{\beta_2}{t_{00}}\right) \quad (5.10)$$

Theorem 5.4. The UMVUE of P is given by

$$\tilde{P}_I = \begin{cases} r_2 \int_0^M (1-Z)^{r_2-1} \left[1 - \left\{ 1 - \frac{\log\left(1 + \beta_1 \left(\beta_2^{-1} \left(1 + \frac{\beta_2}{t_{00}}\right)^{mz} - 1\right)\right)}{n \log\left(1 + \frac{\beta_1}{t_0}\right)} \right\}^{r_1} \right] dz \\ \quad \text{if } \beta_1 \left(\left(1 + \frac{\beta_1}{t_0}\right)^n - 1 \right)^{-1} > \beta_2 \left(\left(1 + \frac{\beta_2}{t_{00}}\right)^m - 1 \right)^{-1} \\ = r_2 \int_0^1 (1-Z)^{r_2-1} \left[1 - \left\{ 1 - \frac{\log\left(1 + \beta_1 \left(\beta_2^{-1} \left(1 + \frac{\beta_2}{t_{00}}\right)^{mz} - 1\right)\right)}{n \log\left(1 + \frac{\beta_1}{t_0}\right)} \right\}^{r_1} \right] dz \\ \quad \text{if } \beta_1 \left(\left(1 + \frac{\beta_1}{t_0}\right)^n - 1 \right)^{-1} > \beta_2 \left(\left(1 + \frac{\beta_2}{t_{00}}\right)^m - 1 \right)^{-1} \end{cases} \quad (5.11)$$

where

$$M = \frac{\log\left(1 + \beta_2 \left(\beta_1^{-1} \left(1 + \frac{\beta_1}{t_0}\right)^n - 1 \right)\right)}{m \log\left(1 + \frac{\beta_2}{t_{00}}\right)}$$

and

$$Z = \frac{\log\left(1 + \frac{\beta_2}{y}\right)}{m \log\left(1 + \frac{\beta_2}{t_{00}}\right)}$$

Proof. From Corollary (5.1)

$$\begin{aligned}
 \tilde{P}_I &= \int_{y=0}^{\infty} \int_{x=y}^{\infty} \tilde{f}_{1I}(x_1; \alpha_1, \beta_1) \tilde{f}_{2I}(y_2; \alpha_2, \beta_2) dy dx \\
 &= \int_{y=0}^{\infty} \tilde{R}_{1I}(y) \left\{ \frac{-d}{dy} \tilde{R}_{2I}(y) \right\} dy \\
 &= \int_{y=\max \beta_1}^{\infty} \left(\left(1 + \frac{\beta_1}{t_0} \right)^n - 1 \right)^{-1} \cdot \beta_2 \left(\left(1 + \frac{\beta_2}{t_0} \right)^m - 1 \right)^{-1} \left\{ 1 - \left[1 - \frac{\log(1 + \frac{\beta_1}{y})}{n \log(1 + \frac{\beta_1}{t_0})} \right]^{r_1} \right\} \\
 &\quad \cdot \left\{ \frac{-r_2 \beta_2 \left[1 - \frac{\log(1 + \frac{\beta_2}{y})}{m \log(1 + \frac{\beta_2}{t_0})} \right]^{r_2 - 1}}{m y^2 \log(1 + \frac{\beta_2}{t_0}) (1 + \frac{\beta_2}{y})} \right\} dy
 \end{aligned} \tag{5.12}$$

The proof follows on putting $Z = \frac{\log(1 + \frac{\beta_2}{y})}{m \log(1 + \frac{\beta_2}{t_0})}$ □

6 Maximum Likelihood Estimation under Bartholomew scheme

In this section, we have derived the expression for MLE of α^p , $R_I(t)$ and P_I in the following given theorems. There arise two cases.

Case1: When β is known and α is unknown

We provide the

Theorem 6.1. For $p \in (-\infty, \infty)$ ($p \neq 0$), the MLE of α^p is given by

$$\hat{\alpha}_I^p = \left(\frac{r}{-n \log(1 + \frac{\beta}{t_0})} \right)^p \tag{6.1}$$

Proof. Taking Logarithm of (5.3) and differentiating it with respect to α , equating the coefficient to zero and solving for α , we get

$$\hat{\alpha} = \frac{r}{-n \log(1 + \frac{\beta}{t_0})} \tag{6.2}$$

The result follows from the invariance property of MLE's. □

Theorem 6.2. The MLE of Reliability function $R(t)$ is given by

$$\hat{R}(t) = 1 - \exp \left(\frac{r \log(1 + \frac{\beta}{t})}{n \log(1 + \frac{\beta}{t_0})} \right) \tag{6.3}$$

Proof. Result follows from the invariance property of MLE's. □

Corollary 6.1 The MLE of the pdf $f(x; \alpha, \beta)$ at specified point x is

$$\hat{f}_I(x, \alpha, \beta) = \left\{ \exp \left(\frac{r \log(1 + \frac{\beta}{x})}{n \log(1 + \frac{\beta}{t_0})} \right) \frac{r \beta}{n x^2 \log(1 + \frac{\beta}{t_0}) (1 + \frac{\beta}{x})} \right\} \tag{6.4}$$

Proof The Proof is similar to that of Corollary (3.1). □

Theorem 6.3. The MLE of P is given by

$$\hat{P}_I = \int_0^\infty \exp \left[\frac{-r_1 \log \left\{ 1 + \beta_1 \left(\beta_2^{-1} \exp \left(\frac{z m \log \left(1 + \frac{\beta_2}{t_0} \right)}{r_2} \right) - 1 \right) \right\}}{n \log \left(1 + \frac{\beta_1}{t_0} \right)} \right] . e^{-z} dz \quad (6.5)$$

Proof. We know that,

$$\begin{aligned} \hat{P}_I &= \int_{y=0}^\infty \int_{x=y}^\infty \hat{f}_{1I}(x_1; \alpha_1, \beta_1) \hat{f}_{2I}(y_2; \alpha_2, \beta_2) dy dx \\ &= \int_{y=0}^\infty \hat{R}_{1I}(y) \left\{ \frac{-d}{dy} \hat{R}_{2I}(y) \right\} dy \end{aligned}$$

using Theorem 6.2 and corollary 6.1, we get

$$= \int_0^\infty \left\{ 1 - \exp \left(\frac{r_1 \log \left(1 + \frac{\beta_1}{y} \right)}{n \log \left(1 + \frac{\beta_1}{t_0} \right)} \right) \right\} \frac{r_2 \beta_2}{m y^2 \log \left(1 + \frac{\beta_2}{t_0} \right) \left(1 + \frac{\beta_2}{y} \right)} \left\{ \exp \left(\frac{r_2 \log \left(1 + \frac{\beta_2}{y} \right)}{m \log \left(1 + \frac{\beta_2}{t_0} \right)} \right) \right\} \quad (6.6)$$

The proof follows on substituting $z = \frac{r_2 \log \left(1 + \frac{\beta_2}{y} \right)}{m \log \left(1 + \frac{\beta_2}{t_0} \right)}$ □

Corollary 6.2 Let X and Y are two independent variables from same the family of distributions β_1 and β_2 .

$$\hat{P}_I = \frac{nr_2}{mr_1 + nr_2} \quad (6.7)$$

Case 2: Point estimation of parameter when both α and β are unknown

Next, we provide the MLE of α^p , $R(t)$ and P in the following given theorems.

The log-likelihood function of α and β given as

$$\text{Log} L = -n \alpha \log \left(1 + \frac{\beta}{t_0} \right) - r \log(n\alpha) - r \log \left(\log \left(1 + \frac{\beta}{t_0} \right) \right) - \log(r!) \quad (6.8)$$

Differentiating (6.8) with respect to α and β and equating to zero , we get

$$-n \log \left(1 + \frac{\beta}{t_0} \right) - \frac{r}{\alpha} = 0 \quad (6.9)$$

$$\frac{-n\alpha}{t_0 \left(1 + \frac{\beta}{t_0} \right)} - \frac{r}{t_0 \left(1 + \frac{\beta}{t_0} \right) \log \left(1 + \frac{\beta}{t_0} \right)} = 0 \quad (6.10)$$

solving the above equations using numerical methods , We can get the MLE of α and β . □

Theorem 6.4. The MLE of Reliability function $R(t)$ is given by

$$\hat{R}(t) = 1 - \exp \left(\frac{-r \log \left(1 + \frac{\hat{\beta}}{t} \right)}{n \log \left(1 + \frac{\hat{\beta}}{t_0} \right)} \right) \quad \square \quad (6.11)$$

Corollary 6.3 The MLE of the pdf $f(x; \alpha, \beta)$ at specified point x is

$$\hat{f}_I(x, \hat{\alpha}, \hat{\beta}) = \left\{ 1 - \exp \left(\frac{-r \log \left(1 + \frac{\hat{\beta}}{t} \right)}{n \log \left(1 + \frac{\hat{\beta}}{t_0} \right)} \right) \right\} \frac{r \hat{\beta}}{n x^2 \log \left(1 + \frac{\hat{\beta}}{t_0} \right) \left(1 + \frac{\hat{\beta}}{x} \right)} \quad \square \quad (6.12)$$

Theorem 6.5. *The MLE of P is given by*

$$\hat{P}_I = \int_0^\infty \exp \left[\frac{-r_1 \log \left\{ 1 + \hat{\beta}_1 \left(\hat{\beta}_2^{-1} \exp \left(\frac{zm \log \left(1 + \frac{\hat{\beta}_2}{t_{00}} \right)}{r_2} \right) - 1 \right) \right\}}{n \log \left(1 + \frac{\hat{\beta}_1}{t_0} \right)} \right] . e^{-z} dz \quad \square \quad (6.13)$$

The proof of Theorem (6.5) is easy to prove using numerical methods.

6.1 Interval Estimation under Bartholomew Scheme

In this section, we have derived asymptotic confidence intervals of this parameters based on expected Fisher information matrix. The Fisher information matrix of the parameter (α, β) is given by

$$I(\hat{\alpha}, \hat{\beta}) = E \begin{vmatrix} -\frac{\partial^2 L}{\partial \alpha^2} & -\frac{\partial^2 L}{\partial \alpha \partial \beta} \\ -\frac{\partial^2 L}{\partial \beta \partial \alpha} & -\frac{\partial^2 L}{\partial \beta^2} \end{vmatrix}_{(\alpha, \beta) = (\hat{\alpha}, \hat{\beta})}$$

We use the observed information matrix obtained by dropping the expectation sign. The variance covariance matrix is the inverse of $I(\hat{\alpha}, \hat{\beta})$. Taking second derivative of (5.3) with respect to α and β respectively we get,

$$\frac{\partial^2 L}{\partial \alpha \partial \beta} = -\frac{r}{\alpha^2} \quad (6.14)$$

$$\frac{\partial^2 L}{\partial \beta \partial \alpha} = \frac{n\alpha}{t_0^2(1 + \frac{\beta}{t_0})^2} - \frac{r}{t_0^2(1 + \frac{\beta}{t_0})^2 \log(1 + \frac{\beta}{t_0})} - \frac{r}{t^2(1 + \frac{\beta}{t_0})^2 \log(1 + \frac{\beta}{t_0})^2} \quad (6.15)$$

Assuming the asymptotic normality of MLE's, the confidence interval for α and β are constructed. Let $\sigma^2(\hat{\alpha})$ and $\sigma^2(\hat{\beta})$ be the estimated variance of the parameters α and β then asymptotic $100(1 - \theta)\%$ confidence interval for the parameter α and β will be $(\alpha \pm Z_{\epsilon/2} \hat{\sigma}(\hat{\alpha}))$ and $(\beta \pm Z_{\epsilon/2} \hat{\sigma}(\hat{\beta}))$ respectively, where $Z_{\epsilon/2}$ is the upper $100(1 - \epsilon)$ percentile point of standard normal distribution.

7 Simulation Study

We have conducted simulation studies using Markov Chain Monte Carlo Simulation techniques. The analysis were performed by using R Software (Version 3.6.1). A comparison of the obtained estimates has been made on the basis of their MSE. We have generated 1000 random samples of different sizes from the distribution defined in (2.4) using MH algorithm. For sample size $n = 200$ and for different combination of the true values of parameters $(\alpha, \beta) = (1.5, 0.5), (1.5, 2.1), (1.5, 3.5)$ computation have been done. For each sample, we sort the data in ascending order and considered a sample of first r ($r \leq n$) observation. For different values of $r = (70, 75, 80)$ using in (2.6) and (5.3), we have computed the average values of $\tilde{\alpha}^p$ and $\hat{\alpha}^p$, and their corresponding average bias, MSE and approximate $100(1 - \theta)\%$, where $\theta = 0.05$ confidence interval. As $p \in (-\infty, \infty)$, $p \neq 0$, we choose a positive and negative values of p for better comparison of the performance of estimators.

Table 1: UMVU and MLE of α^p based on Type-II censoring scheme for $(n = 200, \alpha = 1.5, \beta = 0.5, 2.5, 3.5)$

α^p	$p \downarrow$	$r \downarrow$	UMVUE	BIAS	MSE	LCL	UCL	MLE	BIAS	MSE	LCL	UCL
0.9602	-0.10	70	0.8459	-0.1144	0.0132	0.8450	0.8460	1.1815	0.2213	0.0491	1.1810	1.8130
		75	0.8542	-0.1061	0.0113	0.8540	0.8550	1.1701	0.2098	0.0442	1.1690	1.1710
		80	0.8631	-0.0972	0.0095	0.863	0.8640	1.15810	0.1979	0.0393	1.1570	1.1590
1.04138	0.10	70	1.1842	0.1428	0.0205	1.1840	1.1850	0.8452	-0.1962	0.0386	0.8450	0.8460
		75	1.1704	0.1290	0.0168	1.1700	1.1710	0.8551	-0.1863	0.0348	0.8550	0.8560
		80	1.1596	0.1182	0.0141	1.1590	1.1600	0.8631	-0.1783	0.0319	0.8630	0.8640
0.9036	-0.25	70	0.8121	-0.0915	0.0085	0.8110	0.8130	1.2300	0.32640	0.1069	1.2290	1.2310
		75	0.8328	-0.0708	0.0052	0.8320	0.8340	1.1996	0.2960	0.0881	1.1910	1.2010
		80	0.8525	-0.0511	0.0028	0.8520	0.8530	1.1720	0.2684	0.0725	1.1710	1.1730
1.1066	0.25	70	1.2329	0.1262	0.0164	1.2320	1.2340	0.8131	-0.2936	0.0864	0.8120	0.8140
		75	1.2015	0.0948	0.0094	1.2000	1.2030	0.8343	-0.2724	0.0744	0.8330	0.8350
		80	1.1737	0.0670	0.0049	1.1720	1.1750	0.8539	-0.2528	0.0641	0.8530	0.8550
0.8164	-0.50	70	0.6841	-0.1324	0.0180	0.6830	0.6850	1.4608	0.6443	0.4173	1.4580	1.4640
		75	0.7186	-0.0979	0.0101	0.7170	0.7200	1.3905	0.574	0.3313	1.3880	1.3930
		80	0.7531	-0.0634	0.0046	0.7520	0.7550	1.3272	0.5107	0.2628	1.3240	1.3330
1.2247	0.50	70	1.4737	0.2490	0.0642	1.4710	1.4770	0.6829	-0.5418	0.294	0.6820	0.6840
		75	1.3970	0.1723	0.0317	1.3940	1.4000	0.7201	-0.5046	0.2552	0.7190	0.7220
		80	1.3354	0.1106	0.0142	1.3330	1.3380	0.7532	-0.4716	0.2230	0.7520	0.7550

From the table 1, it is seen that for negative values of p , UMVUE and MLE shows better results as compare to the positive values of p . It can also be seen that MSE of UMVUE is much lower then the MSE of MLE for both negative as well as positive values of p . Thus, we can say that the performance of UMVE of α^p under Type-II censoring is better than the performance of MLE. It may be noted that also as r increases, performance of the both the estimators gets better and estimates come closer to the true value.

To compare the performance of the two estimators of $R(t)$ under Type-II censoring scheme, we have generated 1000 random samples from (2.4) of size 60 with fixed values $\alpha = 0.5$ and $\beta = 1.5$. For the different values of $r = (10, 15, 20 \text{ and } 25)$, using (2.8) and (3.6), the estimates $\hat{R}_{II}(t)$ and $\hat{R}_{II}(t)$ and their corresponding Bias, MSE and approximation $100(1 - \theta)\%$ confidence interval have been computed.

Table 2: UMVU and MLE of $R(t)$ based on Type-II censoring scheme for $(n = 60, \alpha = 0.5, \beta = 1.5)$

$r \downarrow$	$t \downarrow$	$R(t) \downarrow$	UMVUE	BIAS	MSE	LCL	UCL	MLE	BIAS	MSE	LCL	UCL
10	0.1	0.7500	0.4278	-0.3222	0.1533	0.4140	0.4420	0.4455	-0.3045	0.1400	0.4320	0.4590
	0.3	0.5917	0.2619	-0.3298	0.1412	0.2510	0.2730	0.2781	-0.3136	0.1311	0.2670	0.2690
	0.5	0.5000	0.2083	-0.2917	0.1005	0.2010	0.2160	0.2240	-0.2760	0.0928	0.2160	0.2320
	0.7	0.4359	0.1839	-0.2521	0.0772	0.1770	0.1910	0.1983	-0.2376	0.0713	0.1910	0.2060
15	0.1	0.7500	0.4816	-0.2684	0.1050	0.4700	0.4930	0.4944	-0.2556	0.0974	0.4830	0.5060
	0.3	0.5917	0.3550	-0.2368	0.0851	0.3440	0.3660	0.3678	-0.2240	0.0792	0.3570	0.3780
	0.5	0.5000	0.2938	-0.2062	0.0626	0.2850	0.3030	0.3062	-0.1938	0.0583	0.2970	0.3150
	0.7	0.4359	0.2531	-0.1829	0.0511	0.2450	0.2610	0.2646	-0.1713	0.0477	0.2560	0.2730
20	0.1	0.7500	0.5650	-0.1850	0.0637	0.5540	0.5760	0.5738	-0.1762	0.0595	0.5630	0.5840
	0.3	0.5917	0.4350	-0.1567	0.0478	0.4260	0.4450	0.4453	-0.1464	0.04450	0.4360	0.4550
	0.5	0.5000	0.3570	-0.1430	0.0432	0.3480	0.3660	0.3670	-0.1330	0.0407	0.3580	0.3760
	0.7	0.4359	0.3260	-0.1099	0.0330	0.3170	0.3350	0.3357	-0.1002	0.0314	0.3270	0.3450
25	0.1	0.7500	0.6388	-0.1112	0.0370	0.629	0.6490	0.6448	-0.1052	0.0346	0.6350	0.6540
	0.3	0.5917	0.4992	-0.0926	0.0360	0.4890	0.5090	0.5069	-0.0848	0.0342	0.4970	0.5170
	0.5	0.5000	0.4214	-0.0786	0.0267	0.4120	0.4300	0.4297	-0.0703	0.0254	0.4210	0.4390
	0.7	0.4359	0.3451	-0.0908	0.0208	0.3380	0.3520	0.3534	-0.0825	0.0195	0.3460	0.360

From table 2, it may be noted that the performance of MLE of $R(t)$ is better than that of UMVUE of $R(t)$ under Type-II censoring as it has lower MSE. It is also observed that with increasing values of r , both the estimators approaches their true value.

In order to investigate performance of estimators of P , we have generated 1000 random samples from the population X and Y with sizes $(n,m)=(100,100),(60,60)$ from (2.4) with fixed values of $\beta_1 = \beta_2 = 1.5$ and $(\alpha_1, \alpha_2) = (1.5,1), (2.5,1.5)$, and $(2.5,2)$. We considered the first (r,s) observations are taken i.e $(r,s)=(10,20), (15,20)$ and $(20,30)$ using (2.17) and (3.11) and have computed $\hat{P}_{II}$ and $\hat{P}_{II}$ and corresponding values of bias and MSE.

Table 3: UMVU and MLE of P based on Type-II censoring scheme when $\beta_1 = \beta_2$

$(\alpha_1, \alpha_2) \rightarrow$ $\hat{P} \rightarrow$ $(n, m) \downarrow$	$(r, s) \downarrow$	$\hat{P}$	$\hat{P}$	$\hat{P}$	$\hat{P}$	$\hat{P}$	$\hat{P}$
		(1.5,1) 0.4000	(2.5,1.5) 0.3750	(2.5,2) 0.4444			
(100,100)	(10,20)	0.4421 0.0421 1e-05 (0.4290,0.4550)	0.4601 0.0601 0.0594 3e-05 (0.4430,0.477)	0.4344 0.0594 3e-05 (0.3430,0.3710)	0.4597 0.0847 6e-05 (0.5700,0.6040)	0.4256 -0.0189 3e-05 (0.4120,0.4390)	0.4838 0.0393 1e-05 (0.4660,0.502)
(100,100)	(15,20)	0.4769 0.0769 4e-05 (0.4620,0.4920)	0.4578 0.0578 6e-05 (0.4400,0.4500)	0.4490 0.0740 5e-05 (0.4310,0.4670)	0.4768 0.1018 1e-04 (0.4620,0.4920)	0.4572 0.0128 1e-06 (0.4420,0.4720)	0.4897 0.0453 2e-05 (0.4720,0.5070)
(100,100)	(20,30)	0.4788 0.0788 6e-05 (0.4640,0.4940)	0.4604 0.0832 7e-05 (0.4420,0.4780)	0.4785 0.0854 6e-05 (0.4630,0.4940)	0.5405 0.1033 6e-05 (0.4680,0.5030)	0.4512 0.0068 5e-05 (0.4350,0.4670)	0.5019 0.0574 7e-05 (0.4840,5200)
(60,60)	(10,20)	0.4338 0.0338 3e-05 (0.4200,0.4470)	0.4874 0.0874 6e-05 (0.4710,0.5040)	0.4344 0.0594 4e-05 (0.4210,0.448)	0.4724 0.0974 7e-05 (0.4550,0.4900)	0.3956 -0.0489 4e-05 (0.3810,0.410)	0.5219 0.0774 7e-05 (0.5040,0.5400)
(60,60)	(15,20)	0.4660 0.0660 4e-05 (0.4520,0.4800)	0.4858 0.0858 5e-05 (0.4690,0.5020)	0.4523 0.0773 5e-05 (0.4350,0.4690)	0.4523 0.1026 1e-04 (0.4630,0.4920)	0.4479 0.0035 5e-07 (0.4330,0.4630)	0.4958 0.0513 7e-05 (0.4780,0.5140)
(60,60)	(20,30)	0.4766 0.0766 5e-05 (0.4620,0.4920)	0.4931 0.0931 8e-05 (0.4760,0.5100)	0.4725 0.0975 5e-05 (0.4570,0.4880)	0.4841 0.1091 8e-04 (0.4660,0.5020)	0.4499 0.0055 5e-07 (0.4340,0.4650)	0.5177 0.0732 6e-05 (0.5000,0.5350)

In each column of table 3, the first row values represents the estimated values of the parameter, second row values represent the bias of the estimator, third row values represents the MSE of the estimator and fourth row value represent the $100(1 - \theta)\%$ confidence interval of the estimated value. From table 3, on comparing MSE of the estimators, we observed that with increasing values of r both the estimators approaches to their true value and compared MSE values are decreases. It is clear that under Type-II censoring, the UMVUE of $\hat{P}$ shows better result than MLE of $\hat{P}$ for all different values of (r,s) and (n,m) .

Next, in order to obtain point estimators of ILD under Bartholomew censored sampling scheme, we have generated 1000 random samples each of size 100 for fixed values of $\alpha = (0.5)$ and $\beta = (2.5)$ from the (2.4). By fixing the termination time at t_0 and replacing the failure by operating X_1 , we have computed the values of $R(t)$ is the number of failures before time t_0 . For different termination times $t_0 = (0.50, 0.70, 0.90 \text{ and } 1)$, we have computed $\hat{R}_T(t)$ and $\hat{R}_T(t)$ and their corresponding bias, MSE and approximation $100(1 - \theta)\%$ confidence interval.

Table 4: UMVU and MLE of $R(t)$ based on Bartholomew censoring scheme

$t_0 \rightarrow$ $t \downarrow$	$R(t) \downarrow$	0.50	0.70	0.90	1
		$\hat{P}$	$\hat{P}$	$\hat{P}$	$\hat{P}$
0.1	0.8039	0.3210 -0.4853 0.00185 (0.3170,0.3250)	0.5271 -0.2768 0.00059 (0.5230,0.5310)	0.7013 -0.1026 0.00010 (0.6970,0.7050)	0.8969 -0.0327 0.00010 (0.8930,0.9010)
0.15	0.7621	0.2938 -0.4683 0.0017 (0.2900,0.2970)	0.4863 -0.2756 0.00056 (0.4820,0.4910)	0.6582 -0.1038 9.1e-05 (0.6540,0.6620)	0.8543 -0.0365 9.7e-05 (0.8500,0.8580)
0.20	0.7278	0.2654 -0.4624 0.0016 (0.2620,0.269)	0.4510 -0.2768 0.00067 (0.4570,0.4550)	0.6164 -0.1114 0.00011 (0.6120,0.6200)	0.8128 -0.0369 (0.8090,0.8190)
0.25	0.6985	0.2471 -0.4514 0.0015 (0.2440,0.2500)	0.4248 -0.2737 0.00063 (0.4210,0.4290)	0.5908 -0.1077 0.00010 (0.5870,0.5950)	0.7589 -0.0396 3e-05 (0.7540,0.7630)
0.35	0.6496	0.2185 -0.4311 0.0013 (0.2160,0.2210)	0.3830 -0.2665 0.00053 (0.3790,0.3870)	0.5439 -0.1056 0.00014 (0.5400,0.5480)	0.7124 -0.0372 3.4e-05 (0.7080,0.7160)
0.50	0.5918	0.1948 -0.3970 0.0011 (0.1920,0.1970)	0.3410 -0.2507 0.00046 (0.3360,0.3440)	0.4883 -0.1024 0.00011 (0.4830,0.4930)	0.6124 -0.0342 3.3e-05 (0.6090,0.6160)
0.75	0.5196	0.1608 -0.3588 0.00096 (0.1590,0.1630)	0.2872 -0.2324 0.00043 (0.2840,0.2900)	0.4199 -0.0997 0.00010 (0.4170,0.4230)	0.4874 -0.0322 3.1e-05 (0.4840,0.4910)
		$\hat{P}$	$\hat{P}$	$\hat{P}$	$\hat{P}$
		0.3186 -0.4853 0.00187 (0.3120,0.3220)	0.5233 -0.2806 0.00060 (0.5190,0.527)	0.7013 -0.1026 0.00010 (0.6970,0.7050)	0.8969 -0.0327 0.00010 (0.8930,0.9010)
		0.2918 -0.4703 0.0017 (0.2880,0.2950)	0.4831 -0.2790 0.00058 (0.4790,0.4870)	0.6582 -0.1038 9.1e-05 (0.6540,0.6620)	0.8543 -0.0365 9.7e-05 (0.8500,0.8580)
		0.2637 -0.4641 0.0016 (0.2600,0.2670)	0.4510 -0.2768 0.00067 (0.4570,0.4550)	0.6164 -0.1114 0.00011 (0.6120,0.6200)	0.8128 -0.0369 (0.8090,0.8190)
		0.2457 -0.4528 0.0016 (0.2430,0.2490)	0.4248 -0.2737 0.00063 (0.4210,0.4290)	0.5908 -0.1077 0.00010 (0.5870,0.5950)	0.7589 -0.0396 3e-05 (0.7540,0.7630)
		0.2174 -0.4322 0.0013 (0.2140,0.2210)	0.3810 -0.2656 0.00053 (0.3770,0.3850)	0.5439 -0.1056 0.00014 (0.5400,0.5480)	0.7124 -0.0372 3.4e-05 (0.7080,0.7160)
		0.1939 -0.3978 0.0011 (0.1910,0.1960)	0.3410 -0.2507 0.00046 (0.3360,0.3430)	0.4883 -0.1024 0.00011 (0.4830,0.4930)	0.6124 -0.0342 3.3e-05 (0.6090,0.6160)
		0.1602 -0.3594 0.00097 (0.1580,0.1620)	0.2872 -0.2324 0.00043 (0.2840,0.2900)	0.4199 -0.0997 0.00010 (0.4170,0.4230)	0.4874 -0.0322 3.1e-05 (0.4840,0.4910)

From table 4, it is observed that under sampling scheme of Bartholomew for different values of t and termination time t_0 , with increasing values of t both estimators approaches to true value and MSE values are decreases. It is also observed that for different termination time t_0 and t the results obtained from UMVUE of $R(t)$ are better than the results obtained from MLE of $R(t)$, but as termination time increases both the estimators of $R(t)$ gives better results as MSE decreasing as t increases.

Next, we have investigated that the performance of estimators of P Bartholomew under scheme. For this we generated 1000 random samples from each population X and Y with sizes (n,m) from (2,4) with $\beta_1 = \beta_2 = 2.5$ and $(\alpha_1, \alpha_2) = (0.5, 0.5)$ and (1,1.5). For each sample by fixing the termination time at $t_{0=t_0}$ in both the population and replacing the failure by operating $X_{1, r}$ and s is the number of failures before time t_0 in X and t_{00} in Y populations respectively. For $t_0 = t_{00} = 0.70, 0.90$ and 1, using (5.8) and (6.8), we have computed average values of $\tilde{P}_I$ and $\hat{P}_I$, their corresponding average bias, MSE and approximation $(1 - \theta)100\%$ confidence interval and for both cases $n > m$ and $m > n$.

Table 5: UMVU and MLE of P based on Type-I censoring scheme

$(\alpha_1, \alpha_2) \rightarrow$	$(0.5, 0.5)$	$(0.5, 1)$	$(1, 1.5)$
$P \rightarrow$	0.5000	0.6666	0.6000
$(t = t_0) \downarrow$	$\hat{P}$	$\hat{P}$	$\hat{P}$
0.70	(50, 30)		
	0.4937	0.7300	0.7466
	-0.0063	0.2300	0.0799
	1e-04	0.0530	0.0064
0.90	(0.4930, 0.4940)	(0.7296, 0.7304)	(0.7460, 0.7470)
	0.4170	0.7359	0.7322
	-0.0421	0.2359	0.0655
	0.0018	0.0557	0.0043
1	(0.4050, 0.4290)	(0.7354, 0.7363)	(0.7320, 0.7330)
	0.3659	0.7373	0.7365
	-0.1341	0.2373	0.0698
	0.0198	0.0563	0.0051
0.70	(0.3520, 0.3800)	(0.7369, 0.7375)	(0.7360, 0.7370)
0.90	(50, 40)		
	0.4453	0.6142	0.7429
	-0.0547	0.1142	0.0762
	0.0030	0.0131	0.0059
1	(0.4350, 0.4550)	(0.6138, 0.6145)	(0.7420, 0.7440)
	0.3130	0.6133	0.7283
	-0.1870	0.1133	0.0616
	0.0349	0.0129	0.0038
0.70	(0.2980, 0.3280)	(0.6126, 0.6139)	(0.7280, 0.7290)
	0.2679	0.6093	0.7285
	-0.2321	0.1093	0.0618
	0.1163	0.0120	0.0039
0.90	(0.2520, 0.2830)	(0.6089, 0.6096)	(0.7280, 0.7290)

In each column of table 5 and 6, the first row represents the estimated value of the parameter, second row represent the bias of the estimator, third row represents the MSE of the estimator and the fourth row represent the $100(1 - \theta)\%$ confidence interval of the estimated value. On comparing MSE of the estimators, it may be observed that with increasing values of t both estimators approaches to their true value so MSE values starts decreasing. It is also observed that for both cases $n > m$ and for $m > n$, UMVUE of P provides better estimates than MLE.

8 Conclusion

In this paper, we have discussed the classical procedures to estimate the Stress-strength reliability (SSR) parameter under the Type-II censoring and Bartholomew sampling scheme. Under these procedures, we have obtained the different parameters viz. MLE and UMVUE of α^P , $R(t)$ and P . Further, we have compared the performance of these estimators on a simulated data. It was established that UMVUE provides either better estimate or as good estimates as MLE under Type-II censoring scheme as well as Bartholomew Scheme.

This work can be further extended by using other methods of estimation new techniques can be developed for the testing the hypothesis for the parameters of the ILD. One can adopt the Bayesian approach to estimate the parameters and comparison can be made vis a vis. classical method of estimation.

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