

Bayes Meets Riemann – Bayesian Characterization of Infinite Series with Application to Riemann Hypothesis

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ABSTRACT

In the classical literature on infinite series there are various tests to determine if a given infinite series converges, diverges, or oscillates. But unfortunately, for very many infinite series all the existing tests can fail to provide definitive answers. In this article we propose a novel Bayesian theory for assessment of convergence properties of any given infinite series. Remarkably, this theory attempts to provide conclusive answers to the question of convergence even where all the existing tests of convergence fail. We apply our ideas to seven different examples, obtaining very encouraging results. Importantly, we also apply our ideas to investigate the Riemann Hypothesis, and obtain results that do not completely support the conjecture. We also extend our ideas to develop a Bayesian theory on oscillating series, where we allow even infinite number of limit points. Analysis of Riemann Hypothesis using Bayesian multiple limit points theory yielded almost identical results as the Bayesian theory of convergence assessment.

Keywords: Bayesian theory; Dirichlet process; Infinite series; Möbius function; Riemann Hypothesis; Tests of series convergence.

2000 Mathematics Subject Classification: 40D05; 60F15; 60F20; 65-04.

1 Introduction

Determination of convergence, divergence or oscillation of infinite series has a very rich tradition in mathematics, and a large number of tests exist for the purpose. Unfortunately, there does not seem to exist any universal test that provides conclusive answers to all infinite series; see, for example, (Ilyin and Poznyak, 1982), (Knopp, 1990), (Bourchtein, Bourchtein, Nornberg and Venzke, 2012). Attempts to resolve the issue as much as possible using hierarchies of

tests, with the successive tests in the hierarchy providing conclusive answers to successively larger ranges of infinite series, are provided by (Knopp, 1990), (Bromwich, 2005), (Bourchtein, Bourchtein, Nornberg and Venzke, 2011) and (Liflyand, Tikhonov and Zeltser, 2011). These tests are based on the Kummer approach for positive series and the chain of the Ermakov tests for positive monotone series. The hierarchy of tests provided in (Bourchtein et al., 2012) are based on (Bromwich, 2005) and are related to the well-known Cauchy's test (see, for example, (Fichtenholz, 1970), (Rudin, 1976), (Spivak, 1994)). However, even these hierarchical tests, in spite of restrictions such as positive series (Bourchtein et al., 2012), fail to provide conclusive answers in many cases.

It is well-known that the famous Riemann Hypothesis is equivalent to convergence of an infinite series on a certain interval. A brief introduction to the problem, along with the necessary background, is provided in Section 6. Studying the relevant infinite series, if at all possible, is then the most challenging problem of mathematics. The existing mathematical literature, however, does not seem to be able to provide any directions in this regard. Hence, innovative theories and methods for analyzing infinite series should be particularly welcome.

Note that direct and successive evaluation of sums of consecutive terms of the deterministic series of interest need not even provide any insight into the convergence behaviour of the series. This is because if the said sum seems to have approximately stabilized after a large number of successive evaluations, a further large number of evaluations may reveal a slow increase of the sums. On the other hand, even though initially the sums might exhibit an increasing nature, eventually they might stabilize. To combat such problems, it would be worthwhile to create some appropriate transformation of the sums such that convergence of the series may be indicated if the transformed sums approach a certain pre-defined value (say, 1), and divergence would be anticipated if the transformed sums approach another pre-defined value (say, 0), in a large number of evaluations. Although these two pre-defined values and the progress of the transformed sums towards these values in a large, but finite number of evaluations do not, in any way, formally settle the question of convergence of the underlying series, strong evidence regarding the convergence behaviour may be gained, when the number of evaluations is considerably large.

In this paper, our approach of characterization of convergence properties of infinite series is based on the aforementioned intuition, which we formalize rigorously through a novel Bayesian procedure. We subsequently extend the idea and formalism to infinite series with multiple or even infinite number of limit points. The main motivation and the idea of Bayesian formalism is illustrated in Section 2.

2 The key concept

Let us assume that the terms $\{x_1, x_2, \dots\}$ of any deterministic infinite series of the form $\sum_{i=1}^{\infty} x_i$ of interest is a realization of some stochastic process $\{X_i : i = 1, 2, \dots\}$, so that $\sum_{i=1}^{\infty} x_i$ is a

realization of the corresponding random infinite series

$$S_{1,\infty} = \sum_{i=1}^{\infty} X_i. \quad (2.1)$$

In the above, we do not assume any distributional form for $\{X_i : i = 1, 2, \dots\}$, signifying the nonparametric nature of our problem. Let $p \in [0, 1]$ denote the probability of convergence the sum $S_{1,\infty}$. In particular, if $\{X_i : i = 1, 2, \dots\}$ are independent, then by Kolmogorov's 0-1 law (see, for example, (Stroock, 1999)), p is either 0 or 1, where 0 stands for divergence of almost all realizations of $S_{1,\infty}$ and 1 is associated with convergence of almost all realizations of $S_{1,\infty}$. Kolmogorov's three series theorem (see, for example, (Stroock, 1999)) helps determine in this case if $p = 0$ or $p = 1$. However, the three series theorem requires parametric specification of the distributions of $\{X_i : i = 1, 2, \dots\}$, and specific choices of the parameters determine if $p = 0$ or $p = 1$. Since our goal is to determine the convergence behaviour of the deterministic series $\sum_{i=1}^{\infty} x_i$, interpreted as a realization of the specified stochastic process, different choices of the parameters would lead to convergence and divergence of the same series, along with almost all other realizations of the stochastic process. In other words, Kolmogorov's three series theorem is inappropriate when it comes to determination of convergence behaviour of deterministic series.

If the random variables are not independent, then it may happen that some of the realizations of $S_{1,\infty}$ are convergent, some are divergent and the rest are oscillatory. Since the above argument regarding Kolmogorov's three series theorem shows that it is inappropriate to assume parametric forms of the distributions of the random variables, we do not assume any particular distributional form of $\{X_i : i = 1, 2, \dots\}$. It then follows that the value of p is unknown, so that from the Bayesian perspective, one must acknowledge uncertainty about p in the form of some appropriate prior.

Now, specifying a prior directly on p associated with the entire infinite series and computing the posterior given $\{X_i : i = 1, 2, \dots\}$, is not a valid proposition, as computing the likelihood would require evaluation of infinite number of terms associated with the infinite series, which amounts to knowing the convergence behaviour of the series of interest. Instead, it makes sense to specify priors on the probabilities associated with the finite partial sums of the form $\sum_{i=m}^n X_i$, for $m \leq n$. Indeed, let

$$P\left(\left|\sum_{i=m}^n X_i\right| \leq c_{m,n}\right) = p_{m,n},$$

where $c_{m,n}$ are non-negative quantities satisfying $c_{m,n} \downarrow 0$ as $m, n \rightarrow \infty$. Thus, the probability depends on how large m and n are.

Now note that, as $m, n \rightarrow \infty$,

$$\mathbb{I}\left\{\left|\sum_{i=m}^n X_i\right| \leq c_{m,n}\right\} \rightarrow \mathbb{I}\left\{\lim_{m,n \rightarrow \infty} \left|\sum_{i=m}^n X_i\right| = 0\right\}$$

almost surely, so that uniform integrability leads to

$$\begin{aligned}
 \lim_{m,n \rightarrow \infty} p_{m,n} &= \lim_{m,n \rightarrow \infty} P \left(\left| \sum_{i=m}^n X_i \right| \leq c_{m,n} \right) \\
 &= \lim_{m,n \rightarrow \infty} E \left(\mathbb{I}_{\left\{ \left| \sum_{i=m}^n X_i \right| \leq c_{m,n} \right\}} \right) = E \left(\mathbb{I}_{\left\{ \lim_{m,n \rightarrow \infty} \left| \sum_{i=m}^n X_i \right| = 0 \right\}} \right) \\
 &= P \left(\lim_{m,n \rightarrow \infty} \left| \sum_{i=m}^n X_i \right| = 0 \right) = \lim_{m,n \rightarrow \infty} p_{m,n} = p,
 \end{aligned} \tag{2.2}$$

so that it is sufficient to deal with $p_{m,n}$ associated with the partial sums rather than p . It is only required to ensure that the priors on $p_{m,n}$ are built such that given any realization $\{x_i : i = 1, 2, \dots\}$ of the stochastic process $\{X_i : i = 1, 2, \dots\}$ associated with the corresponding series of interest $\sum_{i=1}^{\infty} x_i$, the posterior corresponding to the prior of $p_{m,n}$, which we denote by $\pi_{m,n}(\cdot | |\sum_{i=m}^n x_i|)$, converges to $\pi(\cdot | |\lim_{m,n \rightarrow \infty} |\sum_{i=m}^n x_i|)$, the posterior corresponding to the prior of p . Since the latter posterior is based on some given, single realization of the underlying stochastic process, the overall probability of convergence p is informed with respect to the conditioned single realization only. Consequently, the overall probability of convergence, given the series of interest, admits interpretation as the probability of convergence of the series of interest. Hence, it is reasonable to require that, $\pi(\cdot | |\lim_{m,n \rightarrow \infty} |\sum_{i=m}^n x_i|) = \delta_{\{z\}}(\cdot)$, the point mass at z , where $z = 1$ or $z = 0$ accordingly as $\lim_{m,n \rightarrow \infty} |\sum_{i=m}^n x_i|$ is zero or positive, that is, accordingly as $\sum_{i=1}^{\infty} x_i$ is convergent or divergent. Thus, it is required to construct the priors on $p_{m,n}$ such that $\pi_{m,n}(\cdot | |\sum_{i=m}^n x_i|) \rightarrow \delta_{\{z\}}(\cdot)$ in some appropriate sense, as $m, n \rightarrow \infty$, for any realization of the stochastic process.

It is important to appreciate that for another realization $\{\tilde{x}_i : i = 1, 2, \dots\}$ of the underlying stochastic process, the corresponding infinite sum $\sum_{i=1}^{\infty} \tilde{x}_i$ may have different convergence behaviour than $\sum_{i=1}^{\infty} x_i$. For instance, $\sum_{i=1}^{\infty} \tilde{x}_i$ may be divergent while $\sum_{i=1}^{\infty} x_i$ may be convergent. Hence, the corresponding posteriors based on the partial sums of $\sum_{i=1}^{\infty} \tilde{x}_i$ will converge to 0, while those associated with $\sum_{i=1}^{\infty} x_i$ will converge to 1. Since p is the probability that $S_{1,\infty}$ converges, at first glance such discrepant posteriors may create the impression that the Bayesian inference procedure regarding p is inconsistent. However, as discussed above, given only the series of interest, the overall probability of convergence p admits interpretability as the probability of convergence of the series at hand. This is exactly what is desired, since our goal is to study the convergence properties of the series of our interest only, not to learn about p . As an aside, note that it is of course possible to learn about p via its posterior distribution which may be obtained by conditioning on adequate number of realizations (instead of a single realization) of the stochastic process as in the usual Bayesian inference problems of learning about unknown parameters.

In Section 3 we devise a recursive Bayesian methodology that achieves the goal discussed above. It is important to remark that no restrictive assumption is necessary for the development of our ideas, not even independence of X_i . With this methodology, we then characterize convergence and divergence of infinite series in Section 4, illustrating in Section 5 our theory

and methods with seven examples. In Section 6 we apply our ideas to Riemann Hypothesis, obtaining results that are not in complete favour of the conjecture. We also extend our theory and methods to infinite series with multiple or infinite number of limit points; details are provided in Section S-3 of the supplement. Illustrations of our Bayesian multiple limit point theory are provided in Sections S-4 and S-5 of the supplement, the latter section detailing the application to Riemann Hypothesis in order to vindicate our results obtained in Section 6. Finally, we make concluding remarks in Section 7.

3 A recursive Bayesian procedure for studying infinite series

Since we view X_i as realizations from some random process, we first formalize the notion in terms of the relevant probability space. Let $(\Omega, \mathcal{A}, \mu)$ be a probability space, where Ω is the sample space, $\mathcal{A}$ is the Borel σ -field on Ω , and μ is some probability measure. Let, for $i = 1, 2, 3, \dots$, $X_i : \Omega \mapsto \mathbb{R}$ be real valued random variables measurable with respect to the Borel σ -field $\mathcal{B}$ on $\mathbb{R}$. As in (Schervish, 1995), we can then define a σ -field of subsets of $\mathbb{R}^\infty$ with respect to which $X = (X_1, X_2, \dots)$ is measurable. Indeed, let us define $\mathbb{B}^\infty$ to be the smallest σ -field containing sets of the form

$$B = \{X : X_{i_1} \leq r_1, X_{i_2} \leq r_2, \dots, X_{i_p} \leq r_p, \text{ for some } p \geq 1, \\ \text{some integers } i_1, i_2, \dots, i_p, \text{ and some real numbers } r_1, r_2, \dots, r_p\}.$$

Since B is an intersection of finite number of sets of the form $\{X : X_{i_j} \leq r_j\}$; $j = 1, \dots, p$, all of which belong to $\mathcal{A}$ (since X_{i_j} are measurable) it follows that $X^{-1}(B) \in \mathcal{A}$, so that X is measurable with respect to $(\mathbb{R}^\infty, \mathbb{B}^\infty, P)$, where P is the probability measure induced by μ .

Alternatively, note that it is possible to represent any stochastic process $\{X_i : i \in \mathcal{I}\}$, for fixed i as a random variable $\omega \mapsto X_i(\omega)$, where $\omega \in \Omega$; Ω being the set of all functions from $\mathcal{I}$ into $\mathbb{R}$. Also, fixing $\omega \in \Omega$, the function $i \mapsto X_i(\omega)$; $i \in \mathcal{I}$, represents a path of X_i ; $i \in \mathcal{I}$. Indeed, we can identify ω with the function $i \mapsto X_i(\omega)$ from $\mathcal{I}$ to $\mathbb{R}$; see, for example, (Øksendal, 2000), for a lucid discussion.

This latter identification will be convenient for our purpose, and we adopt this in this article. Note that the σ -algebra $\mathcal{F}$ induced by X is generated by sets of the form

$$\{\omega : \omega(i_1) \in B_1, \omega(i_2) \in B_2, \dots, \omega(i_k) \in B_k\},$$

where $B_j \subset \mathbb{R}$; $j = 1, \dots, k$, are Borel sets in $\mathbb{R}$.

3.1 Development of the stage-wise likelihoods

For $j = 1, 2, 3, \dots$, let

$$S_{j, n_j} = \sum_{i=\sum_{k=0}^{j-1} n_k+1}^{\sum_{k=0}^j n_k} X_i, \quad (3.1)$$

where $n_0 = 0$ and $n_j \geq 1$ for all $j \geq 1$. Also let $\{c_j\}_{j=1}^{\infty}$ be a non-negative decreasing sequence and

$$Y_{j,n_j} = \mathbb{I}\{|S_{j,n_j}| \leq c_j\}. \quad (3.2)$$

Let, for $j \geq 1$,

$$P(Y_{j,n_j} = 1) = p_{j,n_j}. \quad (3.3)$$

Hence, the likelihood of p_{j,n_j} , given y_{j,n_j} , is given by

$$L(p_{j,n_j}) = p_{j,n_j}^{y_{j,n_j}} (1 - p_{j,n_j})^{1-y_{j,n_j}} \quad (3.4)$$

It is important to relate p_{j,n_j} to convergence or divergence of the underlying series. Note that p_{j,n_j} is the probability that $|S_{j,n_j}|$ falls below c_j . Thus, p_{j,n_j} can be interpreted as the probability that the series $S_{1,\infty}$ is convergent when the data observed is S_{j,n_j} . If $S_{1,\infty}$ is convergent, then it is to be expected *a posteriori*, that

$$p_{j,n_j} \rightarrow 1 \quad \text{as } j \rightarrow \infty. \quad (3.5)$$

Note that the above is expected to hold even for $n_j = n$ for all $j \geq 1$, and for all $n \geq 1$. This is related to Cauchy's criterion of convergence of partial sums: for every $\epsilon > 0$ there exists a positive integer N such that for all $n \geq m \geq N$, $|\sum_{i=m}^n X_i| < \epsilon$. Indeed, as we will formally show, condition (3.5) is both necessary and sufficient for convergence of the series.

On the other hand, if the series is divergent, then there exist $j_0 \geq 1$ such that for every $j > j_0$ there exists $n_j \geq 1$ satisfying $|S_{j,n_j}| > c_j$. Here we expect, *a posteriori*, that

$$p_{j,n_j} \rightarrow 0 \quad \text{as } j \rightarrow \infty. \quad (3.6)$$

Again, we will prove formally that the above condition is both necessary and sufficient for divergence.

In this work we call the series $S_{1,\infty}$ oscillating if the sequence $\{S_{1,n}; n = 1, 2, \dots\}$ has more than one limit points. Thus, these are non-convergent series, and so, the probability of convergence of these series must tend to zero in our Bayesian framework, which is in fact ensured by our theoretical developments. But it is also important to be able to categorize and learn about the limit points. A general theory, which encompasses finite as well as infinite number of limit points, with perhaps unequal frequencies of occurrences, is developed in Section S-3 of the supplement.

In what follows we shall first construct a recursive Bayesian methodology that formally characterizes convergence and divergence in terms of formal posterior convergence related to (3.5) and (3.6).

3.2 Development of recursive Bayesian posteriors

We assume that $\{y_{j,n_j}; j = 1, 2, \dots\}$ is observed successively at stages indexed by j . That is, we first observe y_{1,n_1} , and based on our prior belief regarding the first stage probability, p_{1,n_1} , compute the posterior distribution of p_{1,n_1} given y_{1,n_1} , which we denote by $\pi(p_{1,n_1}|y_{1,n_1})$.

Based on this posterior we construct a prior for the second stage, and compute the posterior $\pi(p_{2,n_2}|y_{1,n_1}, y_{2,n_2})$. We continue this procedure for as many stages as we desire. Details follow.

Consider the sequences $\{\alpha_j\}_{j=1}^{\infty}$ and $\{\beta_j\}_{j=1}^{\infty}$, where $\alpha_j = \beta_j = 1/j^2$ for $j = 1, 2, \dots$. At the first stage of our recursive Bayesian algorithm, that is, when $j = 1$, let us assume that the prior is given by

$$\pi(p_{1,n_1}) \equiv \text{Beta}(\alpha_1, \beta_1), \quad (3.7)$$

where, for $a > 0$ and $b > 0$, $\text{Beta}(a, b)$ denotes the Beta distribution with mean $a/(a+b)$ and variance $(ab)/\{(a+b)^2(a+b+1)\}$. Combining this prior with the likelihood (3.4) (with $j = 1$), we obtain the following posterior of p_{1,n_1} given y_{1,n_1} :

$$\pi(p_{1,n_1}|y_{1,n_1}) \equiv \text{Beta}(\alpha_1 + y_{1,n_1}, \beta_1 + 1 - y_{1,n_1}). \quad (3.8)$$

At the second stage (that is, for $j = 2$), for the prior of p_{2,n_2} we consider the posterior of p_{1,n_1} given y_{1,n_1} associated with the $\text{Beta}(\alpha_1 + \alpha_2, \beta_1 + \beta_2)$ prior. That is, our prior on p_{2,n_2} is given by:

$$\pi(p_{2,n_2}) \equiv \text{Beta}(\alpha_1 + \alpha_2 + y_{1,n_1}, \beta_1 + \beta_2 + 1 - y_{1,n_1}). \quad (3.9)$$

The reason for such a prior choice is that the uncertainty regarding convergence of the series is reduced once we obtain the posterior at the first stage, so that at the second stage the uncertainty regarding the prior is expected to be lesser compared to the first stage posterior. With our choice, it is easy to see that the prior variance at the second stage, given by

$$\{(\alpha_1 + \alpha_2 + y_{1,n_1})(\beta_1 + \beta_2 + 1 - y_{1,n_1})\} / \{(\alpha_1 + \alpha_2 + \beta_1 + \beta_2 + 1)^2(\alpha_1 + \alpha_2 + \beta_1 + \beta_2 + 2)\},$$

is smaller than the first stage posterior variance, given by

$$\{(\alpha_1 + y_{1,n_1})(\beta_1 + 1 - y_{1,n_1})\} / \{(\alpha_1 + \beta_1 + 1)^2(\alpha_1 + \beta_1 + 2)\}.$$

The posterior of p_{2,n_2} given y_{2,n_2} is then obtained by combining the second stage prior (3.9) with (3.4) (with $j = 2$). The form of the posterior at the second stage is thus given by

$$\pi(p_{2,n_2}|y_{2,n_2}) \equiv \text{Beta}(\alpha_1 + \alpha_2 + y_{1,n_1} + y_{2,n_2}, \beta_1 + \beta_2 + 2 - y_{1,n_1} - y_{2,n_2}). \quad (3.10)$$

Continuing this way, at the k -th stage, where $k > 1$, we obtain the following posterior of p_{k,n_k} :

$$\pi(p_{k,n_k}|y_{k,n_k}) \equiv \text{Beta}\left(\sum_{j=1}^k \alpha_j + \sum_{j=1}^k y_{j,n_j}, k + \sum_{j=1}^k \beta_j - \sum_{j=1}^k y_{j,n_j}\right). \quad (3.11)$$

It follows from (3.11) that

$$E(p_{k,n_k}|y_{k,n_k}) = \frac{\sum_{j=1}^k \alpha_j + \sum_{j=1}^k y_{j,n_j}}{k + \sum_{j=1}^k \alpha_j + \sum_{j=1}^k \beta_j}; \quad (3.12)$$

$$\text{Var}(p_{k,n_k}|y_{k,n_k}) = \frac{(\sum_{j=1}^k \alpha_j + \sum_{j=1}^k y_{j,n_j})(k + \sum_{j=1}^k \beta_j - \sum_{j=1}^k y_{j,n_j})}{(k + \sum_{j=1}^k \alpha_j + \sum_{j=1}^k \beta_j)^2(1 + k + \sum_{j=1}^k \alpha_j + \sum_{j=1}^k \beta_j)}. \quad (3.13)$$

Since $\sum_{j=1}^k \alpha_j = \sum_{j=1}^k \beta_j = \sum_{j=1}^k \frac{1}{j^2}$, (3.12) and (3.13) admit the following simplifications:

$$E(p_{k,n_k} | y_{k,n_k}) = \frac{\sum_{j=1}^k \frac{1}{j^2} + \sum_{j=1}^k y_{j,n_j}}{k + 2 \sum_{j=1}^k \frac{1}{j^2}}; \quad (3.14)$$

$$Var(p_{k,n_k} | y_{k,n_k}) = \frac{(\sum_{j=1}^k \frac{1}{j^2} + \sum_{j=1}^k y_{j,n_j})(k + \sum_{j=1}^k \frac{1}{j^2} - \sum_{j=1}^k y_{j,n_j})}{(k + 2 \sum_{j=1}^k \frac{1}{j^2})^2 (1 + k + 2 \sum_{j=1}^k \frac{1}{j^2})}. \quad (3.15)$$

4 Characterization of convergence properties of the underlying infinite series

Based on our recursive Bayesian theory we have the following theorem that characterizes convergence of $S_{1,\infty}$ in terms of the limit of the posterior probability of p_{k,n_k} , as $k \rightarrow \infty$. Note that the sample space of $S_{1,\infty}$ is also given by $\mathfrak{S}$. We also assume, for the sake of generality, that for any $\omega \in \mathfrak{S} \cap \mathfrak{N}^c$, where $\mathfrak{N} (\subset \mathfrak{S})$ has zero probability measure, the non-negative monotonically decreasing sequence $\{c_j\}_{j=1}^\infty$ depends upon ω , so that we shall denote the sequence by $\{c_j(\omega)\}_{j=1}^\infty$. In other words, we allow $\{c_j(\omega)\}_{j=1}^\infty$ to depend upon the corresponding series $S_{1,\infty}(\omega)$. Note that if $S_{1,\infty}(\omega) < \infty$, then the sequence $\{|S_{j,n_j}(\omega)|\}_{j=1}^\infty$ is uniformly bounded, for all sequences $\{n_j\}_{j=1}^\infty$, and converges to zero for all sequences $\{n_j\}_{j=1}^\infty$, which implies that there exists a monotonically decreasing sequence $\{c_j(\omega)\}_{j=1}^\infty$ independent of the choice of $\{n_j\}_{j=1}^\infty$ such that for some $j_0(\omega) \geq 1$,

$$|S_{j,n_j}(\omega)| \leq c_j(\omega), \text{ for } j \geq j_0(\omega). \quad (4.1)$$

Indeed, in most of our illustrations presented in this paper, including the Riemann Hypothesis, we choose $\{c_j(\omega)\}_{j=1}^\infty$ in a way that depends upon the infinite series at hand.

Theorem 4.1. *For any $\omega \in \mathfrak{S} \cap \mathfrak{N}^c$, where $\mathfrak{N}$ is some null set having probability measure zero, $S_{1,\infty}(\omega) < \infty$ if and only if there exists a non-negative monotonically decreasing sequence $\{c_j(\omega)\}_{j=1}^\infty$ such that for any choice of the sequence $\{n_j\}_{j=1}^\infty$,*

$$\pi(\mathcal{N}_1 | y_{k,n_k}(\omega)) \rightarrow 1, \quad (4.2)$$

as $k \rightarrow \infty$, where $\mathcal{N}_1$ is any neighborhood of 1 (one).

Proof. Let, for $\omega \in \mathfrak{S} \cap \mathfrak{N}^c$, $S_{1,\infty}(\omega)$ be convergent. Then, by (4.1), $|S_{j,n_j}(\omega)| \leq c_j(\omega)$ for all n_j , so that $y_{j,n_j}(\omega) = 1$ for all $j > j_0(\omega)$, for all n_j . Hence, in this case, $\sum_{j=1}^k y_{j,n_j}(\omega) = k - k_0(\omega)$, where $k_0(\omega) \geq 0$. Also, $\sum_{j=1}^k \frac{1}{j^2} \rightarrow \frac{\pi^2}{6}$, as $k \rightarrow \infty$. Consequently, it is easy to see that

$$\mu_k = E(p_{k,n_k} | y_{k,n_k}(\omega)) \sim \frac{\frac{\pi^2}{6} + k - k_0(\omega)}{k + \frac{\pi^2}{3}} \rightarrow 1, \text{ as } k \rightarrow \infty, \text{ and,} \quad (4.3)$$

$$\sigma_k^2 = Var(p_{k,n_k} | y_{k,n_k}(\omega)) \sim \frac{(\frac{\pi^2}{6} + k)(\frac{\pi^2}{6})}{(k + \frac{\pi^2}{3})^2 (1 + k + \frac{\pi^2}{3})} \rightarrow 0 \text{ as } k \rightarrow \infty. \quad (4.4)$$

In the above, for any two sequences $\{a_k\}_{k=1}^\infty$ and $\{b_k\}_{k=1}^\infty$, $a_k \sim b_k$ indicates $\frac{a_k}{b_k} \rightarrow 1$, as $k \rightarrow \infty$. Now let $\mathcal{N}_1$ denote any neighborhood of 1, and let $\epsilon > 0$ be sufficiently small such that $\mathcal{N}_1 \supseteq \{1 - p_{k,n_k} < \epsilon\}$. Combining (4.3) and (4.4) with Chebychev's inequality ensures that (4.2) holds.

Now assume that (4.2) holds. Then for any given $\epsilon > 0$,

$$\pi(p_{k,n_k} > 1 - \epsilon | y_{k,n_k}(\omega)) \rightarrow 1, \text{ as } k \rightarrow \infty. \quad (4.5)$$

Hence, it can be seen, using Markov's inequality, that

$$E(p_{k,n_k} | y_{k,n_k}(\omega)) \rightarrow 1; \quad (4.6)$$

$$Var(p_{k,n_k} | y_{k,n_k}(\omega)) \rightarrow 0, \quad (4.7)$$

as $k \rightarrow \infty$. If $S_{1,\infty}(\omega)$ does not converge then there exists $j_0(\omega)$ such that for each $j \geq j_0(\omega)$, there exists $n_j(\omega)$ satisfying $|S_{j,n_j(\omega)}(\omega)| > c_j(\omega)$, for any choice of non-negative sequence $\{c_j(\omega)\}_{j=1}^{\infty}$ monotonically converging to zero. Hence, in this situation, $0 \leq \sum_{j=1}^k y_{j,n_j(\omega)}(\omega) \leq j_0(\omega)$. Substituting this in (3.14) and (3.15), it is easy to see that, as $k \rightarrow \infty$,

$$E(p_{k,n_k(\omega)} | y_{k,n_k(\omega)}(\omega)) \rightarrow 0; \quad (4.8)$$

$$Var(p_{k,n_k(\omega)} | y_{k,n_k(\omega)}(\omega)) \rightarrow 0, \quad (4.9)$$

so that (4.6) is contradicted. □

We now prove the following theorem that provides necessary and sufficient conditions for divergence of $S_{1,\infty}(\omega)$ in terms of the limit of the posterior probability of $p_{k,n_k(\omega)}$, as $k \rightarrow \infty$.

Theorem 4.2. *For any $\omega \in \mathfrak{S} \cap \mathfrak{N}^c$, where $\mathfrak{N}$ is some null set having probability measure zero, $S_{1,\infty}(\omega)$ is divergent if and only if there exists a sequence $\{n_j(\omega)\}_{j=1}^{\infty}$ such that*

$$\pi(\mathcal{N}_0 | y_{k,n_k(\omega)}(\omega)) \rightarrow 1, \quad (4.10)$$

$k \rightarrow \infty$, where $\mathcal{N}_0$ is any neighborhood of 0 (zero).

Proof. Assume that $S_{1,\infty}(\omega)$ is divergent. Then then there exist $j_0(\omega) \geq 1$ such that for every $j \geq j_0(\omega)$, one can find $n_j(\omega)$ satisfying $|S_{j,n_j(\omega)}(\omega)| > c_j(\omega)$, for any choice of non-negative sequence $\{c_j(\omega)\}_{j=1}^{\infty}$ monotonically converging to zero. From the proof of the sufficient condition of Theorem 4.1 it follows that (4.8) and (4.9) hold. Let $\epsilon > 0$ be small enough so that $\mathcal{N}_0 \supseteq \{p_{k,n_k(\omega)} < \epsilon\}$. Then combining Chebychev's inequality with (4.8) and (4.9) it is easy to see that (4.10) holds.

Now assume that (4.10) holds. Then for any given $\epsilon > 0$,

$$\pi(p_{k,n_k(\omega)} < \epsilon | y_{k,n_k(\omega)}(\omega)) \rightarrow 1, \text{ as } k \rightarrow \infty. \quad (4.11)$$

It can be seen, now using Markov's inequality with respect to $1 - p_{k,n_k(\omega)}$, that

$$E(p_{k,n_k(\omega)} | y_{k,n_k(\omega)}(\omega)) \rightarrow 0; \quad (4.12)$$

$$Var(p_{k,n_k(\omega)} | y_{k,n_k(\omega)}(\omega)) \rightarrow 0, \quad (4.13)$$

as $k \rightarrow \infty$.

If $S_{1,\infty}(\omega)$ is convergent, then by Theorem 4.1, $\pi(\mathcal{N}_1 | y_{k,n_k(\omega)}(\omega)) \rightarrow 1$ as $k \rightarrow \infty$, for all sequences $\{n_j\}_{j=1}^{\infty}$, so that $E(p_{k,n_k(\omega)} | y_{k,n_k(\omega)}(\omega)) \rightarrow 1$, which is a contradiction to (4.12). □

Note that Theorem 4.2 encompasses even oscillatory series. For instance, if for some $\omega \in \mathfrak{N}^c$, $S_{1,\infty}(\omega) = \sum_{i=1}^{\infty} (-1)^{i-1}$, then the sequence $n_j(\omega) = 1 + 2(j-1)$ ensures that $|S_{j,n_j}(\omega)| > c_j(\omega)$ for all $j \geq j_0(\omega)$, for some $j_0(\omega) \geq 1$, for any monotonically decreasing non-negative sequence $\{c_j(\omega)\}_{j=1}^{\infty}$. This of course forces declaration of divergence of this particular series, as per Theorem 4.2. We show in Section S-4.1 of the supplement, with the help of our Bayesian idea of studying oscillatory series, how to identify the number and proportions of the limit points of this oscillatory series.

4.1 Characterization of infinite series using non-recursive Bayesian posteriors

Observe that it is not strictly necessary for the prior at any stage to depend upon the previous stage. Indeed, we may simply assume that $\pi(p_{j,n_j}) \equiv \text{Beta}(\alpha_j, \beta_j)$, for $j = 1, 2, \dots$. In this case, the posterior of p_{k,n_k} given y_{k,n_k} is simply $\text{Beta}(\alpha_k + y_{k,n_k}, 1 + \beta_k - y_{k,n_k})$. The posterior mean and variance are then given by

$$E(p_{k,n_k} | y_{k,n_k}(\omega)) = \frac{\alpha_k + y_{k,n_k}(\omega)}{1 + \alpha_k + \beta_k}; \quad (4.14)$$

$$\text{Var}(p_{k,n_k} | y_{k,n_k}(\omega)) = \frac{(\alpha_k + y_{k,n_k}(\omega))(1 + \beta_k - y_{k,n_k}(\omega))}{(1 + \alpha_k + \beta_k)^2(2 + \alpha_k + \beta_k)}. \quad (4.15)$$

Since $y_{k,n_k}(\omega)$ converges to 1 or 0 as $k \rightarrow \infty$, accordingly as $S_{1,\infty}(\omega)$ is convergent or divergent, it is easily seen, provided that $\alpha_k \rightarrow 0$ and $\beta_k \rightarrow 0$ as $k \rightarrow \infty$, that (4.14) converges to 1 (respectively, 0) if and only if $S_{1,\infty}(\omega)$ is convergent (respectively, divergent).

Thus, characterization of convergence or divergence of infinite series is possible even with the non-recursive approach. Indeed, note that the prior parameters α_k and β_k are more flexible compared to those associated with the recursive approach. This is because, in the non-recursive approach we only require $\alpha_k \rightarrow 0$ and $\beta_k \rightarrow 0$ as $k \rightarrow \infty$, so that convergence of the series $\sum_{j=1}^{\infty} \alpha_j$ and $\sum_{j=1}^{\infty} \beta_j$ are not necessary, unlike the recursive approach. However, choosing α_k and β_k to be of sufficiently small order ensures much faster convergence of the posterior mean and variance as compared to the recursive approach.

Unfortunately, an important drawback of the non-recursive approach is that it does not admit extension to the case of general oscillatory series with multiple limit points, where blocks of partial sums can not be used; see Section S-3 of the supplement. On the other hand, as we show in Section S-3 of the supplement, the principles of our recursive theory can be easily adopted to develop a Bayesian characterization of oscillating series, which also includes the characterization of non-oscillating series as a special case. In other words, the recursive approach seems to be more powerful from the perspective of development of a general characterization theory. Moreover, as our examples on convergent and divergent series demonstrate, the recursive posteriors converge sufficiently fast to the correct degenerate distributions, obviating the need to consider the non-recursive approach. Consequently, we do not further pursue the non-recursive approach in this article but reserve the topic for further investigation in the future.

Remark 4.1. An important issue associated with our characterization results is that the terms $\{x_1, x_2, \dots\}$ of the underlying deterministic series of interest $\sum_{i=1}^{\infty} x_i$ is assumed to lie in the

complement of the null set. For appropriately specified stochastic processes this need not be difficult to verify. However, for the sake of sufficient generality we have not assumed any specific form of the underlying stochastic process, which makes the question of null sets relevant in our case. The solution is that, even if $\{x_1, x_2, \dots\}$ falls in some null set, we can still compute a pseudo-posterior distribution of p_{k,n_k} conditional on $\{x_1, x_2, \dots\}$, which has exactly the same form as before. This pseudo-posterior may not admit interpretability as a *bona fide* posterior distribution, but characterizes the convergence property of $\sum_{i=1}^{\infty} x_i$ in exactly the same way as before. In other words, interestingly and very importantly, all our results of characterization presented in our paper hold for *all* $\omega \in \mathfrak{S}$.

5 Illustrations

We now illustrate our ideas with seven examples. These seven examples can be categorized into three categories in terms of construction of the upper bound c_j . With the first example we demonstrate that it may sometimes be easy to devise an appropriate upper bound. In Examples 2 – 5, we show that usually simple bounds such as that in Example 1, are not adequate in practice, but appropriate bounds may be constructed if convergence and divergence of the series in question is known for some values of the parameters; the resultant bounds can be utilized to learn about convergence or divergence of the series for the remaining values of the parameters. In Examples 6 and 7, the series in question are stand-alone in the sense they are not defined by parameters with known convergence/divergence for some of their values which might have aided our construction of c_j . However, we show that these series can be embedded into appropriately parameterized series, facilitating similar analysis as Examples 2 – 5.

For these examples, we consider $n_j = n$ for $j = 1, \dots, K$, with $n = 10^6$ and $K = 10^5$. Since n seems to be sufficiently large, in the case of divergence we expect $|S_{j,n}|$ to exceed the monotonically decreasing c_j for all $j \geq j_0$, for sufficiently large j_0 . Our experiments demonstrate that this is indeed the case. For further justification we conducted some experiments with larger values of n , but the results remained unchanged. Hence, for relative computational ease we set $n = 10^6$ for the illustrations in this work.

Since we needed to sum 10^6 terms at each step of 10^5 stages, the associated computation is extremely demanding. For the purpose of efficiency, we parallelized the computation of the sums of 10^6 terms, splitting the job on many processors, using the Message Passing Interface (MPI) protocol. In more details, we implemented our parallelized codes, written in C, in VMware consisting of 60 double-threaded, 64-bit physical cores, each running at 2793.269 MHz. Parallel computation of our methods associated with Examples 1 to 5 take, respectively, 1 minute, 4 minutes, 7 minutes, 6 minutes, and 9 minutes. Examples 6 and 7 require about 6 minutes and 4 minutes of computational time.

For space issues we present our applications to the first four examples here; the applications of the remaining examples are provided in Section S-2 of the supplement.

5.1 Example 1

In their first example (Bourchtein et al., 2012) study the following divergent series with their methods:

$$S = \sum_{i=2}^{\infty} \frac{1}{\log(i)}. \quad (5.1)$$

We test our Bayesian idea on this series choosing the monotonically decreasing sequence as $c_{j,n} = 1/\sqrt{nj}$, where we represent c_j as $c_{j,n}$ to reflect dependence on n . Figure 1, a plot of the posterior means of $\{p_{k,n}; k = 1, \dots, 10^5\}$, clearly and correctly indicates that the series is divergent. We also constructed approximate 95% highest posterior density credible intervals at each recursive step; however, thanks to very less variances at each stage, the intervals turned out to be too small to be clearly distinguishable from the plot of the stage-wise posterior means.

5.2 Example 2

Example 2 of (Bourchtein et al., 2012) deals with the following series:

$$S^a = \sum_{i=2}^{\infty} \left(1 - \left\{ \frac{\log(i)}{i} \right\} - a \frac{\log \log(i)}{i} \right)^i, \quad (5.2)$$

where $a \in \mathbb{R}$. (Bourchtein et al., 2012) prove that the series converges for $a > 1$ and diverges for $a \leq 1$.

5.2.1 Choice of $c_{j,n}$

Now, however, selecting the monotone sequence as $c_{j,n} = 1/\sqrt{nj}$ turn out to be inappropriate for this series, the behaviour of which is quite sensitive to the parameter a , particularly around $a = 1$. Hence, any appropriate sequence $\{c_{j,n}\}_{j=1}^{\infty}$ must depend on the parameter a of the series (5.2).

Denoting $c_{j,n}$ by $c_{j,n}^a$ to reflect the dependence on a as well, we first set

$$u_{j,n}^a = S_{j,n}^{a_0} + \frac{(a - 1 - 9 \times 10^{-11})}{\log(j+1)}, \quad (5.3)$$

and then let

$$c_{j,n}^a = \begin{cases} u_{j,n}^a, & \text{if } u_{j,n}^a > 0; \\ S_{j,n}^{a_0}, & \text{otherwise.} \end{cases} \quad (5.4)$$

where $a_0 = 1 + 10^{-10}$. The reason behind such a choice of $c_{j,n}^a$ is provided below.

Let, for $\epsilon > 0$,

$$\tilde{S} = \sup \{S^a : a \geq 1 + \epsilon\}. \quad (5.5)$$

Thus, $\tilde{S}$ may be interpreted as the convergent series which is closest to divergence given the convergence criterion $a \geq 1 + \epsilon$. Since S^a is decreasing in a , it easily follows that equality of (5.5) is attained at $a_0 = 1 + \epsilon$.

Since the terms of the series S^a are decreasing in i , it follows that $S_{j,n}^{a_0}$ in (5.4) is decreasing in j . We assume that ϵ is chosen to be so small that convergence properties of the series for $\{a \leq 1\} \cup \{a \geq 1 + \epsilon\}$ are only desired. Indeed, since $\left(1 - \left\{\frac{\log(i)}{i}\right\} - a \frac{\log \log(i)}{i}\right)^i$ is decreasing in a for any given $i \geq 3$, our method of constructing $c_{j,n}^a$ need not be able to correctly identify the convergence properties of the series for $1 < a < 1 + \epsilon$.

For the purpose of illustrations we choose $\epsilon = 10^{-10}$. Note that for $a > 1$ the term $\frac{(a-1-9 \times 10^{-11})}{\log(j+1)}$ inflates $c_{j,n}^a$ making $S_{j,n}^a$ more likely to fall below $c_{j,n}^a$ for increasing a , thus paving the way for diagnosing convergence. The same term also ensures that for $a \leq 1$, $c_{j,n}^a < S_{j,n}^{a_0}$, so that $S_{j,n}^a$ is likely to exceed $c_{j,n}^a$, thus providing an inclination towards divergence. The term -9×10^{-11} is an adjustment for the case $a = 1 + 10^{-10}$, ensuring that $c_{j,n}^a$ marginally exceeds $S_{j,n}^a$ to ensure convergence. The scaling factor $\log(j+1)$ ensures that the part $\frac{(a-1-9 \times 10^{-11})}{\log(j+1)}$ of (5.4) tends to zero at a slow rate so that $c_{j,n}^a$ is decreasing with j and n even if $a - 1 - 9 \times 10^{-11}$ is negative.

Figure 2, depicting our Bayesian results for this series, is in agreement with the results of (Bourchtein et al., 2012). In fact, we have applied our methods to many more values of $a \in A_\epsilon$ with $\epsilon = 10^{-10}$, and in every case the correct result is vindicated.

5.3 Example 3

Let us now consider the following series analysed by (Bourchtein et al., 2012):

$$S = \sum_{i=3}^{\infty} \left(1 - \left(\frac{\log(i)}{i}\right) a \frac{\log \log(i)}{\log(i)}\right)^i, \quad (5.6)$$

where $a > 0$. As is shown by (Bourchtein et al., 2012), the series converges for $a > e$ and diverges for $a \leq e$.

5.3.1 Choice of $c_{j,n}$

Here we first set

$$u_{j,n}^a = S_{j,n}^{a_0} + \frac{(a - e - 9 \times 10^{-11})}{\log(j+1)}, \quad (5.7)$$

and then let $c_{j,n}^a$ defined by (5.4). Again, it is easily seen that $S_{j,n}^{a_0}$ is decreasing in j . In this example we set $a_0 = e + 10^{-10}$. The rationale behind the choice remains the same as detailed in Section 5.2.1.

As before, the results obtained by our Bayesian theory, as displayed in Figure 3, are in complete agreement with the results obtained by (Bourchtein et al., 2012).

5.4 Example 4

(Bourchtein et al., 2012) consider the following series:

$$S_1 = \sum_{i=3}^{\infty} \left(1 - \frac{\log i}{i} - \frac{\log \log i}{i} \left\{ \cos^2 \left(\frac{1}{i} \right) \right\} (a + (-1)^i b) \right)^i, \quad (5.8)$$

where $a \geq 0$ and $b \geq 0$. It has been proved by (Bourchtein et al., 2012) that the series is convergent for $a - b > 1$ and divergent for $a + b < 1$. However, for $a = b = 1$, the hierarchy of tests fails to provide definitive answer to the question of convergence. With our Bayesian procedure, we investigate the convergence properties of the series, and in particular, seek to address the case of $a = b = 1$.

In this example we denote the partial sums by $S_{j,n}^{a,b}$ and the actual series S by $S^{a,b}$ to reflect the dependence on both the parameters a and b .

$$S_{j,n}^{a,b} = \sum_{i=3+n(j-1)}^{3+nj-1} \left(1 - \frac{\log i}{i} - \frac{\log \log i}{i} \left\{ \cos^2 \left(\frac{1}{i} \right) \right\} (a + (-1)^i b) \right)^i, \quad (5.9)$$

We then have the following lemma, the proof of which is presented in Section S-1 of the supplement.

Lemma 5.1. *For series (5.8), for $j \geq 1$ and n even, $S_{j,n}^{a,b}$ given by (5.9) is decreasing in a but increasing in b .*

Since $S^{a,b}$ is just summation of the partial sums, it follows that

Corollary 5.2. *$S^{a,b}$ is decreasing in a and increasing in b .*

We let

$$A_\epsilon = \{a : 0 \leq a \leq 1\} \cup \{a : a \geq 1 + \epsilon\}, \quad (5.10)$$

and

$$\tilde{S} = \inf_{a \in A_\epsilon} \sup_{b \geq 0} \left\{ S^{a,b} : a - b > 1 \right\}. \quad (5.11)$$

It is easy to see in this case, due to Corollary 5.2 and the convergence criterion $a - b > 1$, that $\tilde{S}$ is attained at $a_0 = 1 + \epsilon$ and $b_0 = 0$. As before, we set $\epsilon = 10^{-10}$. Hence, arguments similar to those in Section 5.2.1 lead to the following choice of the upper bound for $S_{j,n}^{a,b}$, which we denote in this example by $c_{j,n}^{a,b}$:

$$c_{j,n}^{a,b} = \begin{cases} u_{j,n}^{a,b}, & \text{if } u_{j,n}^{a,b} > 0; \\ S_{j,n}^{a_0, b_0}, & \text{otherwise,} \end{cases} \quad (5.12)$$

where $a_0 = 1 + 10^{-10}$, $b_0 = 0$, and

$$u_{j,n}^{a,b} = S_{j,n}^{a_0, b_0} + \frac{(a - 1 - b - 9 \times 10^{-11})}{\log(j + 1)}. \quad (5.13)$$

As before, it is easily seen that $S_{j,n}^{a_0, b_0}$ is decreasing in j . Also note that $-b$ in (5.13) takes account of the fact that the partial sums are increasing in b , thus favouring divergence for increasing b .

Setting aside panel (c) of Figure 5, observe that the remaining panels of Figures 4 and 5 are in agreement with the results of (Bourchtein et al., 2012), but in the case $a = b = 1$, the tests of (Bourchtein et al., 2012) turned out to be inconclusive. Panel (c) of Figure 5 demonstrates that the series is divergent for $a = b = 1$.

6 Application to Riemann Hypothesis

6.1 Brief background

Consider the Riemann zeta function given by

$$\zeta(a) = \frac{1}{1 - 2^{1-a}} \sum_{n=0}^{\infty} \frac{1}{2^{n+1}} \sum_{k=0}^n (-1)^k \frac{n!}{k!(n-k)!} (k+1)^{-a}, \quad (6.1)$$

where a is complex. The above function is formed by first considering Euler's function

$$Z(a) = \sum_{n=1}^{\infty} \frac{1}{n^a}, \quad (6.2)$$

then by multiplying both sides of (6.2) by $(1 - \frac{2}{2^a})$ to obtain

$$\left(1 - \frac{2}{2^a}\right) Z(a) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^a}, \quad (6.3)$$

and then dividing the right hand side of (6.3) by $(1 - \frac{2}{2^a})$. The advantage of the function $\zeta(a)$ in comparison with the parent function $Z(a)$ is that, $Z(a)$ is divergent if the real part of a , which we denote by $Re(a)$, is less than or equal to 1, while $\zeta(a)$ is convergent for all a with $Re(a) > 0$. Importantly, $\zeta(a) = Z(a)$ whenever $Z(a)$ is convergent.

Whenever $0 < Re(a) < 1$, $\zeta(a)$ satisfies the following identity:

$$\zeta(a) = 2^a \pi^{a-1} \sin\left(\frac{\pi a}{2}\right) \Gamma(1-a) \zeta(1-a), \quad (6.4)$$

where $\Gamma(\cdot)$ is the gamma function. This can be extended to the set of complex numbers by defining a function with non-positive real part by the right hand side of (6.4); abusing notation, we denote the new function by $\zeta(a)$. Because of the sine function, it follows that the trivial zeros of the above function occur when the values of a are negative even integers. Hence, the non-trivial zeros must satisfy $0 < Re(a) < 1$.

(Riemann, 1859) conjectured that all the non-trivial zeros have the real part $1/2$, which is the famous Riemann Hypothesis. For accessible account of the Riemann Hypothesis, see (Borwein, Choi, Rooney and Weirathmueller, 2006), (Derbyshire, 2004).

One equivalent condition for the Riemann Hypothesis is related to sums of the Möbius function, given by

$$\mu(n) = \begin{cases} -1 & \text{if } n \text{ is a square-free positive integer with an odd number of prime factors;} \\ 0 & \text{if } n \text{ has a squared prime factor;} \\ 1 & \text{if } n \text{ is a square-free positive integer with an even number of prime factors,} \end{cases} \quad (6.5)$$

where, by square-free integer we mean that the integer is not divisible by any perfect square other than 1. Specifically, the condition

$$\sum_{n=1}^x \mu(n) = O\left(x^{\frac{1}{2}+\epsilon}\right) \quad (6.6)$$

for any $\epsilon > 0$, is equivalent to Riemann Hypothesis. This condition implies that the Dirichlet series for the Möbius function, given by

$$M(a) = \sum_{n=1}^{\infty} \frac{\mu(n)}{n^a} = \frac{1}{\zeta(a)}, \quad (6.7)$$

is analytic in $\text{Re}(a) > 1/2$. This again ensures that $\zeta(a)$ is meromorphic in $\text{Re}(a) > 1/2$ and that it has no zeros in this region. Using the functional equation (6.4) it follows that there are no zeros of $\zeta(a)$ in $0 < \text{Re}(a) < 1/2$ either. Hence, (6.6) implies Riemann Hypothesis. The converse is also certainly true.

The above arguments also imply that convergence of $M(a)$ in (6.7) for $\text{Re}(a) > 1/2$ is equivalent to Riemann Hypothesis, and it is this criterion that is of our interest in this paper. Now, $M(a)$ converges absolutely for $\text{Re}(a) > 1$; moreover, $M(1) = 0$. The latter is equivalent to the prime number theorem stating that the number of primes below x is asymptotically $x/\log(x)$, as $x \rightarrow \infty$ ((Landau, 1906)). Thus, $M(a)$ converges for $\text{Re}(a) \geq 1$. That $M(a)$ diverges for $\text{Re}(a) \leq 1/2$ can be seen as follows. Note that if $M(a)$ converged for any a^* such that $\text{Re}(a^*) \leq 1/2$, then analytic continuation for Dirichlet series of the form $M(a)$ would guarantee convergence of $M(a)$ for all a with $\text{Re}(a) > \text{Re}(a^*)$. But $\zeta(a)$ is not analytic on $0 < \text{Re}(a) < 1$ because of its non-trivial zeros on the strip. This would contradict the analytic continuation leading to the identity $M(a) = 1/\zeta(a)$ on the entire set of complex numbers. Hence, $M(a)$ must be divergent for $\text{Re}(a) \leq 1/2$.

In this paper, we apply our ideas to particularly investigate convergence of $M(a)$ when $1/2 < a < 1$.

6.2 Choice of the upper bound and implementation details

To form an idea of the upper bound we first plot the partial sums $S_{j,n}^a$, for $j = 1000$ and $n = 10^6$, with respect to a . In this regard, panel (a) of Figure 6 shows the decreasing nature of the partial sums with respect to a , and panel (b) magnifies the plot in the domain $1/2 < a < 1$ that we are particularly interested in. The latter shows that the partial sums decrease sharply till about 0.7, getting appreciably close to zero around that point, after which the rate of decrease diminishes. Thus, one may expect a change point around 0.7 regarding convergence. Specifically, divergence may be expected below a point slightly larger than 0.7 and convergence above it.

Since $M(1) < \infty$, we consider this series as the basis for our upper bound, with the value of a also taken into account. Specifically, we choose the upper bound as

$$c_{j,n} = \left| S_{j,n}^1 + \frac{a}{j+1} \right|. \quad (6.8)$$

Since Figure 6 shows that the partial sums are of monotonically decreasing nature, the above choice of upper bound facilitates detection of convergence for relatively large values of a . The part $\frac{a}{j+1}$, which tends to zero as $j \rightarrow \infty$, takes care of the fact that the series may be convergent if $a < 1$, by slightly inflating $S_{j,n}^1$.

For our purpose, we compute the first 10^9 values of the Möbius function using an efficient algorithm proposed in (Lioen and van de Lune, 1994), which is based on the Sieve of Eratosthenes ((Horsley, 1772)). We set $K = 1000$ and $n = 10^6$. A complete analysis with our VMware with our parallel implementation takes about 2 minutes.

6.3 Results of our Bayesian analysis

Panels (a)–(e) of Figure 7 and panels (d)–(f) of Figure 8 show that $M(a)$ diverges for $a = 0.1, 0.2, 0.3, 0.4, 0.5$, but converges for $a = 1 + 10^{-10}, 2$ and 3 . In fact, for many other values that we experimented with, $M(a)$ converged for $a > 1$ and diverged for $a < 1/2$, demonstrating remarkable consistency with the known, existing results.

Certainly far more important are the results for $1/2 < a < 1$. Indeed, panel (f) of Figure 7 and panels (a)–(c) of Figure 8 show that $M(a)$ diverged for $a = 0.6$ and 0.7 and converged for $a = 0.8$ and 0.9 . It thus appears that $M(a)$ diverges for $a < a^*$ and converges for $a \geq a^*$, for some $a^* \in (0.7, 0.8)$. Figure 9 displays results of our further experiments in this regard. Panels (a) and (b) of Figure 9 show the posterior means for the full set of iterations and the last 500 iterations, respectively, for $a = 0.71$. Note that from panel (a), convergence seems to be attained, although towards the end, the plot seems to be slightly tilted downwards. Panel (b) magnifies this, clearly showing divergence. Panels (c) and (d) of Figure 9 depict similar phenomenon for $a = 0.715$, but as per panel (d), divergence seems to ensue all of a sudden, even after showing signs of convergence for the major number of iterative stages. Convergence of $M(a)$ begins at $a = 0.72$ (approximately); panels (e) and (f) of Figure 9 take clear note of this.

Thus, as per our methods, $M(a)$ diverges for $a < 0.72$ and converges for $a \geq 0.72$. This is remarkably in keeping with the wisdom gained from panel (b) of Figure 6 that convergence is expected to occur for values of a exceeding 0.7 . Note that neither the upper bound (6.8), nor our methodology, is in any way biased towards $a \approx 0.7$; hence, our result is perhaps not implausible.

6.4 Implications of our result

As per our results, $M(a)$ does not converge for all $a > 1/2$, and hence does not completely support Riemann Hypothesis. However, convergence of $M(a)$ fails only for the relatively small region $0.5 < a < 0.72$, which perhaps is the reason why there exists much evidence in favour of Riemann Hypothesis.

7 Summary and conclusion

In this paper, we proposed and developed a novel Bayesian methodology for assessment of convergence of infinite series; we further extended the theory to enable detection of multiple or even infinite number of limit points of the underlying infinite series. Our developments do not

require any restrictive assumption, not even independence of the elements X_i of the infinite series.

We demonstrated the reliability and efficiency of our methods with varieties of examples, the most important one being associated with Riemann Hypothesis.

Both methods proposed in this paper, namely the convergence assessment method and the multiple limit points method are almost completely in agreement that the Riemann Hypothesis can not be completely supported. Indeed, both the methods agree that there exists some a^* in the neighborhood of 0.7 such that the infinite series based on the Möbius function diverges for $a < a^*$ and converges for $a \geq a^*$. The results that we obtained by our Bayesian analyses are also supported by informal plots of the partial sums depicted in Figure 6. Further support of our Riemann hypothesis results can be obtained by exploiting the characterization of Riemann hypothesis by convergence of certain infinite series based on Bernoulli numbers; the details are presented in Section S-6 of the supplement.

In fine, it is worth reminding the reader that although our work attempts to provide insights regarding Riemann hypothesis, we did not develop our Bayesian approach keeping Riemann hypothesis in mind. Indeed, our primary objective is to develop Bayesian approaches to studying convergence properties of infinite series in general. From this perspective, Riemann hypothesis is just an example where it makes sense to learn about convergence properties of a certain class of infinite series. Further development of our approach is of course in the cards. Note that the theory that we developed for deterministic series remains valid for random series as well, but since the forms of the terms of random series are unknown, direct application of our methods is not possible. We are currently developing new theories and methods for random series where the terms have unknown distributions and/or not independent. We shall carry out a detailed investigation including comparisons with existing theories on random infinite series. We then intend to extend these works to complex infinite series, both deterministic and random.

Acknowledgment

We thank Arun Kumar Kuchibhotla and Debapratim Banerjee for their very useful feedback on the first draft of this paper.

Example 1: Divergence

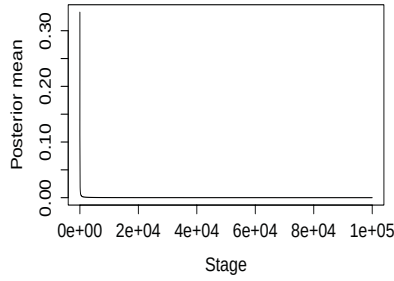


Figure 1: Example 1: The series (5.1) is divergent.

Supplementary Material

S-1 Proof of Lemma 5.1

Since each term of the series (1) is decreasing in a , it is clear that $S_{j,n}^{a,b}$ is decreasing in a . We need to show that $S_{j,n}^{a,b}$ is increasing in b .

Let, for $i \geq 3$,

$$g(i) = \left(1 - \frac{\log i}{i} - \frac{\log \log i}{i} \left\{ \cos^2 \left(\frac{1}{i} \right) \right\} (a + (-1)^i b) \right)^i. \quad (\text{S-1.1})$$

Observe that all our partial sums of the form $S_{j,n}^{a,b}$ for $j \geq 3$ admit the form

$$S_{j,n}^{a,b} = \sum_{i=r}^{r+n-1} g(i), \quad (\text{S-1.2})$$

where $r = 3 + n(j-1)$, which is clearly odd because n is even. Now,

$$\sum_{i=r}^{r+n-1} g(i) = \{g(r) + g(r+1)\} + \{g(r+2) + g(r+3)\} + \cdots + \{g(r+n-2) + g(r+n-1)\}, \quad (\text{S-1.3})$$

where the sums of the consecutive terms within the parentheses have the form

$$\begin{aligned} & g(r+\ell) + g(r+\ell+1) \\ &= \left(1 - \frac{\log(r+\ell)}{r+\ell} - \frac{\log \log(r+\ell)}{r+\ell} \left\{ \cos^2 \left(\frac{1}{r+\ell} \right) \right\} (a + (-1)^{(r+\ell)} b) \right)^{(r+\ell)} \\ & \quad + \left(1 - \frac{\log(r+\ell+1)}{r+\ell+1} - \frac{\log \log(r+\ell+1)}{r+\ell+1} \left\{ \cos^2 \left(\frac{1}{r+\ell+1} \right) \right\} (a + (-1)^{(r+\ell+1)} b) \right)^{(r+\ell+1)}. \end{aligned} \quad (\text{S-1.4})$$

Since r is odd, and since the terms are represented pairwise in (S-1.3) it follows that in (S-1.4), $r+\ell$ is odd and $r+\ell+1$ is even. That is, in (S-1.4), $a + (-1)^{(r+\ell)} b = a - b$ and $a + (-1)^{(r+\ell+1)} b = a + b$. Since $\cos^2(\theta)$ is decreasing on $[0, \frac{\pi}{2}]$, and since $\frac{1}{i} \leq \frac{\pi}{2}$ for $i \geq 3$,

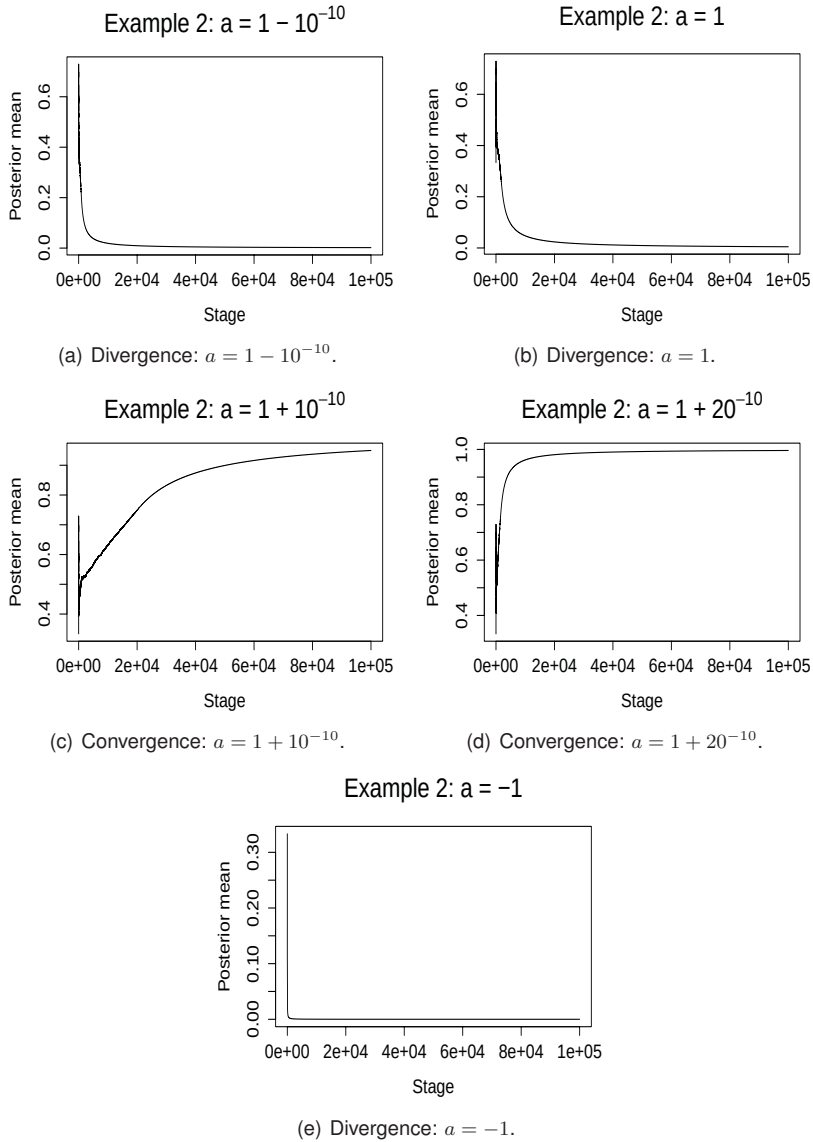


Figure 2: Example 2: The series (5.2) converges for $a > 1$ and diverges for $a \leq 1$.

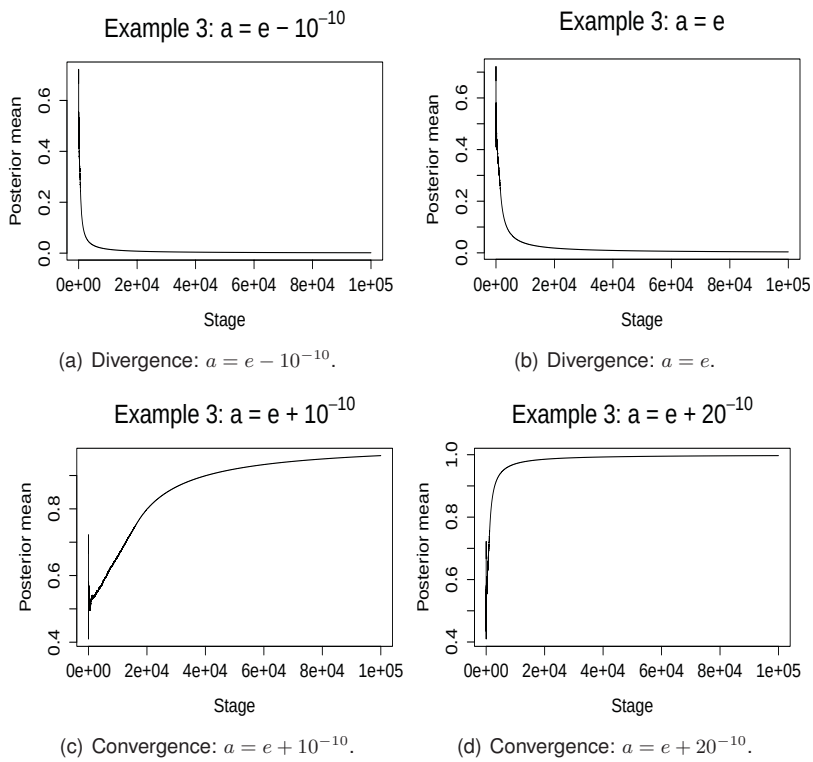


Figure 3: Example 3: The series (5.6) converges for $a > e$ and diverges for $a \leq e$.

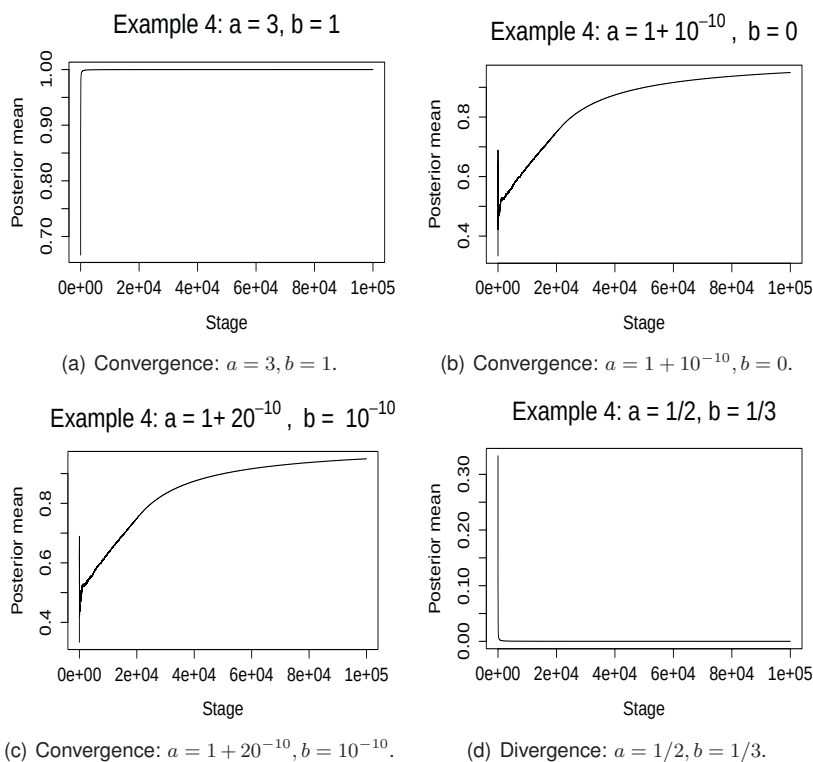


Figure 4: Example 4: The series (5.8) converges for $(a = 3, b = 1)$, $(a = 1 + 10^{-10}, b = 0)$, $(a = 1 + 20^{-10}, b = 10^{-10})$ and diverges for $(a = 1/2, b = 1/3)$.

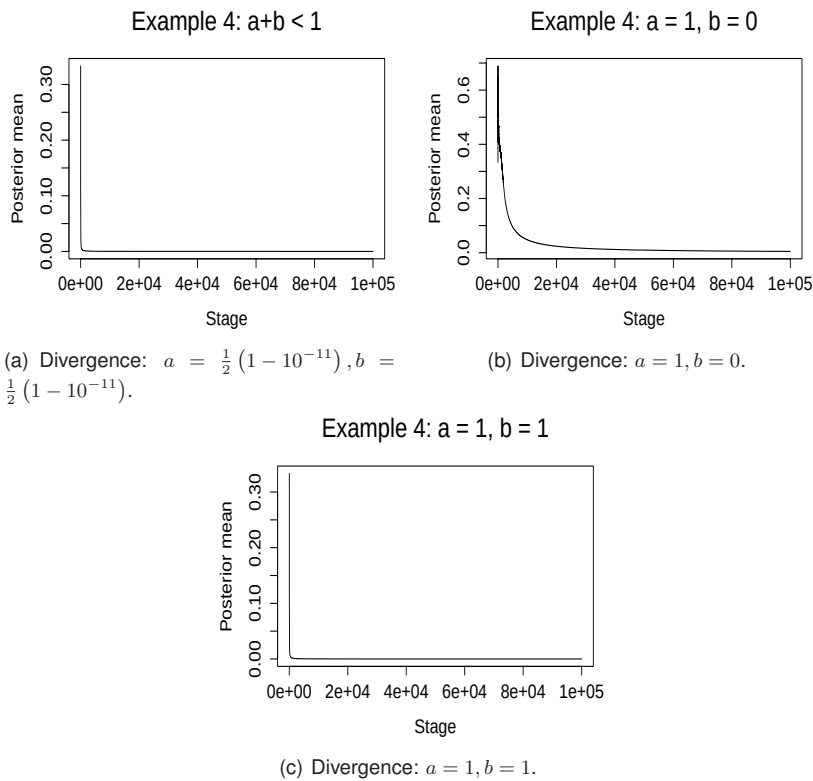


Figure 5: Example 4: The series (5.8) diverges for $(a = \frac{1}{2}(1 - 10^{-11}), b = \frac{1}{2}(1 - 10^{-11}))$, $(a = 1, b = 0)$ and $(a = 1, b = 1)$.

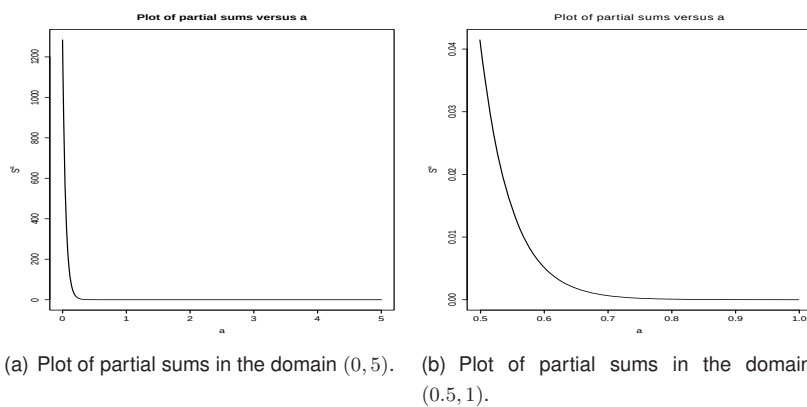


Figure 6: Plot of the partial sums $S_{1000,1000000}^a$ versus a . Panel (a) shows the plot in the domain $[0, 5]$ while panel (b) magnifies the same in the domain $(0.5, 1)$.

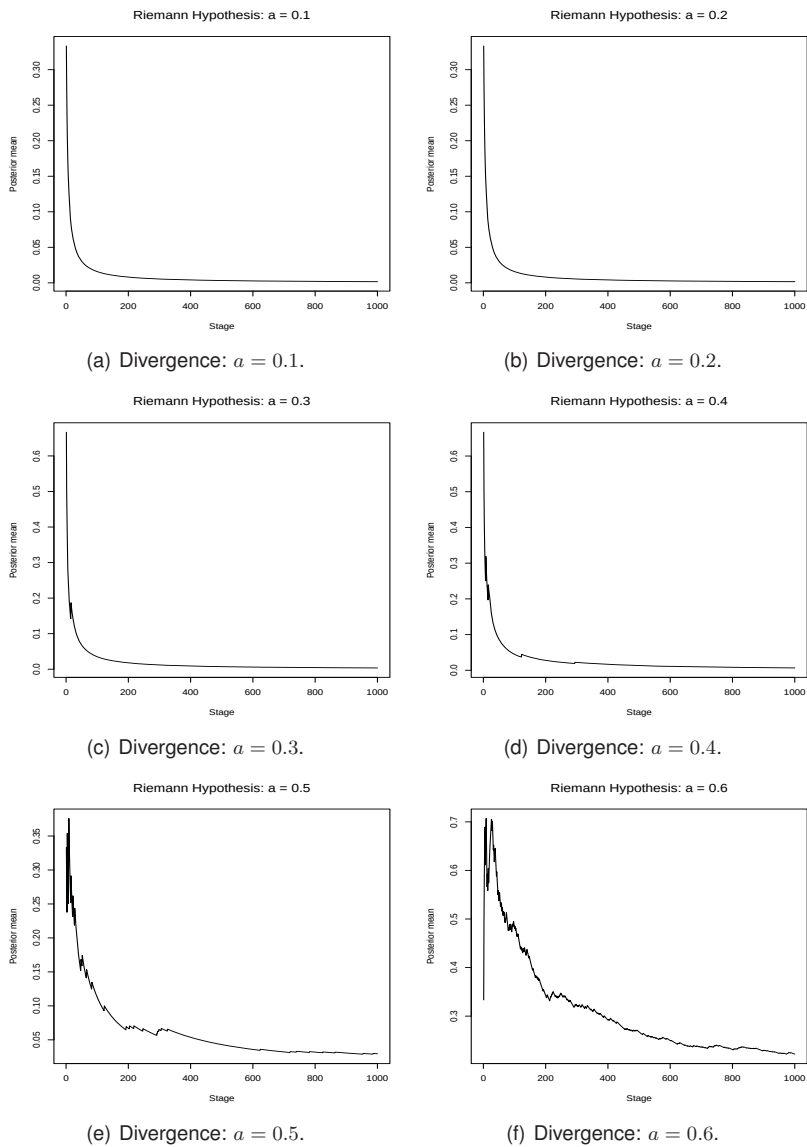


Figure 7: Riemann Hypothesis: The Möbius function based series diverges for $a = 0.1, 0.2, 0.3, 0.4, 0.5, 0.6$.

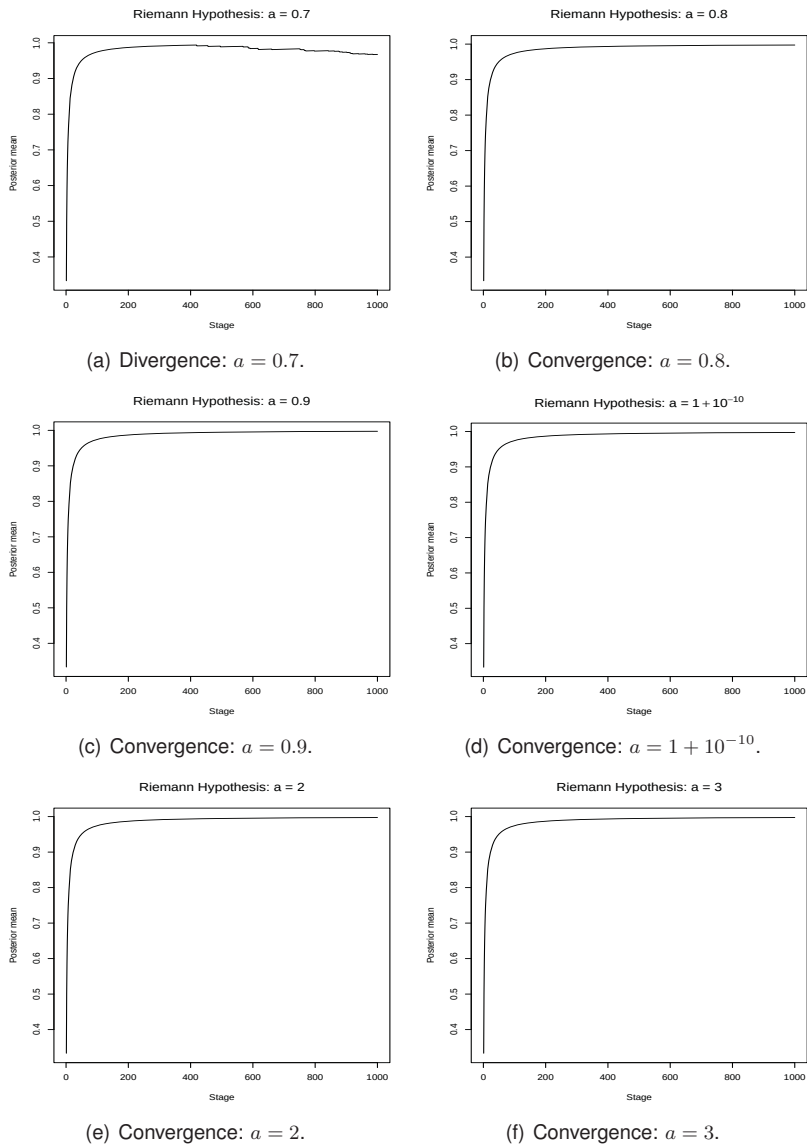


Figure 8: Riemann Hypothesis: The Möbius function based series diverges for $a = 0.7$ but converges for $a = 0.8, 0.9, 1 + 10^{-10}, 2, 3$.

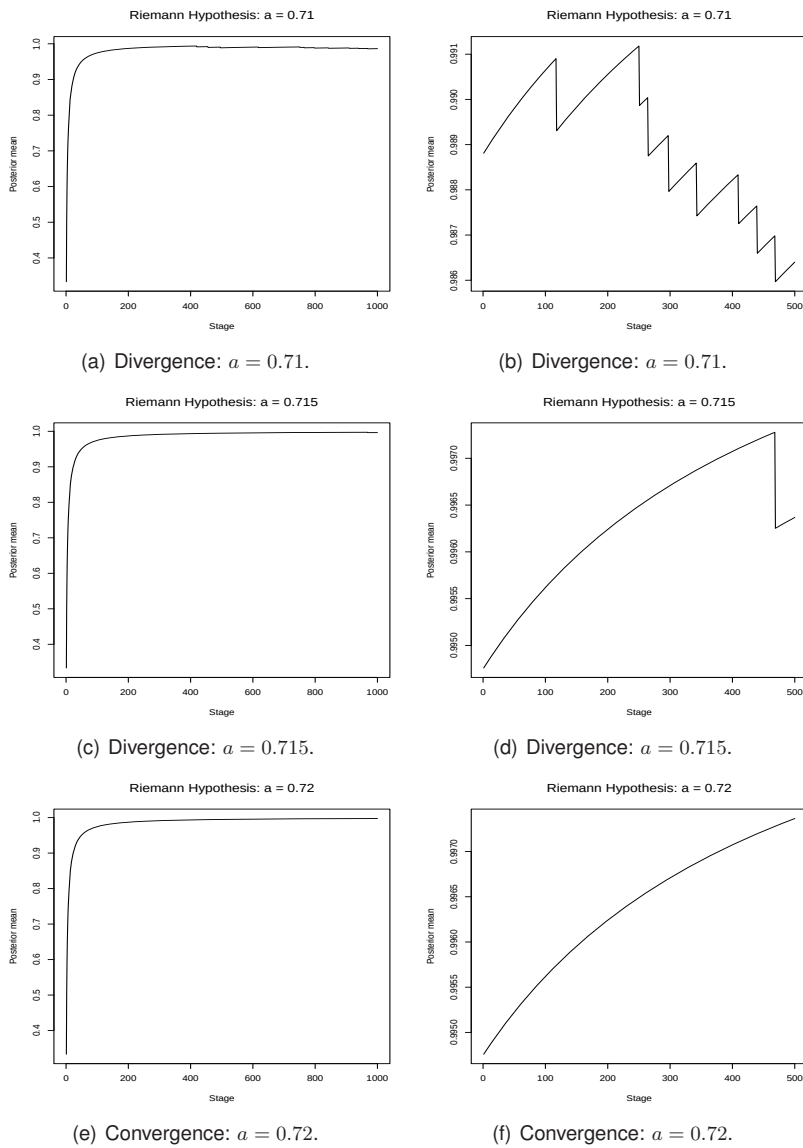


Figure 9: Riemann Hypothesis: The left panels show the posterior means for the full set of iterations, while the right panels depict the posterior means for the last 500 iterations, for $a = 0.71$, 0.715 and 0.72 . It is evident that the Möbius function based series diverges for $a = 0.71$ and 0.715 but converges for $a = 0.72$.

it follows that $\cos^2\left(\frac{1}{i}\right)$ is increasing in i . Moreover, $\frac{\log \log i}{i}$ decreases in i at a rate faster than $\cos^2\left(\frac{1}{i}\right)$ increases, so that $\frac{\log \log i}{i} \times \cos^2\left(\frac{1}{i}\right)$ decreases in i . It follows that

$$\frac{\log \log(r+\ell)}{r+\ell} \cos^2\left(\frac{1}{r+\ell}\right) > \frac{\log \log(r+\ell+1)}{r+\ell+1} \cos^2\left(\frac{1}{r+\ell+1}\right). \quad (\text{S-1.5})$$

Note that in $g(r+\ell)+g(r+\ell+1)$, $\frac{\log \log(r+\ell)}{r+\ell} \cos^2\left(\frac{1}{r+\ell}\right)$ is associated with $-b$ while $\frac{\log \log(r+\ell+1)}{r+\ell+1} \cos^2\left(\frac{1}{r+\ell+1}\right)$ involves b . Hence, increasing b increases $g(r+\ell)$ but decreases $g(r+\ell+1)$, and because of (S-1.5), $g(r+\ell)+g(r+\ell+1)$ increases in b . This ensures that $\sum_{i=r}^{r+n-1} g(i)$, given by (S-1.3), is increasing in b . In other words, partial sums of the form (S-1.2) are increasing in b , proving Lemma 5.1 when n is even.

S-2 Further examples on detection of series convergence and divergence using our Bayesian method

S-2.1 Example 5

Now consider the following series presented and analysed in (Bourchtein et al., 2012):

$$S = \sum_{i=3}^{\infty} \left(1 - \left(\frac{\log(i)}{i} \right) \left(a \left(1 + \sin^2 \left(\sqrt{\left(\frac{\log(\log(i))}{\log(i)} \right)} \right) \right) + b \sin \left(\frac{i\pi}{4} \right) \right) \right)^i; \quad a > 0, b > 0. \quad (\text{S-2.1})$$

(Bourchtein et al., 2012) show that the series converges when $a - b > 1$ and diverges when $a + b < 1$. Again, as in the case of Example 4, the following lemma holds in Example 5. Note that for mathematical convenience we consider partial sums from the 5-th term onwards. We also assume n to be a multiple of 4.

Lemma S-2.1. *For the series (S-2.1), let*

$$S_{j,n}^{a,b} = \sum_{i=5+n(j-1)}^{5+nj-1} \left(1 - \left(\frac{\log(i)}{i} \right) \left(a \left(1 + \sin^2 \left(\sqrt{\left(\frac{\log(\log(i))}{\log(i)} \right)} \right) \right) + b \sin \left(\frac{i\pi}{4} \right) \right) \right)^i, \quad (\text{S-2.2})$$

for $j \geq 1$ and n , a multiple of 4. Then $S_{j,n}^{a,b}$ is decreasing in a and increasing in b .

Proof. That $S_{j,n}^{a,b}$ is decreasing in a follows trivially since each term of (S-2.1) is decreasing in a . We need to show that $S_{j,n}^{a,b}$ is increasing in b .

Let, for $i \geq 5$,

$$g(i) = \left(1 - \left(\frac{\log(i)}{i} \right) \left(a \left(1 + \sin^2 \left(\sqrt{\left(\frac{\log(\log(i))}{\log(i)} \right)} \right) \right) + b \sin \left(\frac{i\pi}{4} \right) \right) \right)^i. \quad (\text{S-2.3})$$

Now note that, with $r = 5 + n(j-1)$,

$$\begin{aligned} \sum_{i=r}^{r+n-1} g(i) &= \sum_{m=1}^{\frac{n}{4}} Z_{r,m} \\ &= \{Z_{r,1} + Z_{r,2}\} + \{Z_{r,3} + Z_{r,4}\} + \cdots + \left\{Z_{r,\frac{n}{4}-1} + Z_{r,\frac{n}{4}}\right\}, \end{aligned} \quad (\text{S-2.4})$$

where

$$Z_{r,m} = \sum_{\ell=5+4(m-1)}^{5+4(m-1)+3} g(r+\ell). \quad (\text{S-2.5})$$

Now, for any $\ell \geq 1$, observe that in $\{Z_{r,\ell} + Z_{r,\ell+1}\}$, the term $Z_{r,\ell}$ consists of only negative signs of the sine-values, while in $Z_{r,\ell+1}$ the corresponding signs are positive, although the magnitudes are the same. Since $\log(i)/i$ is decreasing in i , it follows that $\{Z_{r,\ell} + Z_{r,\ell+1}\}$ is increasing in b for $\ell \geq 1$. Hence, it follows that (S-2.4), and $S_{j,n}^{a,b}$, defined by (S-2.2), are increasing in b for $j \geq 1$ and n , a multiple of 4, proving Lemma S-2.1. $\square$

The following corollary with respect to $S^{a,b}$ again holds:

Corollary S-2.2. $S^{a,b}$ is decreasing in a and increasing in b .

Thus, we follow the same method as in Example 4 to determine $c_{j,n}^{a,b}$, but we need to note that in this example $a > 0$ and $b > 0$ instead of $a \geq 0$ and $b \geq 0$ of Example 4. Consequently, here we define $b \geq \epsilon$, for $\epsilon > 0$, the set A_ϵ given by $A_\epsilon = \{a : 0 \leq a \leq 1\} \cup \{a : a \geq 1 + \epsilon\}$ and

$$\tilde{S} = \inf_{a \in A_\epsilon} \sup_{b \geq \epsilon} \{S^{a,b} : a - b > 1\}. \quad (\text{S-2.6})$$

In this case, Corollary S-2.2 and the convergence criterion $a - b > 1$ ensure that $\tilde{S}$ is attained at $a_0 = 1 + \epsilon$ and $b_0 = \epsilon$. As before, we set $\epsilon = 10^{-10}$. The rest of the arguments leading to the choice of $c_{j,n}^{a,b}$ remains the same as in Example 4, and hence in this example $c_{j,n}^{a,b}$ has the form

$$c_{j,n}^{a,b} = \begin{cases} u_{j,n}^{a,b}, & \text{if } u_{j,n}^{a,b} > 0; \\ S_{j,n}^{a_0,b_0}, & \text{otherwise,} \end{cases} \quad (\text{S-2.7})$$

with $a_0 = 1 + 10^{-10}$, $b_0 = 10^{-10}$, where $S_{j,n}^{a_0,b_0}$ is decreasing in j as before.

Figure S-1 depicts the results of our Bayesian analysis of the series (S-2.1) for various values of a and b . All the results are in accordance with those of (Bourchtein et al., 2012).

S-2.2 Example 6

We now investigate whether or not the following series converges:

$$S = \sum_{i=1}^{\infty} \frac{1}{i^3 |\sin i|}. \quad (\text{S-2.8})$$

This series is a special case of the generalized form of the Flint Hills series (see (Pickover, 2002) and (Alekseyev, 2011)).

For our purpose, we first embed the above series into

$$S^{a,b} = \sum_{i=1}^{\infty} \frac{i^{b-3}}{a + |\sin i|}, \quad (\text{S-2.9})$$

where $b \in \mathbb{R}$ and $|a| \leq \eta$, for some $\eta > 0$, specified according to our purpose. Note that, $S = S^{0,0}$, and we set $\eta = 10^{-10}$ for our investigation of (S-2.8).

Note that for any fixed $a \neq 0$, $S^{a,b}$ converges if $b < 2$ and diverges if $b \geq 2$. Since $S^{a,b}$ increases in b it follows that the equality in

$$\tilde{S} = \sup \left\{ S^{a,b} : a = \epsilon, b \leq 2 - \epsilon \right\} \quad (\text{S-2.10})$$

is attained at $(a_0, b_0) = (\epsilon, 2 - \epsilon)$.

Arguments in keeping with those in the previous examples lead to the following choice of the upper bound for $S_{j,n}^{a,b}$, which we again denote by $c_{j,n}^{a,b}$:

$$c_{j,n}^{a,b} = \begin{cases} u_{j,n}^{a,b} & \text{if } b < 2; \\ v_{j,n}^{a,b} & \text{otherwise,} \end{cases} \quad (\text{S-2.11})$$

where

$$u_{j,n}^{a,b} = S_{j,n}^{a_0,b_0} + \frac{(|a| - b + 2 - 2\epsilon + 10^{-5})}{\log(j+1)}; \quad (\text{S-2.12})$$

$$v_{j,n}^{a,b} = S_{j,n}^{a_0,b_0} + \frac{(|a| - b + 2 - 2\epsilon - 10^{-5})}{\log(j+1)}. \quad (\text{S-2.13})$$

It can be easily verified that the upper bound is decreasing in j . Notice that we add the term 10^{-5} when $b < 2$ so that our Bayesian method favours convergence and subtract the same when $b \geq 2$ to facilitate detection of divergence. Since convergence or divergence of $S^{a,b}$ does not depend upon $a \in [-\eta, \eta] \setminus \{0\}$, we use $|a|$ in (S-2.12) and (S-2.13).

Setting $\epsilon = 10^{-10}$, Figures S-2 and S-3 depict convergence and divergence of $S^{a,b}$ for various values of a and b . In particular, panel (e) of Figure S-3 shows that our main interest, the series S , given by (S-2.8), converges.

S-2.3 Example 7

We now consider

$$S = \sum_{i=1}^{\infty} \frac{|\sin i| i^i}{i}. \quad (\text{S-2.14})$$

We embed this series into

$$S^{a,b} = \sum_{i=1}^{\infty} \frac{|\sin a\pi i| i^i}{i^b}, \quad (\text{S-2.15})$$

where $a \in \mathbb{R}$ and $b \geq 1$. The above series converges if $b > 1$, for all $a \in \mathbb{R}$. But for $b = 1$, it is easy to see that the series diverges if $a = \ell/2m$, where ℓ and m are odd integers.

Letting $a_0 = \pi^{-1}$ and $b_0 = 1 + \epsilon$, with $\epsilon = 10^{-10}$, we set the following upper bound that is decreasing in j :

$$c_{j,n}^{a,b} = S_{j,n}^{a_0,b_0} + \frac{\epsilon}{j}. \quad (\text{S-2.16})$$

Thus, $c_{j,n}^{a,b}$ corresponds to a convergent series which is also sufficiently close to divergence. Addition of the term $\frac{\epsilon}{j}$ provides further protection from erroneous conclusions regarding divergence.

Panel(a) of Figure S-4 demonstrates that the series of our interest, given by (S-2.14), diverges. Panel (b) confirms that for $a = 5/(2 \times 7)$ and $b = 1$, the series indeed diverges, as it should.

S-3 Oscillatory series with multiple limit points

In this section we assume that the sequence $\{S_{1,n}\}_{n=1}^{\infty}$ has multiple limit points, including the possibility that the number of limit points is countably infinite.

S-3.1 Finite number of limit points

Let us assume that there are $M (> 1)$ limit points of the sequence $\{S_{1,n}\}_{n=1}^{\infty}$. Then there exist sequences $\{c_{m,j}\}_{j=1}^{\infty}$; $m = 0, \dots, M$, such that $\{(c_{m-1,j}, c_{m,j}]; m = 1, \dots, M\}$ partition the real line $\mathbb{R}$ for every $j \geq 1$ and that there exists $j_0 \geq 1$ such that for all $j \geq j_0$, the interval $(c_{m-1,j}, c_{m,j}]$ contains at most one limit point of the sequence $\{S_{1,n}\}_{n=1}^{\infty}$, for every $m = 1, \dots, M$. With these sequences we define

$$Y_j = m \text{ if } c_{m-1,j} < S_{1,j} \leq c_{m,j}; m = 1, 2, \dots, M, \quad (\text{S-3.1})$$

Recall that in Section 4 of our main manuscript we allowed the sequence $\{c_j\}_{j=1}^{\infty}$ to depend upon the underlying series $S_{1,\infty}$. Likewise, here also we allow the quantities $c_{0,j}, c_{1,j}, \dots, c_{M,j}$ to depend upon $S_{1,\infty}$. In other words, for $\omega \in \mathfrak{S}$, for $m = 0, 1, 2, \dots, M$, and $j = 1, 2, 3, \dots$, $c_{m,j} = c_{m,j}(\omega)$ corresponds to $S_{1,\infty}(\omega)$.

Note that unlike our ideas appropriate for non-oscillating series, here do not consider blocks of partial sums, $S_{j,n_j} = \sum_{i=\sum_{k=0}^{j-1} n_k+1}^{\sum_{k=0}^j n_k} X_i$, but $S_{1j} = \sum_{i=1}^j X_i$. In other words, for Bayesian analysis of non-oscillating series we compute sums of n_j terms in each iteration, whereas for oscillating series we keep adding a single term at every iteration. Thus, computationally, the latter is a lot simpler.

We assume that

$$(\mathbb{I}(Y_j = 1), \dots, \mathbb{I}(Y_j = M)) \sim \text{Multinomial}(1, p_{1,j}, \dots, p_{M,j}), \quad (\text{S-3.2})$$

where $p_{m,j}$ can be interpreted as the probability that $S_{1,j} \in (c_{m-1,j}, c_{m,j}]$. As $j \rightarrow \infty$ it is expected that $c_{m-1,j}$ and $c_{m,j}$ will converge to appropriate constants depending upon m , and that $p_{m,j}$ will tend to the correct proportion of the limit point indexed by m . Indeed, let $\{p_{m,0}; m = 1, \dots, M\}$ denote the actual proportions of the limit points indexed by $\{1, \dots, M\}$, as $j \rightarrow \infty$.

Following the same principle discussed in Section 3 of our main manuscript, and extending the Beta prior to the Dirichlet prior, at the k -th stage we arrive at the following posterior of $\{p_{m,k}; m = 1, \dots, M\}$:

$$\pi(p_{1,k}, \dots, p_{M,k} | y_k) \equiv \text{Dirichlet} \left(\sum_{j=1}^k \frac{1}{j^2} + \sum_{j=1}^k \mathbb{I}(y_j = 1), \dots, \sum_{j=1}^k \frac{1}{j^2} + \sum_{j=1}^k \mathbb{I}(y_j = M) \right). \quad (\text{S-3.3})$$

The posterior mean and posterior variance of $p_{m,k}$, for $m = 1, \dots, M$, are given by:

$$E(p_{m,k}|y_k) = \frac{\sum_{j=1}^k \frac{1}{j^2} + \sum_{j=1}^k \mathbb{I}(y_j = m)}{M \sum_{j=1}^k \frac{1}{j^2} + k}; \quad (\text{S-3.4})$$

$$Var(p_{m,k}|y_k) = \frac{\left(\sum_{j=1}^k \frac{1}{j^2} + \sum_{j=1}^k \mathbb{I}(y_j = m)\right) \left((M-1) \sum_{j=1}^k \frac{1}{j^2} + k - \sum_{j=1}^k \mathbb{I}(y_j = m)\right)}{\left(M \sum_{j=1}^k \frac{1}{j^2} + k\right)^2 \left(M \sum_{j=1}^k \frac{1}{j^2} + k + 1\right)}. \quad (\text{S-3.5})$$

Let $k = M\tilde{k}$, where $\tilde{k} \rightarrow \infty$. Then, from (S-3.4) and (S-3.5) it is easily seen, using $\frac{\sum_{j=1}^k \mathbb{I}(y_j(\omega)=m)}{k} \rightarrow p_{m,0}$ as $k \rightarrow \infty$, that,

$$E(p_{m,k}|y_k) \rightarrow p_{m,0}, \text{ and} \quad (\text{S-3.6})$$

$$Var(p_{m,k}|y_k) = O\left(\frac{1}{k}\right) \rightarrow 0, \quad (\text{S-3.7})$$

as $k \rightarrow \infty$.

We can now characterize the m limit points of $S_{1,\infty}(\omega)$ in terms of the limits of the marginal posterior probabilities of $p_{m,k}$, denoted by $\pi_m(\cdot|y_k(\omega))$, as $k \rightarrow \infty$.

Theorem S-3.1. For $\omega \in \mathfrak{S} \cap \mathfrak{N}^c$, where $\mathfrak{N}$ has zero probability measure, $\{S_{1,n}(\omega)\}_{n=1}^\infty$ has $M (> 1)$ limit points almost surely if and only if

- (1) There exist sequences $\{c_{m,j}(\omega)\}_{j=1}^\infty$; $m = 0, \dots, M$, such that $(c_{m-1,j}(\omega), c_{m,j}(\omega))$ partition the real line $\mathbb{R}$ for every $j \geq 1$ and $m = 1, \dots, M$.
- (2) There exists $j_0(\omega) \geq 1$ such that for all $j \geq j_0(\omega)$, for $m = 1, \dots, M$, $(c_{m-1,j}(\omega), c_{m,j}(\omega))$ contains at most one limit point of $\{S_{1,n}(\omega)\}_{n=1}^\infty$.
- (3) With Y_j defined as in (S-3.1),

$$\pi_m(\mathcal{N}_{p_{m,0}}|y_k(\omega)) \rightarrow 1, \quad (\text{S-3.8})$$

as $k \rightarrow \infty$. In the above, $\mathcal{N}_{p_{m,0}}$ is any neighborhood of $p_{m,0}$, with $p_{m,0}$ satisfying $0 < p_{m,0} < 1$ for $m = 1, \dots, M$ such that $\sum_{m=1}^M p_{m,0} = 1$.

Proof. For $\omega \in \mathfrak{S} \cap \mathfrak{N}^c$, where $\mathfrak{N}$ has zero probability measure, let $S_{1,\infty}(\omega)$ be oscillatory with M limit points having proportions $\{p_{m,0}; m = 1, \dots, M\}$. Conditions (1) and (2) then clearly hold. Then with our definition of Y_j provided in (S-3.1), the results (S-3.6) and (S-3.7) hold with $k = M\tilde{k}$, where $\tilde{k} \rightarrow \infty$. Now let $\mathcal{N}_{p_{m,0}}$ be any neighborhood of $p_{m,0}$. Let $\epsilon > 0$ be sufficiently small so that $\mathcal{N}_{p_{m,0}} \supseteq \{|p_{m,k} - p_{m,0}| < \epsilon\}$. Then by Chebychev's inequality, using (S-3.6) and (S-3.7), it is seen that $\pi_m(\mathcal{N}_{p_{m,0}}|y_k(\omega)) \rightarrow 1$, as $k \rightarrow \infty$. Thus, (S-3.8) holds. In fact, more generally, condition (3) holds.

Now assume that conditions (1), (2), (3) hold. Then $\pi_m(|p_{m,k} - p_{m,0}| < \epsilon|y_k(\omega)) \rightarrow 1$, as $k \rightarrow \infty$. Combining this with Chebychev's inequality it follows that (S-3.6) and (S-3.7) hold with $0 < p_{m,0} < 1$ for $m = 1, \dots, M$ such that $\sum_{m=1}^M p_{m,0} = 1$. If $\{S_{1,n}(\omega)\}_{n=1}^\infty$ has less than M limit points, then at least one $p_{m,0} = 0$, providing a contradiction. Hence $\{S_{1,n}(\omega)\}_{n=1}^\infty$ must have M limit points. $\square$

S-3.2 Choice of $c_{0,j}, \dots, c_{M,j}$ for a given series

Let us define, for $j = 1, 2, \dots, k$,

$$\tilde{p}_{\ell,j} = \begin{cases} 0 & \text{if } \ell = 0; \\ E(p_{\ell,j}|y_j) & \text{if } \ell = 1, 2, \dots, M. \end{cases} \quad (\text{S-3.9})$$

We also define, for $\ell = 1, 2, \dots, M$,

$$\tilde{p}_{\ell,0} = E(p_{\ell,1}), \quad (\text{S-3.10})$$

the prior mean at the first stage, before observing any data.

We then set $c_{0,j} \equiv 0$ for all $j = 1, 2, \dots, k$, and, for $m \geq 1$, define

$$c_{m,j} = \log \left[\frac{(\sum_{\ell=1}^m \tilde{p}_{\ell,j-1})^{1/\rho(\theta)}}{1 - (\sum_{\ell=1}^m \tilde{p}_{\ell,j-1})^{1/\rho(\theta)}} \right], \quad (\text{S-3.11})$$

for $j = 1, 2, \dots, k$. Thus, the inequality $c_{m-1,j} < S_{1,j} \leq c_{m,j}$ in (S-3.1) is equivalent to

$$\sum_{\ell=1}^{m-1} \tilde{p}_{\ell,k} < \left(\frac{\exp(S_{1,j})}{1 + \exp(S_{1,j})} \right)^{\rho(\theta)} \leq \sum_{\ell=1}^m \tilde{p}_{\ell,k}, \quad (\text{S-3.12})$$

where $\rho(\theta)$ is some relevant power depending upon the set of parameters θ of the given series, responsible for appropriately inflating or contracting the quantity $\frac{\exp(S_{1,j})}{1 + \exp(S_{1,j})}$ for properly diagnosing the limit points. Thus, given the series $S_{1,\infty}(\omega)$, $\theta = \theta(\omega)$ is allowed to depend upon the underlying series. If $\left(\frac{\exp(S_{1,j})}{1 + \exp(S_{1,j})} \right)^{\rho(\theta)} \geq 1$, we set $Y_j = M$. By (S-3.8), for large k , $\tilde{p}_{\ell,k}$ and $S_{1,j}$ adaptively adjust themselves so that the correct proportions of the limit points are achieved in the long run.

S-3.3 Infinite number of limit points

We now assume that the number of limit points of $\{S_{1,n}(\omega)\}_{n=1}^{\infty}$ is countably infinite, and that $\{p_{m,0}; m = 1, 2, 3, \dots\}$, where $0 \leq p_{m,0} \leq 1$ and $\sum_{m=1}^{\infty} p_{m,0} = 1$, are the true proportions of the limit points.

Now we define

$$Y_j = m \text{ if } c_{m-1,j} < S_{1,j} \leq c_{m,j}; m = 1, 2, \dots, \infty, \quad (\text{S-3.13})$$

where the sequences $\{c_{m,j}\}_{j=1}^{\infty}; m \geq 1$, are such that $(c_{m-1,j}, c_{m,j}]; m \geq 1$, partition $\mathbb{R}$ for every $j \geq 1$, and that there exists $j_0 \geq 1$ such that for all $j \geq j_0$, these intervals contain at most one limit point of $\{S_{1,n}\}_{n=1}^{\infty}$.

Let $\mathcal{X} = \{1, 2, \dots\}$ and let $\mathcal{B}(\mathcal{X})$ denote the Borel σ -field on $\mathcal{X}$ (assuming every singleton of $\mathcal{X}$ is an open set). Let $\mathcal{P}$ denote the set of probability measures on $\mathcal{X}$. Then, at the j -th stage,

$$[Y_j|P_j] \sim P_j, \quad (\text{S-3.14})$$

where $P_j \in \mathcal{P}$. We assume that P_j is the following Dirichlet process (see (Ferguson, 1973)):

$$P_j \sim DP\left(\frac{1}{j^2}G\right), \quad (\text{S-3.15})$$

where, the probability measure G is such that, for every $j \geq 1$,

$$G(Y_j = m) = \frac{1}{2^m}. \quad (\text{S-3.16})$$

It then follows using the same previous principles that, at the k -th stage, the posterior of P_k is again a Dirichlet process, given by

$$[P_k|y_k] \sim DP \left(\sum_{j=1}^k \frac{1}{j^2} G + \sum_{j=1}^k \delta_{y_j} \right), \quad (\text{S-3.17})$$

where δ_{y_j} denotes point mass at y_j . It follows from (S-3.17) that

$$E(p_{m,k}|y_k) = \frac{\frac{1}{2^m} \sum_{j=1}^k \frac{1}{j^2} + \sum_{j=1}^k \mathbb{I}(y_j = m)}{\sum_{j=1}^k \frac{1}{j^2} + k}; \quad (\text{S-3.18})$$

$$Var(p_{m,k}|y_k) = \frac{\left(\sum_{j=1}^k \frac{1}{j^2} + \sum_{j=1}^k \mathbb{I}(y_j = m) \right) \left(\left(1 - \frac{1}{2^m} \right) \sum_{j=1}^k \frac{1}{j^2} + k - \sum_{j=1}^k \mathbb{I}(y_j = m) \right)}{\left(\sum_{j=1}^k \frac{1}{j^2} + k \right)^2 \left(\sum_{j=1}^k \frac{1}{j^2} + k + 1 \right)}. \quad (\text{S-3.19})$$

As before, it easily follows from (S-3.18) and (S-3.19) that for $m = 1, 2, 3, \dots$,

$$E(p_{m,k}|y_k) \rightarrow p_{m,0}, \text{ and} \quad (\text{S-3.20})$$

$$Var(p_{m,k}|y_k) = O\left(\frac{1}{k}\right) \rightarrow 0, \quad (\text{S-3.21})$$

almost surely, as $k \rightarrow \infty$.

The theorem below characterizes countable number of limit points of $S_{1,\infty}$ in terms of the limit of the marginal posterior probabilities of $p_{m,k}$, as $k \rightarrow \infty$.

Theorem S-3.2. For $\omega \in \mathfrak{S} \cap \mathfrak{N}^c$, where $\mathfrak{N}$ has zero probability measure, $\{S_{1,n}(\omega)\}_{n=1}^{\infty}$ has countable limit points almost surely if and only if

- (1) There exist sequences $\{c_{m,j}(\omega)\}_{j=1}^{\infty}; m = 0, 1, 2, \dots$, such that $(c_{m-1,j}(\omega), c_{m,j}(\omega))$ partition the real line $\mathbb{R}$ for every $j \geq 1$ and $m \geq 1$.
- (2) There exists $j_0(\omega) \geq 1$ such that for all $j \geq j_0(\omega)$, $(c_{m-1,j}(\omega), c_{m,j}(\omega))$ contains at most one limit point of $\{S_{1,n}(\omega)\}_{n=1}^{\infty}$, for every $m \geq 1$.
- (3) With Y_j defined as in (S-3.13),

$$\pi_m(\mathcal{N}_{p_{m,0}}|y_k(\omega)) \rightarrow 1, \quad (\text{S-3.22})$$

as $k \rightarrow \infty$. In the above, $\mathcal{N}_{p_{m,0}}$ is any neighborhood of $p_{m,0}$, with $p_{m,0}$ satisfying $0 \leq p_{m,0} \leq 1$ for $m = 1, 2, \dots$ such that $\sum_{m=1}^{\infty} p_{m,0} = 1$, with at most finite number of m such that $p_{m,0} = 0$.

Proof. Follows using the same ideas as the proof of Theorem S-3.1. □

As regards the choice of the quantities $c_{m,j}$, we simply extend the construction detailed in Section S-3.2 by only letting $M \rightarrow \infty$, and with obvious replacement of the posterior means with those associated with the posterior Dirichlet process.

It is useful to remark that our theory with countably infinite number of limit points is readily applicable to situations where the number of limit points is finite but unknown. In such cases, only a finite number of the probabilities $\{p_{m,j}; m = 1, 2, 3 \dots\}$ will have posterior probabilities around positive quantities, while the rest will concentrate around zero. For known finite number of limit points, it is only required to specify G such that it gives positive mass to only a specific finite set.

S-3.4 Characterization of convergence and divergence with our approach on limit points

Note that for convergent series, $\pi_m(\mathcal{N}_1|y_k(\omega)) \rightarrow 1$ as $k \rightarrow \infty$ for smaller values of m , while for divergent series with $S_{1,\infty}(\omega) = \infty$ or $S_{1,\infty}(\omega) = -\infty$, $\pi_m(\mathcal{N}_1|y_k(\omega)) \rightarrow 1$ as $k \rightarrow \infty$ for much larger values of m and the smallest value of m , respectively. We formalize these statements below as the following theorems.

Theorem S-3.3. *Let there be M number of possible limit points of $S_{1,\infty}(\omega)$, where M may be infinite. Then for any $\omega \in \mathfrak{S} \cap \mathfrak{N}^c$, where $\mathfrak{N}$ has zero probability measure, $S_{1,\infty}(\omega) = \infty$ if and only if, for any sequences $\{c_{m,j}(\omega)\}_{j=1}^\infty; m = 1, 2, \dots, M$, such that $(c_{m-1,j}(\omega), c_{m,j}(\omega)); m = 1, \dots, M$, partitions the real line $\mathbb{R}$ for every $j \geq 1$, it holds that*

$$\pi_{m,k}(\mathcal{N}_1|y_k(\omega)) \rightarrow 1, \quad (\text{S-3.23})$$

as $k \rightarrow \infty$ and $m \rightarrow M$.

Proof. For $\omega \in \mathfrak{S} \cap \mathfrak{N}^c$, where $\mathfrak{N}$ has zero probability measure, let $S_{1,\infty}(\omega) = \infty$. Then as $k \rightarrow \infty$,

$$\left(\frac{\exp(S_{1,k}(\omega))}{1 + \exp(S_{1,k}(\omega))} \right)^{\rho(\theta(\omega))} \rightarrow 1. \quad (\text{S-3.24})$$

In other words, for any fixed $M (> 1)$, $y_k(\omega) \rightarrow M$, as $k \rightarrow \infty$. Hence, as $k \rightarrow \infty$ and $m \rightarrow M$, it easily follows using the same techniques as before, that (S-3.23) holds. Consequently, for infinite number of limit points, (S-3.23) holds as $m \rightarrow \infty$.

Now assume that (S-3.23) holds. It then follows from the formula of the posterior mean that $y_k(\omega) \rightarrow M$, as $k \rightarrow \infty$, for fixed M . Hence, (S-3.24) holds, from which it follows that $S_{1,\infty}(\omega) = \infty$. $\square$

Theorem S-3.4. *Let there be M number of possible limit points of $S_{1,\infty}(\omega)$, where M may be infinite. Then for any $\omega \in \mathfrak{S} \cap \mathfrak{N}^c$, where $\mathfrak{N}$ has zero probability measure, $S_{1,\infty}(\omega) = -\infty$ almost surely if and only if for any sequences $\{c_{m,j}(\omega)\}_{j=1}^\infty; m = 1, 2, \dots, M$, such that $(c_{m-1,j}(\omega), c_{m,j}(\omega)); m = 1, \dots, M$, partitions the real line $\mathbb{R}$ for every $j \geq 1$, it holds that*

$$\pi_{m,k}(\mathcal{N}_1|y_k(\omega)) \rightarrow 1, \quad (\text{S-3.25})$$

as $k \rightarrow \infty$ and $m \rightarrow 1$.

Proof. For $\omega \in \mathfrak{S} \cap \mathfrak{N}^c$, where $\mathfrak{N}$ has zero probability measure, let $S_{1,\infty}(\omega) = -\infty$. Then as $k \rightarrow \infty$,

$$\left(\frac{\exp(S_{1,k}(\omega))}{1 + \exp(S_{1,k}(\omega))} \right)^{\rho(\theta(\omega))} \rightarrow 0. \quad (\text{S-3.26})$$

In other words, for any fixed $M (> 1)$, $y_k(\omega) \rightarrow 1$, as $k \rightarrow \infty$. Hence, as $k \rightarrow \infty$ and $m \rightarrow 1$, it is easily seen that (S-3.25) holds.

Also, if (S-3.25) holds, then it follows from the formula of the posterior mean that $y_k(\omega) \rightarrow 1$, as $k \rightarrow \infty$. Hence, (S-3.26) holds, from which it follows that $S_{1,\infty}(\omega) = -\infty$. $\square$

Theorem S-3.5. *For $\omega \in \mathfrak{S} \cap \mathfrak{N}^c$, where $\mathfrak{N}$ has zero probability measure, $S_{1,\infty}(\omega)$ is convergent if and only if for any sequences $\{c_{m,j}(\omega)\}_{j=1}^\infty$; $m = 1, 2, \dots, M$, such that $(c_{m-1,j}(\omega), c_{m,j}(\omega))$; $m = 1, \dots, M$, partitions the real line $\mathbb{R}$ for every $j \geq 1$, it holds for some finite $m_0(\omega) \geq 1$, that*

$$\pi_{m_0(\omega),k}(\mathcal{N}_1|y_k(\omega)) \rightarrow 1, \quad (\text{S-3.27})$$

as $k \rightarrow \infty$.

Proof. Let $S_{1,\infty}(\omega)$ be convergent. Then as $k \rightarrow \infty$,

$$\left(\frac{\exp(S_{1,k}(\omega))}{1 + \exp(S_{1,k}(\omega))} \right)^{\rho(\theta(\omega))} \rightarrow c(\omega), \quad (\text{S-3.28})$$

for some constant $0 \leq c(\omega) < 1$. Hence, there exists some finite $m_0(\omega) \geq 1$ such that $y_k(\omega) \rightarrow m_0(\omega)$, as $k \rightarrow \infty$. Using the same techniques as before, it is seen that that (S-3.27) holds.

Now assume that (S-3.27) holds. It then follows from the formula of the posterior mean, that $y_k(\omega) \rightarrow m_0(\omega)$, as $k \rightarrow \infty$. Hence, (S-3.28) holds, from which it follows that $S_{1,\infty}(\omega)$ is convergent. $\square$

According to Theorems S-3.4 and S-3.5, m tends to 1 and a finite quantity greater than or equal to 1, accordingly as the series diverges to $-\infty$ or converges. If the finite quantity in the latter case turns out to be 1, then it is not possible to distinguish between convergence and divergence to $-\infty$ by this method. However, Theorem 4.1 of our main manuscript can be usefully exploited in this case. If this method based on oscillating series yields $m = 1$, then we suggest checking for convergence using Theorem 4.1, which would then help us confirm if the series is truly convergent.

S-3.5 A rule of thumb for diagnosis of convergence, divergence and oscillations

Based on the above theorems we propose the following rule of thumb for detecting convergence and divergence when M is finite: if $\frac{m}{M} > 0.9$ such that $\pi_{m,k}(\mathcal{N}_1|y_k(\omega)) \rightarrow 1$ as $k \rightarrow \infty$, then declare the series as divergent to ∞ . If $0.1 < \frac{m}{M} \leq 0.9$ such that $\pi_{m,k}(\mathcal{N}_1|y_k(\omega)) \rightarrow 1$, then declare the series as convergent. On the other hand, if $\frac{m}{M} \leq 0.1$, use Theorem 4.1 to check for convergence; in the case of negative result, declare the series as divergent to $-\infty$.

If, instead, there exist m_ℓ ; $\ell = 1, \dots, L$ ($L > 1$) such that $\pi_{m_\ell, k}(\mathcal{N}_{p_{m_\ell, 0}}|y_k(\omega)) \rightarrow 1$ as $k \rightarrow \infty$, where $0 < p_{m_\ell, 0} < 1$ for $\ell = 1, \dots, L$ and $\sum_{\ell=1}^L p_{m_\ell, 0} = 1$, then say that the sequence $\{S_{1,n}(\omega)\}_{n=1}^\infty$ has L limit points.

S-4 Illustration of our Bayesian theory on oscillation

We first consider a simple oscillatory series to illustrate our Bayesian idea on detection of limit points (Section S-4.1). Next, in Section S-4.2, we illustrate our theory on limit points with Example 5, arguably the most complex series in our set of examples (other than Riemann Hypothesis) and in Section S-5, validate our result on Riemann Hypothesis with our Bayesian limit point theory.

S-4.1 Illustration with a simple oscillatory series

Let us re-consider the series $S_{1,\infty}(\omega) = \sum_{i=1}^\infty (-1)^{i-1}$, which we already introduced after Theorem 4.2 of our main manuscript. We consider the theory based on Dirichlet process developed in Section S-3.3, assuming for the sake of illustrations that G is concentrated on M values, with $G(Y_j = m) = \frac{1}{M}$; $m = 1, 2, \dots, M$. We set $M = 10$ and $K = 10^5$ for our experiments. With $\rho(\theta) = 2$, the results are depicted in Figure S-5. Two explicit limit points, with proportions 0.5 each, are correctly recognized. The limit points are obviously 0 and 1 for this example. Implementation takes just a fraction of a second, even on an ordinary 32-bit laptop.

S-4.2 Illustration of the Bayesian limit point theory with Example 5

Since there is at most one limit point in the cases that we investigated, application of our ideas to these cases must be able to re-confirm this. As before we consider the theory based on Dirichlet process with $G(Y_j = m) = \frac{1}{M}$; $m = 1, 2, \dots, M$, where we set $M = 10$. Thus, by our rule of thumb, divergence is to be declared only if $\pi_{m=10,k}(\mathcal{N}_1|y_k) \rightarrow 1$, as $k \rightarrow \infty$.

As regards implementation, notice that here there is no scope for parallelization since at the j -th step only y_j is added to the existing $S_{1,j-1}$ to form $S_{1,j} = S_{1,j-1} + y_j$. As such, on our VMware, using a single processor, only about two seconds are required for 10^5 iterations associated with the series (S-2.1), for various values of a (> 0) and b (> 0).

S-4.2.1 Choice of $\rho(\theta)$ in $\left(\frac{\exp(S_{1,k})}{1 + \exp(S_{1,k})}\right)^{\rho(\theta)}$

In our example, $\theta = (a, b)$. We choose, for $j \geq 1$,

$$\tilde{\rho}(\theta) = a - b + \epsilon, \quad (\text{S-4.1})$$

and set

$$\left(\frac{\exp(S_{1,j})}{1 + \exp(S_{1,j})}\right)^{\rho(\theta)} = \min \left\{ 1, \left(\frac{\exp(S_{1,j})}{1 + \exp(S_{1,j})}\right)^{\tilde{\rho}(\theta)} \right\} \quad (\text{S-4.2})$$

Recall that the series (S-2.1), defined for $a > 0$ and $b > 0$, converges for $a - b > 1$ and diverges for $a + b < 1$. In keeping with this result, (S-4.2) decreases as $(a - b)$ increases, so that the chance of correctly diagnosing convergence increases. Moreover, if both a and b are between 0 and 1 such that $a + b < 1$, then (S-4.2) tends to be inflated, thereby increasing the chance of correctly detecting divergence. The term ϵ in (S-4.2) prevents the power from becoming zero when $a = b$. It is important to note here that for $a + b = 1$ convergence or divergence is not guaranteed, but if $\epsilon = 0$ in (S-4.2), then $a = b$ would trivially indicate divergence, even if the series is actually convergent. A positive value of ϵ provides protection from such erroneous decision. Note that if $a < b - \epsilon$, the convergence criterion $a - b > 1$ is not met but the divergence criterion $a + b < 1$ may still be satisfied. Thus, for such instances, greater weight in favour of divergence is indicated. In our illustration, we set $\epsilon = 10^{-10}$.

S-4.2.2 Results

Figure S-6 shows the results of our Bayesian analysis of the series (S-2.1) based on our Dirichlet process model. Based on the rule of thumb proposed in Section S-3.5 all the results are in agreement with the results based on Figure S-1.

S-5 Application of the Bayesian multiple limit points theory to Riemann Hypothesis

To strengthen our result on Riemann Hypothesis presented in Section 6 of our main manuscript we consider application of our Bayesian multiple limit points theory to Riemann Hypothesis.

S-5.1 Choice of $\rho(\theta)$ in $\left(\frac{\exp(S_{1,k})}{1 + \exp(S_{1,k})} \right)^{\rho(\theta)}$

For Riemann Hypothesis, $\theta = a$; we choose, for $j \geq 1$,

$$\tilde{\rho}(\theta) = a^6. \quad (\text{S-5.1})$$

The reason for such choice with a relatively large power is to allow discrimination between $\left(\frac{\exp(S_{1,k})}{1 + \exp(S_{1,k})} \right)^{\rho(\theta)}$ for close values of a . However, substantially large powers of a are not appropriate because that would make the aforementioned term too small to enable detection of divergence. In fact, we have chosen the power after much experimentation. Implementation of our methods takes about 2 seconds on our VMWare, with 10^5 iterations.

S-5.2 Results

The results of application of our ideas on multiple limit points are depicted in Figures S-7, S-8 and S-9. The values of m/M and the thumb rule proposed in Section S-3.5 show that all the results are consistent with those obtained in Section 6. For $a = 2$ and $a = 3$ we obtained $m/M = 0.1$, but the existing theory and our results reported in Section 6 confirm that the series is convergent, and not oscillating, for these values. There seems to be a slight discrepancy

only regarding the location of the change point of convergence. In this case, unlike $a = 0.72$ as obtained in Section 6, we obtained $a = 0.7$ as the change point (see panel (b) of Figure S-8).

This (perhaps) negligible difference notwithstanding, both of our methods are remarkably in agreement with each other, emphasizing our point that Riemann Hypothesis can not be completely supported.

S-6 Characterization of Riemann Hypothesis based on Bernoulli numbers

Characterization of Riemann Hypothesis by convergence of infinite sums associated with Bernoulli numbers are provided in (Carey, 2003) (unpublished, according to our knowledge). In particular, it has been shown that Riemann hypothesis is true if and only if the following series is convergent:

$$\tilde{S}_1 = \sum_{m=1}^{\infty} \frac{\pi(4m+3)}{2^{4m+1}} \sum_{k=0}^m (-1)^k \frac{\binom{2m+1}{k} \binom{4m+2-2k}{2m+1}}{2m+2-2k} \log \left(\frac{(2\pi)^{2m+2-2k} |B_{2m+2-2k}|}{2(2m+2-2k)^2 (2m-2k)!} \right), \quad (\text{S-6.1})$$

where $\{B_n; n = 0, 1, \dots\}$ are Bernoulli numbers characterized by their generating function $\sum_{n=0}^{\infty} B_n x^n / n! = x / (\exp(x) - 1)$. The Bernoulli numbers are related to the Riemann zeta function by (see, for example (Sury, 2003))

$$B_{2m} = (-1)^{m-1} \frac{2(2m)!}{(2\pi)^{2m}} \zeta(2m). \quad (\text{S-6.2})$$

(Carey, 2003) further showed that convergence of the related series

$$\tilde{S}_2 = \sum_{m=1}^{\infty} \frac{\pi(4m+3)}{2^{4m+1}} \sum_{k=0}^m (-1)^k \frac{\binom{2m+1}{k} \binom{4m+2-2k}{2m+1}}{2m+2-2k} \log \left((2m+1-2k) \frac{|B_{2m+2-2k}|}{|B_{2m+4-2k}|} \right), \quad (\text{S-6.3})$$

is also equivalent to the assertion that Riemann hypothesis is correct. However, the terms of both the series (S-6.1) and (S-6.3) tend to explode very quickly. Stirlings's approximation of the factorials involved in the summands facilitates computation of larger number of summands compared to the original terms. In this context, note that Stirling's approximation applied to the factorials in (S-6.2), along with the approximation $\zeta(2m) \sim 1$, as $m \rightarrow \infty$, lead the following asymptotic form of B_{2m} as $m \rightarrow \infty$:

$$B_{2m} \sim (-1)^{m-1} 4\sqrt{\pi m} \left(\frac{m}{\pi e} \right)^{2m}. \quad (\text{S-6.4})$$

Figure S-10 shows the logarithms of the first few terms a_m of the above two series, based on the actual terms a_m and the Stirling-approximated a_m (ignoring a multiplicative constant); the rest of the terms become too large to be reliably computed, even with Stirling's approximation. The bottomline that emerges from (S-10) is that the series $\tilde{S}_1$ and $\tilde{S}_2$ appear to be clearly divergent, providing some support to our result on Riemann hypothesis.

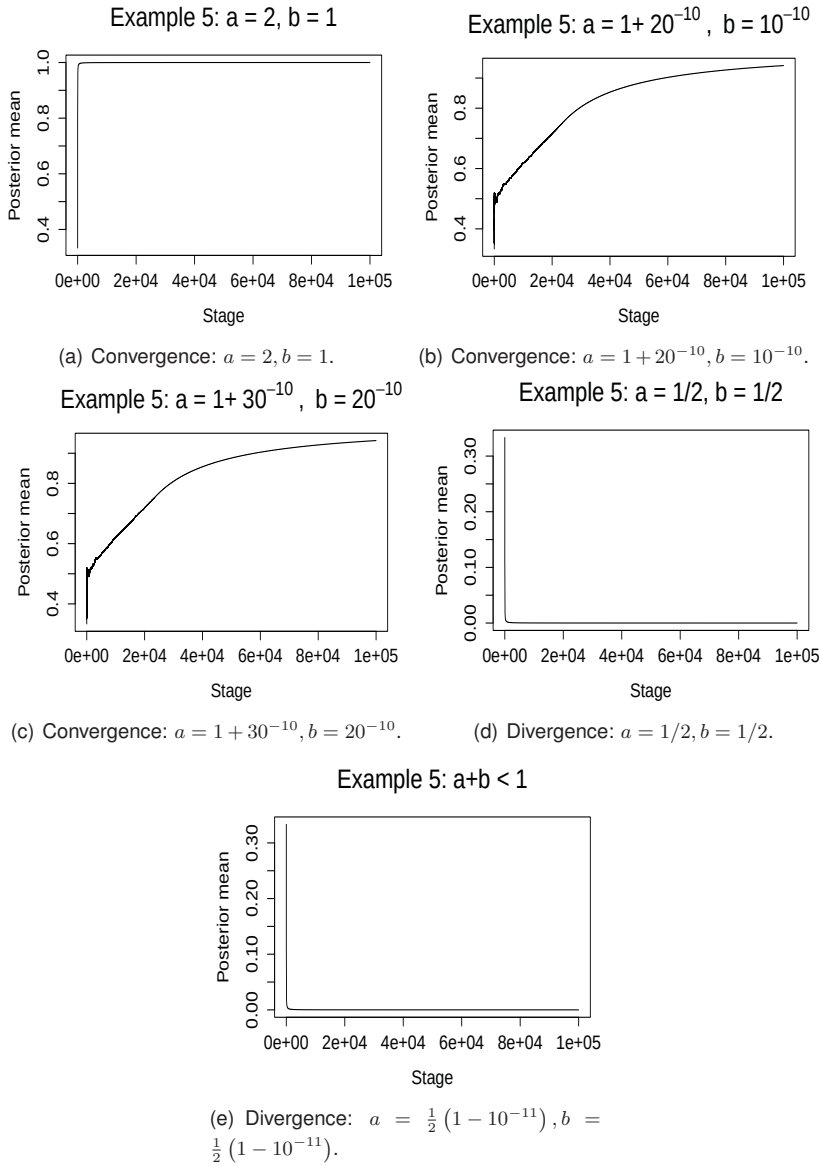


Figure S-1: Example 5: The series (S-2.1) converges for $(a = 2, b = 1)$, $(a = 1 + 20^{-10}, b = 10^{-10})$, $(a = 1 + 30^{-10}, b = 20^{-10})$ and diverges for $(a = 1/2, b = 1/2)$ and $(a = \frac{1}{2}(1 - 10^{-11}), b = \frac{1}{2}(1 - 10^{-11}))$.

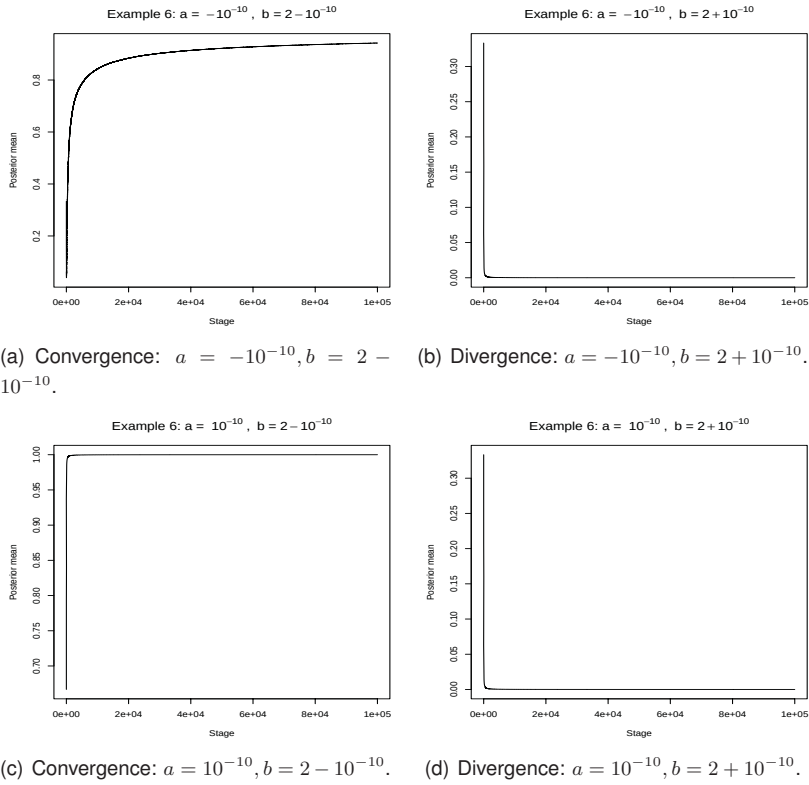


Figure S-2: Example 6: The series (S-2.9) converges for $(a = -10^{-10}, b = 2 - 10^{-10})$, $(a = 10^{-10}, b = 2 - 10^{-10})$, and diverges for $(a = -10^{-10}, b = 2 + 10^{-10})$, $(a = 10^{-10}, b = 2 + 10^{-10})$.

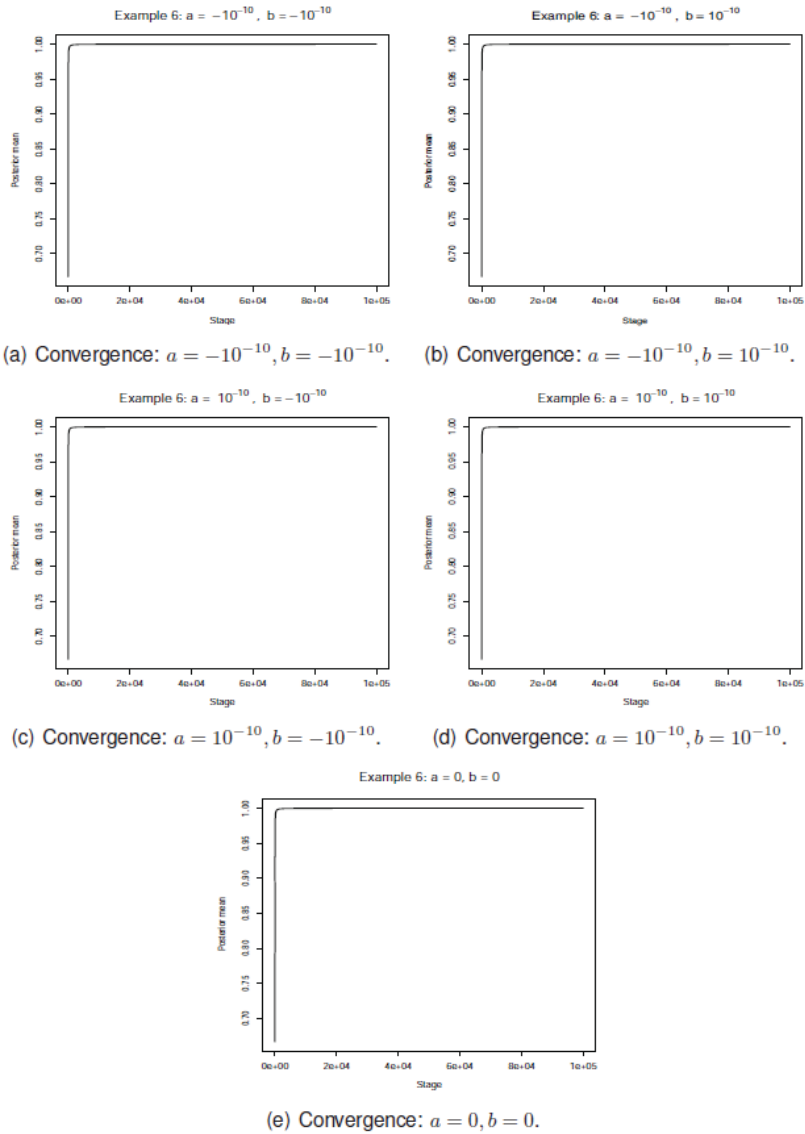


Figure S-3: Example 6: The series (S-2.9) converges for $(a = -10^{-10}, b = -10^{-10})$, $(a = -10^{-10}, b = 10^{-10})$, $(a = 10^{-10}, b = -10^{-10})$, $(a = 10^{-10}, b = 10^{-10})$, and $(a = 0, b = 0)$.

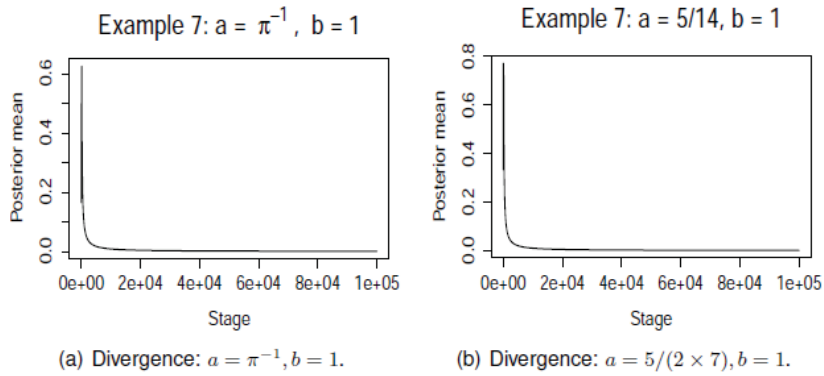


Figure S-4: Example 7: The series (S-2.15) diverges for $(a = \pi^{-1}, b = 1)$, $(a = 5/7, b = 1)$.

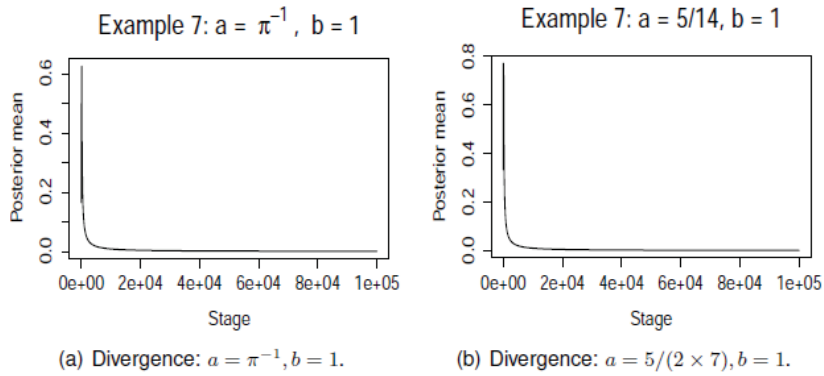


Figure S-4: Example 7: The series (S-2.15) diverges for $(a = \pi^{-1}, b = 1)$, $(a = 5/7, b = 1)$.

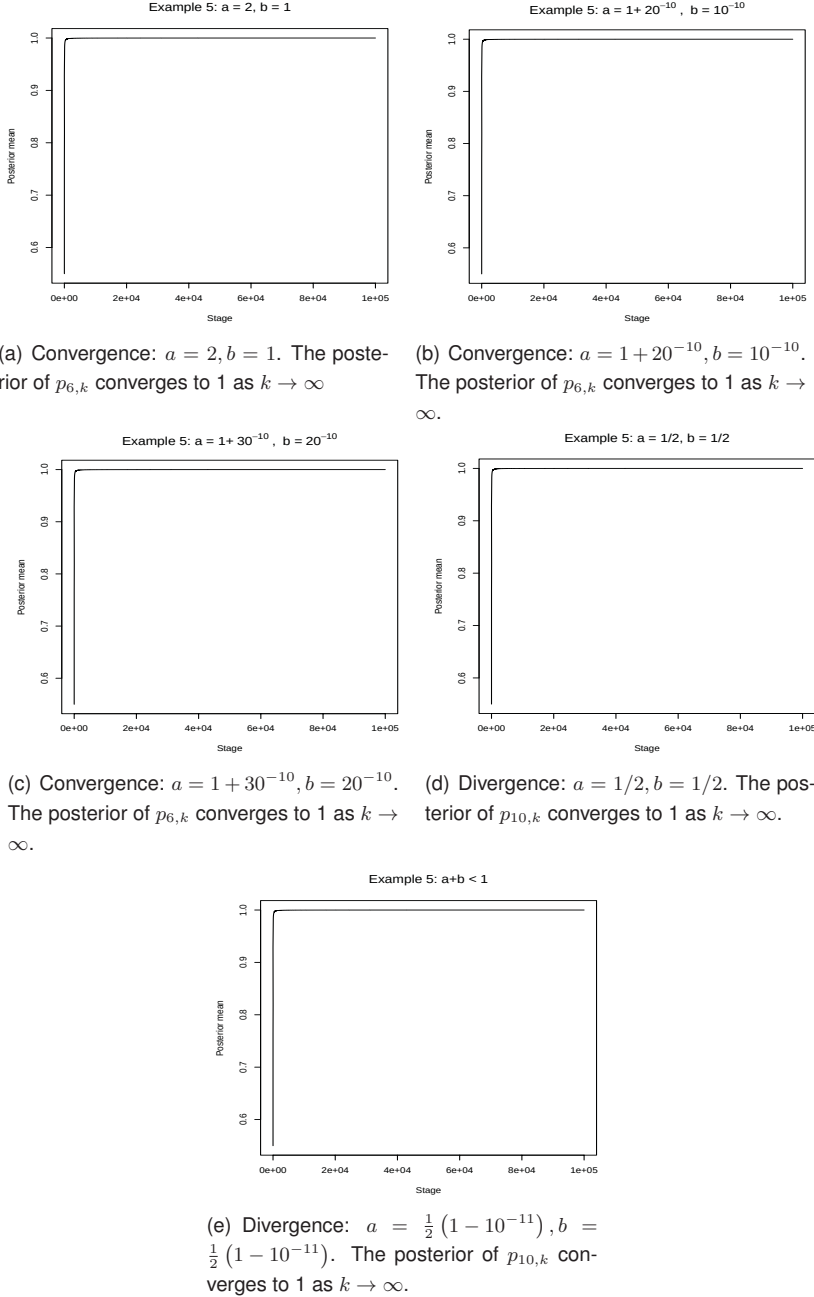


Figure S-6: Illustration of the Dirichlet process based theory with Example 5: For $(a = 2, b = 1)$ in the series (S-2.1), $\frac{m}{M} = \frac{6}{10} < 0.9$, indicating convergence, for $(a = 1 + 20^{-10}, b = 10^{-10})$, $\frac{m}{M} = \frac{6}{10} < 0.9$, indicating convergence, for $(a = 1 + 30^{-10}, b = 20^{-10})$, $\frac{m}{M} = \frac{6}{10} < 0.9$, indicating convergence, for $(a = 1/2, b = 1/2)$, $\frac{m}{M} = \frac{10}{10} > 0.9$, indicating divergence, and for $(a = \frac{1}{2}(1 - 10^{-11}), b = \frac{1}{2}(1 - 10^{-11}))$, $\frac{m}{M} = \frac{10}{10} > 0.9$, indicating divergence.

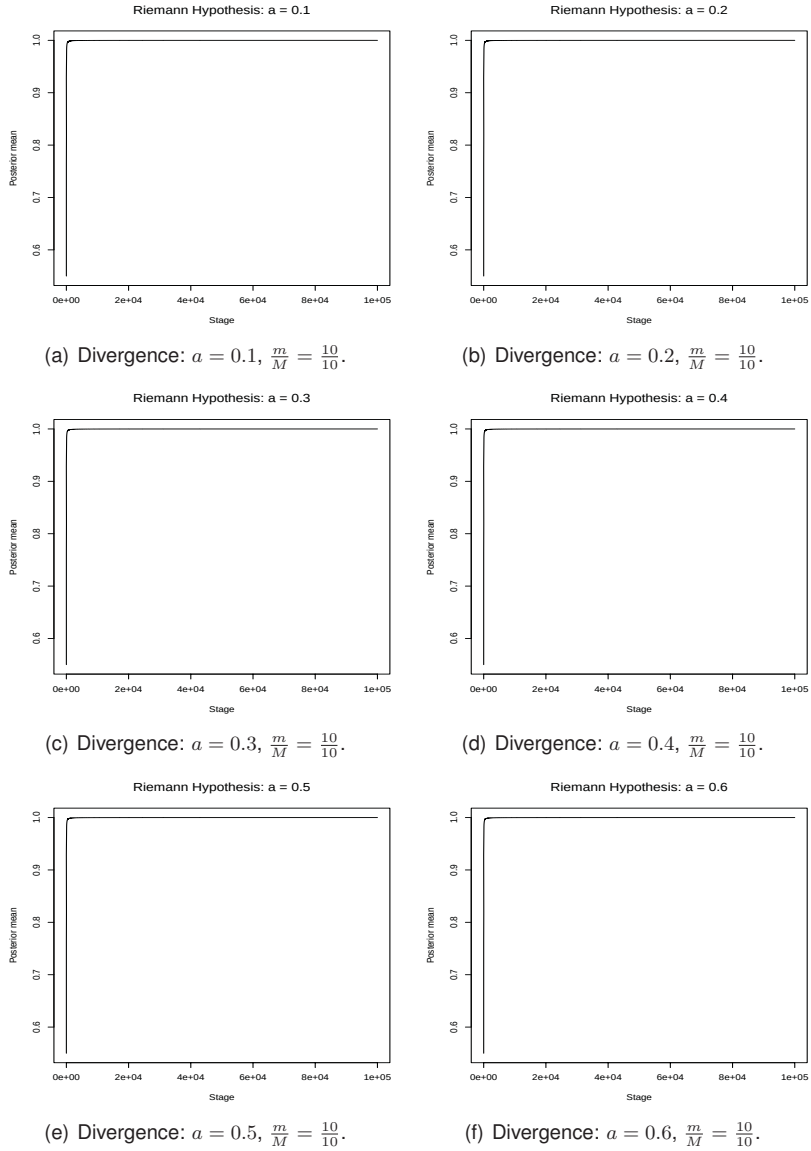


Figure S-7: Riemann Hypothesis based on Bayesian multiple limit points theory: Divergence for $a = 0.1, 0.2, 0.3, 0.4, 0.5, 0.6$.

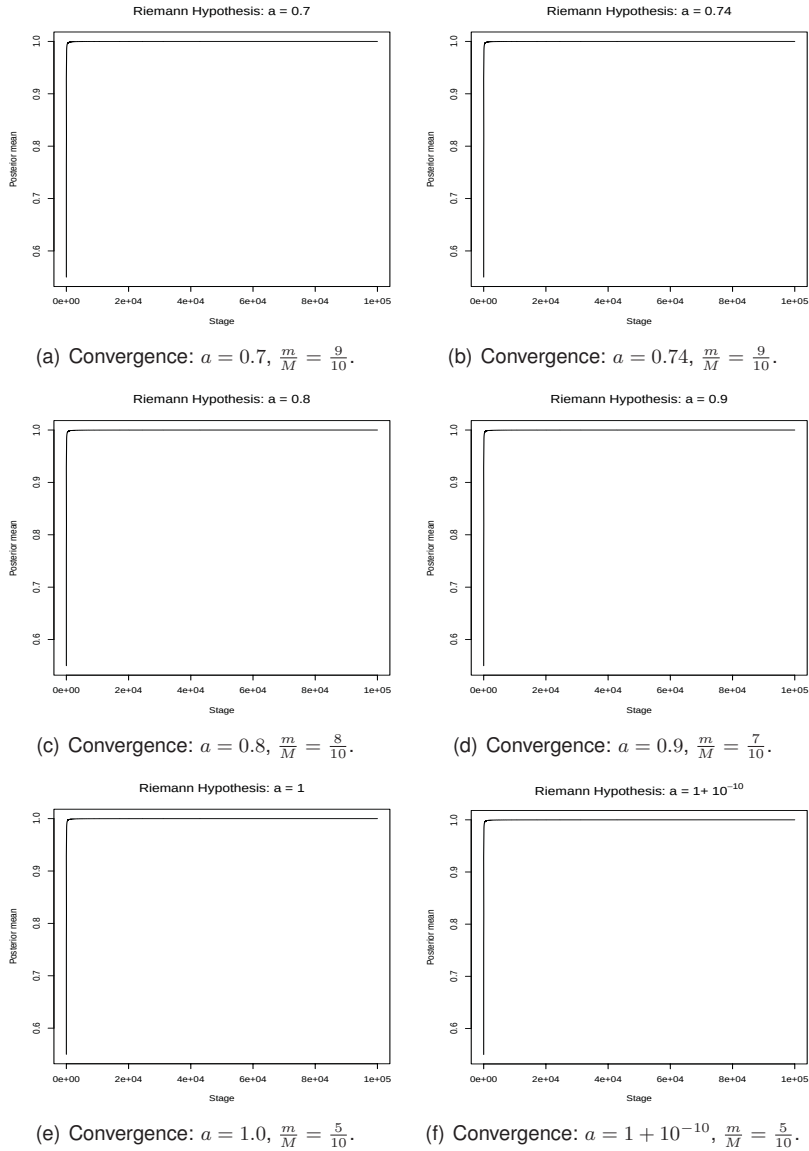


Figure S-8: Riemann Hypothesis based on Bayesian multiple limit points theory: Divergence for $a = 0.7$ but convergence for $a = 0.74, 0.8, 0.9, 1, 1 + 10^{-10}$.

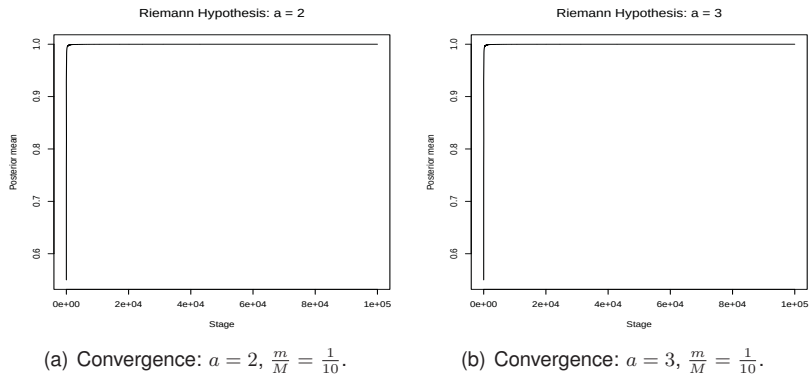


Figure S-9: Riemann Hypothesis based on Bayesian multiple limit points theory: Convergence for $a = 2, 3$.

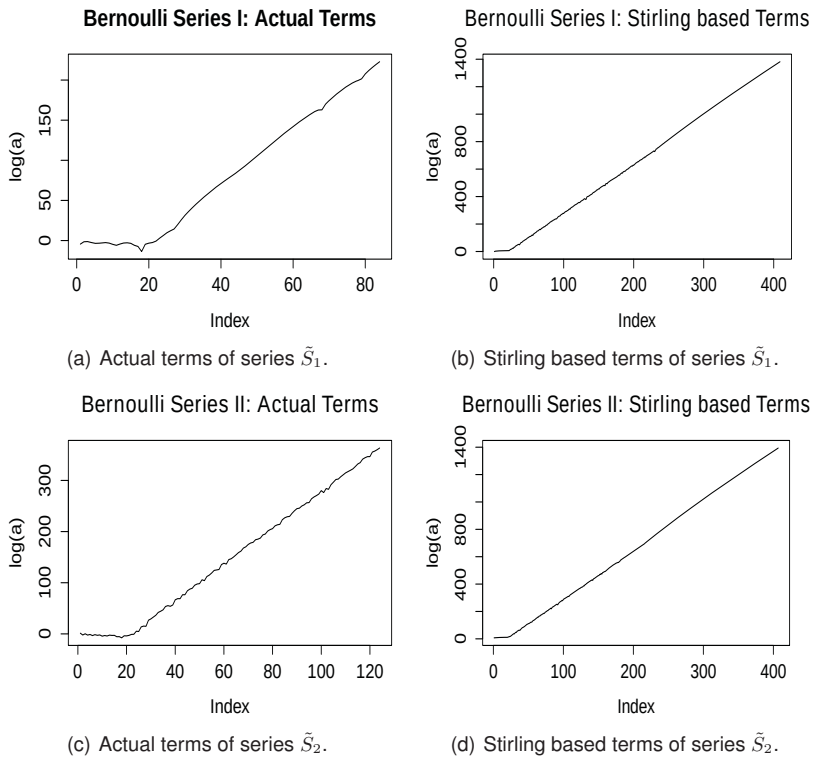


Figure S-10: Actual and Stirling-approximated terms a_m of the series $\tilde{S}_1$ and $\tilde{S}_2$.

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