

A Strong law of large numbers for sequences of set-valued random variables with a martingale difference selection

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ABSTRACT

In this paper, we shall present a strong law of large numbers (SLLN) for sequences of set-valued random variables with a martingale difference selection taking values in a p -smoothable Banach space ($1 \leq p \leq 2$).

Keywords: strong law of large numbers, set-valued random variables, martingale, martingale differences, p -smoothable Banach space.

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1 Introduction

Strong laws of large numbers (SLLN) and martingale limit theorems play a very important role in probability theory and especially in statistics and related fields. In the case of real numbers and on Banach space, these issues have been widely studied (e.g. (Hall and Heyde, 2014), (Wołczyński, 1976), (Wołczyński, 1975), (Wołczyński, 1980), (Elton, 1981), (Hoeffding, Robbins et al., 1948), (Taylor, 1985), (Pisier, 1975), (Hoffmann-Jorgensen, Pisier et al., 1976), (Giné and Zinn, 1983), (Giné, 1978), etc.). For the set-valued case, there are variety of works of the issues for independent random elements (see (Hess, 1994), (Castaing and Valadier, 1977), (Li, Ogura and Kreinovich, 2013), (Li and Ogura, 2006), (Shen and Guan, 2016), etc.). In practical areas, martingale methods have been widely used in financial field for establishing option pricing (e.g. (Zheng and Yang, 2016), (Lin and Jiang, 2015)), finding optimal portfolio (e.g. (Yang and Jin, 2013)) and so on. In another aspect, testing for martingale difference hypothesis has been also applied to assess the efficiency of the market in term of single-valued martingale difference sequences (Escanciano and Lobato, 2009). It is the fact that people may sometimes want to know the predictability of the trend of the market in terms of “very low”, “low”, “normal”, “high” or “very high” through an interval-valued random variables sequence of time series’ returns. For instance, let $\{p_n, n \geq 1\}$ be a sequence of prices of a financial asset then $\{d_n = \frac{p_n - p_{n-1}}{p_{n-1}}, n \geq 1\}$ is a sequence of its returns. It is commonly that $d_n \in (-1, 1)$. Constructing a set-valued random variable sequence of financial returns $\{D_n, n \geq 1\}$ with natural filtration $\mathcal{F}_n, n \geq 1$ taking its values in $\{A_1, A_2, A_3, A_4, A_5\}$ where $A_1 = [-M, -0.02]$,

$A_2 = [-0.02, -0.01]$, $A_3 = [-0.01, 0.01]$, $A_4 = [0.01, 0.02]$ and $A_5 = [0.02, M]$ for some positive number $M < 1$, then the main concern is that whether the set-valued conditional expectation $E[D_n|\mathcal{F}_{n-1}] \ni 0$, almost surely (a.s) for all $n \geq 1$? Notice that D_n is not necessary to contain zero for this condition. For example, $D_n = A_1$ with probability of 0.5 and A_5 with probability of 0.5 then $E[D_n|\mathcal{F}_{n-1}] \ni 0$. If this condition is satisfied, we even can conclude that the trend of the market is unable to predict. In this case, a definition for such type of set-valued random variable sequences needs to be proposed. For this situation, two questions arise:

- What is the necessary and sufficient condition of the definition?
- Is there any kind of strong laws of large numbers for these type of sequences?

This work aims to propose a definition of what to be called *weak set-valued martingale difference sequence* taking values on a p -smoothable separable Banach space ($1 < p \leq 2$). Then the necessary and sufficient condition of weak set-valued martingale differences is proved and its law of large numbers in Mosco convergence sense is stated.

This paper is organized as follows. Section 2 introduces some basic notations, definitions and some related results in the case of single-valued random variables. In Section 3, we prove the necessary and sufficient condition and treat the SLLN for weak set-valued martingale difference on a p -smoothable separable Banach space, $1 < p \leq 2$, based on Mosco convergence conditions.

2 Definitions and Preliminaries

Throughout this paper, assume that $(\Omega, \mathcal{F}, P)$ is a complete probability space, $\mathcal{X}$ is a separable Banach space with the dual space $\mathcal{X}^*$ and $2^{\mathcal{X}}$ is the set of all subsets of $\mathcal{X}$. We denote by

$$L \log^+ L = \{f \in L^1_{\mathcal{X}} : E[\|f\| \log^+ \|f\|] < \infty\} \text{ where } \log^+ x = \log x \vee 0, x \in \mathbb{R}^+,$$

$$L^p = \{f \in L^1_{\mathcal{X}} : E[\|f\|^p] < \infty\}, 1 \leq p < 2,$$

$$\mathcal{P}_c(\mathcal{X}) = \{A \in 2^{\mathcal{X}} : A \text{ is non-empty, closed and convex}\},$$

$$\mathcal{P}_{cb}(\mathcal{X}) = \{A \in \mathcal{P}_c(\mathcal{X}) : A \text{ is bounded}\},$$

$$\mathcal{P}_{wkc}(\mathcal{X}) = \{A \in \mathcal{P}_c(\mathcal{X}) : A \text{ is weakly compact}\}$$

where $L^1_{\mathcal{X}}$ is the set of all Bochner-integrable random variables from $\Omega \rightarrow \mathcal{X}$.

For each $A \in 2^{\mathcal{X}} \setminus \emptyset$, $\text{cl}A$ and $\overline{\text{co}}A$ denote the closure and closed convex hull of A respectively, where the closure is taken with respect to the norm in $L^1_{\mathcal{X}}$ and define

$$|A| = \sup \{\|x\| : x \in A\},$$

$$d(x, A) = \inf \{\|x - y\| : y \in A\}, d(x, \emptyset) = \infty,$$

$d(x, A)$ is called distance function of A .

Suppose that $\mathcal{A}$ is a sub- σ -fields of $\mathcal{F}$. A multi-valued random function F from $\Omega \rightarrow \mathcal{P}_c(\mathcal{X})$ is called $\mathcal{A}$ -measurable if there exists a $\mathcal{A}$ -measurable random variables sequence $\{f_n\}$ such that $f_n : \Omega \rightarrow \mathcal{X}$, $F(\omega) = cl\{f_n(\omega), n \geq 1, \omega \in \Omega\}$ (see (Castaing and Valadier, 1977)).

Denote by $\mathcal{M}(\mathcal{F})$ is the family of all $\mathcal{F}$ -measurable multi-valued random functions. Denote by

$$\begin{aligned}\mathcal{L}_c &= \{F \in \mathcal{M}(\mathcal{F}) : F(\omega) \in \mathcal{P}_c(\mathcal{X}) \text{ a.s.}\}, \\ \mathcal{L}_{wkc} &= \{F \in \mathcal{L}_c : F(\omega) \in \mathcal{P}_{wkc}(\mathcal{X}) \text{ a.s.}\}, \\ \mathcal{L}_c^{d1} &= \{F \in \mathcal{L}_c : E[d(0, F)] < \infty\}, \\ \mathcal{L}_c^1 &= \{F \in \mathcal{L}_c : E[|F|] < \infty\}\end{aligned}$$

For each $F \in \mathcal{L}_c$, define

$$S_F^1(\mathcal{F}) = \{f \in L_{\mathcal{X}}^1 : f \text{ is } \mathcal{F}\text{-measurable, } f(\omega) \in F(\omega) \text{ a.s.}\},$$

$S_F^1(\mathcal{F})$ is denoted by S_F^1 for short. It is easy to find that S_F^1 is non-empty if only if $F \in \mathcal{L}_c^{d1}$ (see Theorem 2.1.7 of (Molchanov and Molchanov, 2005)).

The Aumann integral $\int F dP$ of F is defined by $\int F dP = \{E[f] : f \in S_F^1\}$, where $E[f]$ is the Bochner integral of f . The set-valued expectation of F is defined by $E[F] = cl(\int F dP)$. The conditional expectation $F \in \mathcal{L}_c^{d1}$ relative to a sub- σ -algebra $\mathcal{A}$ of $\mathcal{F}$, $E[F|\mathcal{A}]$ (different from a P -null set) is an unique $\mathcal{A}$ -measurable multi-valued function in $\mathcal{L}_c^{d1}$ such that

$$S_{E[F|\mathcal{A}]}^1(\mathcal{A}) = cl\{E[f|\mathcal{A}] : f \in S_F^1\}$$

Notice these notations about convergence for sequences of subsets which introduced by Mosco (Mourier, 1972). Let τ be a topology on $\mathcal{X}$ and $\{X_n, n \geq 1\}$ a sequence in $\mathcal{P}_c(\mathcal{X})$. We put

$$\tau\text{-li}X_n = \{x \in \mathcal{X}, x = \tau\text{-}\lim x_n : x_n \in X_n, \forall n \geq 1\}, \text{ and}$$

$$\tau\text{-ls}X_n = \{x \in \mathcal{X}, x = \tau\text{-}\lim x_k : x_k \in X_{n_k}, \forall k \geq 1\}$$

for some subsequence $\{X_{n_k}, k \geq 1\}$ of $\{X_n\}$. Let us denote by s (resp. w) the strong (resp. weakly) topology on $\mathcal{X}$. A subset X_∞ is said to be the Mosco limit of $\{X_n, n \geq 1\}$ if

$$X_\infty = s\text{-li}X_n = w\text{-ls}X_n.$$

Suppose that $\{\mathcal{F}_n, n \geq 1\}$ is an increasing family sub- σ -fields of $\mathcal{F}$ and $\mathcal{F}_\infty = \sigma\left(\bigcup_{n=1}^{\infty} \mathcal{F}_n\right)$. $\{D_n, \mathcal{F}_n, n \geq 1\}$ denotes a sequence of set-valued random variables that $D_n \in \mathcal{L}_c^{d1}$ and $\mathcal{F}_n$ -measurable.

Definition 2.1. (Measurable selection) A random variable f with its values in $\mathcal{X}$ is call a (measurable) selection of a set-valued random variable X if $f(\omega) \in X(\omega)$ for almost all $\omega \in \Omega$. The family of all selections of X is denoted by $S(X)$. A countable family $\{f_n \in S(X), n \geq 1\}$ is said to be a Castaing representation of X if

$$X = cl\{f_n, n \geq 1\}.$$

It is known that an adapted sequence $\{f_n, \mathcal{F}_n, n \geq 1\}$ in $L^1_{\mathcal{X}}$ such that for all $n \geq 1$, $E[f_{n+1}|\mathcal{F}_n] = f_n$ a.s. (resp. $E[f_{n+1}|\mathcal{F}_n] = 0$ a.s.) is called a martingale (resp. martingale difference sequence). It is also known in Theorem 2.1.72 of (Molchanov and Molchanov, 2005) that for X be integrable random closed set and $\mathcal{A}$ be a sub- σ -algebra of $\mathcal{F}$ such that $E[X|\mathcal{A}] = \{0\}$ then X is a singleton. Therefore, these suggests us a new definition of “weak set-valued martingale difference” as below:

Definition 2.2. A sequence of set-valued random variables $\{D_n, \mathcal{F}_n, n \geq 1\}$ is called a weak set-valued martingale difference if

$$0 \in S^1_{E[D_n|\mathcal{F}_{n-1}]}(\mathcal{F}_{n-1}), \forall n \geq 2.$$

An adapted sequence $\{f_n, \mathcal{F}_n, n \geq 1\}$ is a martingale difference selection of weak set-valued martingale difference $\{D_n, \mathcal{F}_n, n \geq 1\}$ if $\{f_n, \mathcal{F}_n, n \geq 1\}$ is a $\mathcal{X}$ -valued martingale difference and $\forall n \geq 1, f_n \in S^1_{D_n}(\mathcal{F}_n)$.

Example Let $\{r_n, \mathcal{F}_n, n \geq 1\}$ be a non-negative $\mathbb{R}$ -valued martingale. Then the sequence of set-valued random variables $D_n(\omega) = [-r_n(\omega), r_n(\omega)]$ is set-valued martingale with respect to the filtration $\{\mathcal{F}_n, n \geq 1\}$. Because of $0 \in D_n$ for all $\omega \in \Omega$, $\{D_n, \mathcal{F}_n, n \geq 1\}$ is a weak set-valued martingale difference.

Proof. Indeed, it is obviously that $0 \in D_n$ a.e. for all $n \geq 1$. We will show that $E[D_{n+1}|\mathcal{F}_n] = D_n, \forall n \geq 1$.

We have

$$S^1_{D_n}(\mathcal{F}_n) = \{f \in L^1_{\mathcal{X}} : -r_n(\omega) \leq f(\omega) \leq r_n(\omega) \text{ a.e.}\},$$

and for each $f \in S^1_{D_{n+1}}(\mathcal{F}_{n+1})$

$$\begin{aligned} E[f|\mathcal{F}_n] &= E[f - r_{n+1}|\mathcal{F}_n] + E[r_{n+1}|\mathcal{F}_n] \\ &\leq E[r_{n+1}|\mathcal{F}_n] = r_n, \text{ a.e.} \end{aligned}$$

and $E[f|\mathcal{F}_n] \geq -r_n, \text{ a.e.}$

We thus obtain

$$cl \left\{ E[f|\mathcal{F}_n] : f \in S^1_{D_{n+1}}(\mathcal{F}_{n+1}) \right\} \subset S_{D_n}(\mathcal{F}_n).$$

Conversely, take $f \in S^1_{D_n}(\mathcal{F}_n)$ and put

$$\lambda_n(\omega) = \begin{cases} \frac{r_n(\omega) - f(\omega)}{2r_n(\omega)} & \text{if } r_n(\omega) > 0, \\ \frac{1}{2} & \text{if } r_n(\omega) = 0. \end{cases}$$

Then the random variable

$$g(\omega) = (1 - 2\lambda_n(\omega))r_{n+1}(\omega)$$

belongs to $S_{D_{n+1}}(\mathcal{F}_{n+1})$ and $E[g|\mathcal{F}_n] = f, \text{ a.e.}$ Hence

$$cl \left\{ E[f|\mathcal{F}_n] : f \in S^1_{D_{n+1}}(\mathcal{F}_{n+1}) \right\} \supset S^1_{D_n}(\mathcal{F}_n).$$

So we have

$$S^1_{D_n}(\mathcal{F}_n) = cl \left\{ E[f|\mathcal{F}_n] : f \in S^1_{D_{n+1}}(\mathcal{F}_{n+1}) \right\}.$$

This means $E[D_{n+1}|\mathcal{F}_n] = D_n$. □

We now present some convergence results of limit theorems on separable p -smoothable Banach space ($1 \leq p \leq 2$). Notice that

Definition 2.3. A separable Banach $\mathcal{X}$ is said to be p -uniformly smooth ($1 \leq p \leq 2$) if

$$\rho_{\mathcal{X}}(t) = \sup \left\{ \frac{\|x + ty\| + \|x - ty\|}{2} - 1 : \|x\| = \|y\| = 1 \right\} \leq Ct^p$$

where C is a some constant for each $t \in [0; \infty]$.

Assouad and Pisier (Pisier, 1975) has showed that a Banach space $\mathcal{X}$ is p -uniformly smooth if and only if there exists a constant $C > 0$ such that

$$E \left\| \sum_{i=1}^n d_i \right\|^p \leq C \sum_{i=1}^n E \|d_i\|^p,$$

for all $\mathcal{X}$ -valued martingale differences $\{d_i\}_{i=1}^n$. This inequality is called Assouad-Pisier's inequality.

Assume $\{f_n, \mathcal{F}_n, n \geq 1\}$ be a $\mathcal{X}$ -valued martingale, denote by $d_n = f_n - f_{n-1}$ then $\{d_n, \mathcal{F}_n, n \geq 1\}$ is a $\mathcal{X}$ -valued martingale difference and

$$f_n = \sum_{i=1}^n d_i, n \geq 1.$$

It is to recall that a sequence $\{f_n, n \geq 1\}$ of $\mathcal{X}$ -valued random variables taking values on p -smoothable separable Banach space $\mathcal{X}$ is called have uniformly bounded tail probabilities by the tail probabilities of a random variable $f \in L^p, p > 0$ ($(f_n) \prec f$) if there exists a positive constant C such that for all $x > 0$ and $n = 1, 2, \dots$, we have

$$P(\|f_n\| > x) \leq CP(\|f\| > x).$$

This suggests the following definition.

Definition 2.4. A sequence of set-valued random variables $\{X_n, n \geq 1\}$ in $\mathcal{L}_c^1$ is said to have uniformly bounded tail probabilities by the tail probabilities of a set-valued random variable $X \in l^p$ ($(X_n) \prec X$) if for all $f_n \in S_{X_n}^1$ there exists an $f \in S_X^1$ such that $(f_n) \prec f$.

Definition 2.5. We say that

- i) The $L \log^+ L$ -Martingale Convergence Theorem holds in a Banach space $\mathcal{X}$ ($L \log^+ L - MCT$ for short) if for each $\mathcal{X}$ -valued martingale $\{f_n, \mathcal{F}_n, n \geq 1\}$ satisfying the condition $\sup_n E [\|f_n\| \log^+ \|f_n\|] < \infty$ then f_n almost surely convergence.
- ii) The L_p -Martingale Convergence Theorem holds in a Banach space $\mathcal{X}$ (MCT_p for short), $p > 1$, if for each martingale $\{f_n, \mathcal{F}_n, n \geq 1\}$ satisfying the condition $\sup_n E [\|f_n\|^p] < \infty$, there exists an element f_∞ such that $f_n \rightarrow f_\infty$ a.s and in L^p .

Here we present some results showing the necessary of p -smoothable Banach space ($1 \leq p \leq 2$) for the convergence of a $\mathcal{X}$ -valued martingale.

Proposition 2.1. *For a separable Banach space $\mathcal{X}$, the following properties are equivalent:*
 (RNP) $\mathcal{X}$ has the Radon-Nykodym property;
 (MCT_p) the MCT_p holds in $\mathcal{X}$;
 ($L \log^+ L$) the $L \log^+ L - MCT$ holds in $\mathcal{X}$.

Proof. We shall prove the following diagram

$$(MCT_p) \Leftrightarrow (RNP) \Leftrightarrow (L \log^+ L).$$

Note that $(RNP) \Rightarrow (MCT_p)$ can be proved similarly to the proof of Theorem 1.1 in (Woyczyński, 1975) (see also (Millet and Sucheston, 1981; Yen, 2004)).

$(MCT_p) \Rightarrow (RNP)$ is a result of Chatterji (see Theorem 1 in (Chatterji, 1969)).

The case $(RNP) \Rightarrow (L \log^+ L)$ is carried out by using the methods of (Chatterji, 1969) (which has been proved in (Yen, 2004), Lemma 1). Similarly, $(L \log^+ L) \Rightarrow (RNP)$ is also a result of Chatterji (see Theorem 1.1 in (Smythe, 1974)) $\square$

Now we give a result related to the laws of large numbers of $\mathcal{X}$ -valued martingale differences.

Proposition 2.2. *If $\mathcal{X}$ is a separable Banach space and $1 \leq p \leq 2$, the following propositions are equivalent:*

- 1) $\mathcal{X}$ is isomorphic to a p -smoothable Banach space;
- 2) There is a constant C_p (depending on p) such that for all $\mathcal{X}$ -valued martingale $\{f_n, \mathcal{F}_n, n \geq 1\}$

$$\sup_n E[\|f_n\|^p] \leq C_p \sum_{i=1}^{\infty} E[\|f_{i+1} - f_i\|^p];$$

- 3) For every $\mathcal{X}$ -valued martingale difference $\{d_n\}$ satisfying one of the following conditions

a) $d_n \in L \log^+ L$, for $p = 1$,

b) $d_n \in L^p$, for $1 < p \leq 2$,

then $f_n = \sum_{i=1}^n d_i$ converge almost everywhere as $n \rightarrow \infty$.

Proof.

1) $\Leftrightarrow$ 2) is a result of Pisier (see (Pisier, 1975)).

2) $\Rightarrow$ 3) can be proved directly, see (Thang and Tien, 1982).

3) $\Rightarrow$ 1) is the result of Proposition 2.1. $\square$

Corollary 2.3. (Law of large numbers)

If $\mathcal{X}$ is a p -smoothable separable Banach space ($1 \leq p \leq 2$). $\{f_n, \mathcal{F}_n\}$ is a $\mathcal{X}$ -valued martingale.

(i) Suppose $\{d_n = f_n - f_{n-1}\} \in L \log^+ L$ (e.g. $\sup_n E[\|d_n\| \log^+ \|d_n\|] < \infty$) then $f_n = o(n)$ a.s. (e.g. $\lim_{n \rightarrow \infty} \frac{f_n}{n} = 0$, a.s.)

(ii) If $d_n \in L^p$, $1 < p < r \leq 2$ then $f_n = o(n^{\frac{1}{r}})$ a.s. (e.g. $\lim_{n \rightarrow \infty} \frac{f_n}{n^{\frac{1}{r}}} = 0$, a.s.)

3 Main results

In what below we give some limit theorems for weak set-valued martingale differences $\{D_n, \mathcal{F}_n, n \geq 1\}$. Let us first show the existence of martingale difference selection of a weak set-valued martingale difference by the following proposition.

Theorem 3.1. *Let $\{D_n, \mathcal{F}_n, n \geq 1\}$ is a sequence of set-valued random variables in $\mathcal{L}_{wkc}^1(\mathcal{F})$ such that*

$$\forall n \geq 1, \int_{\Omega} \sup_{f \in S_{D_{n+1}}^1(\mathcal{F}_{n+1})} \|E[f|\mathcal{F}_n]\| dP < \infty. \quad (3.1)$$

Then $\{D_n, \mathcal{F}_n, n \geq 1\}$ is a weak set-valued martingale difference if only if there exists a martingale difference selection of $\{D_n, \mathcal{F}_n, n \geq 1\}$.

Proof. If $\{D_n, \mathcal{F}_n, n \geq 1\}$ has a martingale difference selection, it is obvious that $0 \in S_{E[D_n|\mathcal{F}_{n-1}]}^1(\mathcal{F}_{n-1})$ because of the closeness of $S_{E[D_n|\mathcal{F}_{n-1}]}^1(\mathcal{F}_{n-1})$. Now we only need prove the remaining part which based on the methods of (Ezzaki, 1996).

Suppose $0 \in cl \left\{ E[f|\mathcal{F}_n] : f \in S_{D_{n+1}}^1(\mathcal{F}_{n+1}) \right\}$, then there exists a sequence $f_{n+1}^i \in S_{D_{n+1}}^1(\mathcal{F}_{n+1})$ such that $\lim_{i \rightarrow \infty} E[f_{n+1}^i|\mathcal{F}_n] = 0$ (the norm of $L_{\mathcal{X}}^1$).

For all $\omega \in \Omega$ we set

$$r_{n+1}(\omega) = d(0, D_{n+1}(\omega)) + \sup_{i \geq 1} \|E[f_{n+1}^i|\mathcal{F}_n](\omega)\|, \quad (3.2)$$

then r_{n+1} is $\mathcal{F}_{n+1}$ -measurable and integrable.

Let

$$R_{n+1}(\omega) = \overline{co} \left\{ f(\omega) : f \in S_{D_{n+1}}^1(\mathcal{F}_{n+1}) \text{ and } \|f(\omega)\| \leq r_{n+1}(\omega) \right\}.$$

Since $r_{n+1}(\omega) > d(0, D_{n+1}(\omega))$, $R_{n+1}(\omega) \neq \emptyset$ and R_{n+1} is also $\mathcal{F}_{n+1}$ -measurable and integrable.

We have

$$\int_{\Omega} \|R_{n+1}\|(\omega) P(d\omega) \leq \int_{\Omega} r_{n+1}(\omega) P(d\omega).$$

This inequality along with equations (3.1) and (3.3) yield $\int_{\Omega} \|R_{n+1}\|(\omega) P(d\omega) < \infty$.

By Theorem 2.1.37 (Molchanov and Molchanov, 2005), it follows that

$$B_n = \left\{ E[f|\mathcal{F}_n] : f \in S_{R_{n+1}}^1(\mathcal{F}_{n+1}) \right\} \text{ is closed in } L_{\mathcal{X}}^1.$$

Furthermore, it is obviously that for all $i \geq 1$, $E[f_{n+1}^i|\mathcal{F}_n] \in B_n$. Since $\lim_{i \rightarrow \infty} E[f_{n+1}^i|\mathcal{F}_n] = 0$, then $0 \in B_n$. Hence there exists $d_{n+1} \in S_{R_{n+1}}^1(\mathcal{F}_{n+1})$ such that $E[d_{n+1}|\mathcal{F}_n] = 0$. Let $d_1 \in S_{D_1}^1$ then $\{d_n, \mathcal{F}_n, n \geq 1\}$ is a martingale difference selection of $\{D_n, \mathcal{F}_n, n \geq 1\}$. $\square$

Theorem 3.2. (Strong law of large numbers) *Assume $\mathcal{X}$ be a separable, r -smoothable Banach space, ($1 < r \leq 2$). Suppose that $\{D_n, \mathcal{F}_n, n \geq 1\}$ is a weak set-valued martingale difference in $\mathcal{L}_{wkc}^1(\mathcal{F})$ such that $(D_n) \prec D \in \mathcal{L}_{wkc}^1(\mathcal{F})$ and (3.1) is hold. If for all $f \in S_D^1(\mathcal{F})$, one of following conditions is satisfied:*

$$(i) \quad f \in L \log^+ L$$

$$(ii) \quad f \in L^p, 1 < p < r \leq 2,$$

then

$$0 \in s\text{-}\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{k=1}^n D_k, \text{ a.s.}$$

Before going to the proof of the theorem, we need prove a lemma for the convergence of a $\mathcal{X}$ -valued martingale difference $\{f_n, \mathcal{F}_n, n \geq 1\}$ as below.

Let $\{a_{nk}\}$ be an array of real numbers such that

$$\begin{aligned} (i) \quad & \lim_{n \rightarrow \infty} a_{nk} = 0 \text{ for each } k \geq 1, \\ (ii) \quad & \sum_k |a_{nk}| \leq M_n \text{ for each } n \geq 1 \end{aligned}$$

where M_n is some constant. Set $d_n = \sum_{k=1}^n a_{nk} f_k$, we have the following lemma.

Lemma 3.3. Suppose that $b_n = \max_{k \leq n} |a_{nk}| \downarrow 0$ as $n \rightarrow \infty$ and let $\{f_n, \mathcal{F}_n, n \geq 1\}$ be a $\mathcal{X}$ -valued martingale difference taking values on p -smoothable separable Banach space $\mathcal{X}$ such that $(f_n) \prec f$, $E[N(\|f\|)] < \infty$, $N(x) = \text{card}\{n : b_n^{-1} \leq x\}$, $x \in \mathbb{R}^+$.

If

$$\int_1^\infty x P(\|f\| > x) \int_{y \geq x} \frac{N(y)}{y^3} dy dx < \infty \quad (3.3)$$

and

$$\int_1^\infty P(\|f\| > x) \int_{1 \leq y \leq x} \frac{N(y)}{y^2} dy dx < \infty \quad (3.4)$$

then $d_n \xrightarrow{\text{a.s.}} 0$ as $n \rightarrow \infty$

Proof. It is easy to see that

$$\begin{aligned} \left\| \sum_{k=1}^n a_{nk} f_k \right\| & \leq \sum_{k=1}^n |a_{nk}| \|f_k\| \\ & \leq \max_{k \leq n} |a_{nk}| \sum_{k=1}^n \|f_k\| = \frac{1}{b_n^{-1}} \sum_{k=1}^n \|f_k\|, \quad b_n^{-1} \uparrow \infty \text{ as } n \rightarrow \infty. \end{aligned}$$

Now, put

$$\begin{aligned} r_n &= b_n \sum_{k=1}^n \|f_k\|, \\ y_n &= f_n I(\|f_k\| \leq b_n^{-1}) \\ t_n &= b_n \sum_{k=1}^n \|y_k\|. \end{aligned}$$

Clearly

$$\begin{aligned}
 \sum_{k=1}^{\infty} P(\|f_k\| \neq \|y_k\|) &\leq \sum_{k=1}^{\infty} P(\|f_k\| > b_k^{-1}) \\
 &\leq C \sum_{k=1}^{\infty} P(\|f\| > b_n^{-1}) \\
 &= C \sum_{k=1}^{\infty} \int_{\{x > b_n^{-1}\}} dP(\|f\| \leq x) \\
 &= C \int_0^{\infty} N(x) dP(\|f\| \leq x) \\
 &= C E[N(\|f\|)] < \infty.
 \end{aligned}$$

Therefore, the sequences $\{t_n\}$ and $\{r_n\}$ converge on the same set and to the same limit. It suffices to show that $t_n \xrightarrow{a.s.} 0$ as $n \rightarrow \infty$.

We have

$$\begin{aligned}
 \sum_{k=1}^n b_k^2 E\|y_k - E[y_k | \mathcal{F}_{k-1}]\|^2 &\leq 2 \sum_{k=1}^{\infty} b_k^2 E[\|y_k\|^2] \\
 &= 4 \sum_{k=1}^{\infty} b_k^2 \int_{\{0 < x \leq b_k^{-1}\}} x P(\|f_k\| > x) dx \\
 &\leq 4C \sum_{k=1}^{\infty} b_k^2 \int_{\{0 < x \leq b_k^{-1}\}} x P(\|f\| > x) dx \\
 &\leq 4C \int_0^{\infty} x P(\|f\| > x) \sum_{\{k: b_k^{-1} \geq x\}} b_k^2 dx \\
 &\leq 8C \int_0^{\infty} x P(\|f\| > x) \int_x^{\infty} \frac{N(y)}{y^3} dy dx < \infty.
 \end{aligned}$$

The last inequality follows from the fact that

$$\begin{aligned}
 \sum_{\{k: b_k^{-1} \geq x\}} b_k^2 &= \lim_{u \rightarrow \infty} \sum_{\{k: x \leq b_k^{-1} < u\}} b_k^2 = \lim_{u \rightarrow \infty} \int_x^u \frac{dN(y)}{y^2} \\
 &= \lim_{u \rightarrow \infty} \left(\frac{N(u)}{u^2} - \frac{N(x)}{x^2} + 2 \int_x^u \frac{N(y)}{y^3} dy \right)
 \end{aligned}$$

and because

$$\frac{N(u)}{u^2} \leq 2 \int_u^{\infty} \frac{N(y)}{y^3} dy \rightarrow 0 \text{ as } u \rightarrow \infty.$$

Hence, in the view of the martingale convergence theorem, we get

$$\sum_{k=1}^n (y_k - E[y_k | \mathcal{F}_{k-1}]) \xrightarrow{a.s.} 0. \quad (3.5)$$

Note that

$$0 = E[f_n | \mathcal{F}_{n-1}] = E[y_n | \mathcal{F}_{n-1}] + E[f_n I(\|f_n\| > b_n^{-1}) | \mathcal{F}_{n-1}],$$

and

$$\begin{aligned}
 \sum_{k=1}^{\infty} b_k E[\|f_k\| I(\|f_k\| > b_k^{-1})] &= \sum_{k=1}^{\infty} b_k \int_{\{x > b_k^{-1}\}} P(\|f_k\| > x) \\
 &\leq C \sum_{k=1}^{\infty} b_k \int_{\{x > b_k^{-1}\}} P(\|f\| > x) dx \\
 &= C \int_1^{\infty} P(\|f\| > x) \sum_{\{k: 1 \leq b_k^{-1}\}} b_k dx \\
 &\leq C \int_1^{\infty} P(\|f\| > x) \int_1^x \frac{N(y)}{y^2} dy dx < \infty.
 \end{aligned}$$

Hence, by Kronecker's lemma we obtain

$$\sum_{k=1}^n E[y_k | \mathcal{F}_{k-1}] \xrightarrow{a.s.} 0. \quad (3.6)$$

which, together with (3.5), complete the proof. $\square$

Proof of Theorem 3.2. Since the result of Theorem 3.1, there exists a $\mathcal{X}$ -valued martingale difference selection $\{d_n, \mathcal{F}_n, n \geq 1\}$ of $\{D_n, \mathcal{F}_n, n \geq 1\}$.

Because of assumptions of $\{D_n, \mathcal{F}_n, n \geq 1\}$, for all $f \in S_D^1 \mathcal{F}$ we have $E[\|f\|] < \infty$ so $E[N(\|f\|)] < \infty$ and (3.3) holds.

On the other hand,

$$\int_1^{\infty} P(\|f\| > x) \int_1^x N(y)/y^2 dy dx = \int_1^{\infty} \log x P(\|f\| > x) dx = E[\|f\| \log \|f\|] < \infty,$$

so the condition (3.4) holds.

In the case of $f \in L^p$, $1 < p < 2$, it is easy to check that $N(y) \sim y^p$ so (3.3) and (3.4) hold.

By Corollary 2.3 and Lemma 3.3 for $\begin{cases} a_{nk} = \frac{1}{n} & \text{for } k \leq n, \\ a_{nk} = 0 & \text{for } k > n, \end{cases}$ we have $\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{i=1}^n d_i = 0$, a.s.

Since $\sum_{i=1}^n d_i \in \sum_{i=1}^n D_i$, a.s. then $0 \in s\text{-li} \frac{1}{n} \sum_{k=1}^n D_k$ a.s. $\square$

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References

- Castaing, C. and Valadier, M. 1977. Lecture notes in mathematics, *Convex analysis and measurable multifunctions* **580**.
- Chatterji, S. D. 1969. Martingale convergence and the radon-nikodym theorem in banach spaces, *Mathematica Scandinavica* **22**(1): 21–41.

- Elton, J. 1981. A law of large numbers for identically distributed martingale differences, *The Annals of Probability* pp. 405–412.
- Escanciano, J. C. and Lobato, I. N. 2009. Testing the martingale hypothesis, *Palgrave handbook of econometrics*, Springer, pp. 972–1003.
- Ezzaki, F. 1996. Mosco convergence of multivalued slln, *Vietnam journal of mathematics* 24(4): 399–416.
- Giné, E. 1978. A survey on the general central limit problem in banach spaces, *Séminaire Analyse fonctionnelle (dit "Maurey-Schwartz")* pp. 1–17.
- Giné, E. and Zinn, J. 1983. Central limit theorems and weak laws of large numbers in certain banach spaces, *Zeitschrift für Wahrscheinlichkeitstheorie und Verwandte Gebiete* 62(3): 323–354.
- Hall, P. and Heyde, C. C. 2014. *Martingale limit theory and its application*, Academic press.
- Hess, C. 1994. Multivalued strong laws of large numbers in the slice topology. application to integrands, *Set-Valued Analysis* 2(1-2): 183–205.
- Hoeffding, W., Robbins, H. et al. 1948. The central limit theorem for dependent random variables, *Duke Mathematical Journal* 15(3): 773–780.
- Hoffmann-Jorgensen, J., Pisier, G. et al. 1976. The law of large numbers and the central limit theorem in banach spaces, *The Annals of Probability* 4(4): 587–599.
- Li, S. and Ogura, Y. 2006. Strong laws of large numbers for independent fuzzy set-valued random variables, *Fuzzy Sets and Systems* 157(19): 2569–2578.
- Li, S., Ogura, Y. and Kreinovich, V. 2013. *Limit theorems and applications of set-valued and fuzzy set-valued random variables*, Vol. 43, Springer Science & Business Media.
- Lin, H. and Jiang, L. 2015. Bayesian option pricing using stochastic volatility models with fat-tailed errors, *International Journal Of Applied Mathematics & Statistics* 53(6): 288–296.
- Millet, A. and Sucheston, L. 1981. On regularity of multiparameter amarts and martingales, *Probability Theory and Related Fields* 56(1): 21–45.
- Molchanov, I. and Molchanov, I. S. 2005. *Theory of random sets*, Vol. 87, Springer.
- Mourier, E. 1972. L-random elements and l^* -random elements in banach spaces, *Proc. Third Berkeley Sympos. on Math. Statist. and Prob.*, Vol. 2, pp. 231–242.
- Pisier, G. 1975. Martingales with values in uniformly convex spaces, *Israel Journal of Mathematics* 20(3-4): 326–350.
- Shen, L. and Guan, L. 2016. Strong laws of large numbers for fuzzy set-valued random variables in g_α space, *Advances in Pure Mathematics* 6(09): 583.

- Smythe, R. 1974. Sums of independent random variables on partially ordered sets, *The Annals of Probability* pp. 906–917.
- Taylor, R. 1985. A strong law of large numbers for random sets in banach spaces, *Bull. Instit. Math. Academia Sinica* 13: 403–409.
- Thang, D. H. and Tien, N. D. 1982. On convergence of martingales and geometric properties of banach spaces, *Theory of Probability & Its Applications* 26(2): 379–385.
- Woyczyński, W. 1975. Geometry and martingales in banach spaces, *Probability-Winter School*, Springer, pp. 229–275.
- Woyczyński, W. 1976. Asymptotic behavior of martingales in banach spaces, *Probability in Banach Spaces*, Springer, pp. 273–284.
- Woyczynski, W. A. 1980. On marciniewicz-zygmund laws of large numbers in banach spaces and related rates of convergence, *Probab. Math. Statist* 1(2): 117–131.
- Yang, Y. and Jin, H. 2013. Minimizing of shortfall risk in a jump-diffusion model with continuous dividends, *International Journal of Applied Mathematics and Statistics* 47(17): 440–448.
- Yen, V. V. 2004. On convergence of multiparameter multivalued martingales, *Acta Mathematica Vietnamica* 29: 177–184.
- Zheng, Y. and Yang, Y. 2016. Option pricing model on dividend-paying securities, *International Journal Of Applied Mathematics & Statistics* 54(2): 123–131.